Tutorial Problems #12

MAT 292 – Calculus III – Fall 2014

Solutions

5.7 - # 15 Consider the initial value problem

$$y'' + \gamma y' + y = k\delta(t-1), \quad y(0) = 0, \quad y'(0) = 0$$

where $|k|$ is the magnitude of an impulse at $t = 1$ and γ is the damping coefficient (or resistance)

(a) Let $\gamma = 1/2$. Find the value of k for which the response has a peak value of 2; call this value k_1

Let's solve the ODE to find the peak value in terms of k. Take the Laplace transform to obtain

$$\mathcal{L}\{y''\} + \gamma\mathcal{L}\{y'\} + \mathcal{L}\{y\} = k\mathcal{L}\{\delta(t-1)\} \implies \mathcal{L}\{y\}(s^2 + \gamma s + 1) = ke^{-s}$$

$$\therefore \mathcal{L}\{y\} = \frac{ke^{-s}}{s^2 + \gamma s + 1}$$

If we complete the square we see

$$\mathcal{L}\{y\} = \frac{ke^{-s}}{\sqrt{1-\gamma^2/4}} \frac{\sqrt{1-\gamma^2/4}}{(s+\gamma/2)^2 + 1 - \gamma^2/4}$$

Recall that

$$\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2 + b^2}$$

Clearly if $\gamma = 1/2$ we have

$$a = -\frac{1}{4} \quad \& \quad b = \frac{\sqrt{15}}{4}$$

Thus we see

$$y(t) = \frac{4k}{\sqrt{15}} e^{-(t-1)/4} \sin\left(\frac{\sqrt{15}}{4}(t-1)\right) H(t-1)$$

So the response should max out around

$$y_{max} = 2 \approx \frac{4k}{\sqrt{15}} e^{-\pi/8} \implies k_1 \approx \frac{\sqrt{15}}{2} e^{\pi/8}$$

(b) Repeat a) for $\gamma = 1/4$ Following the previous question, we see this implies

$$a = -\frac{1}{8} \quad \& \quad b = \frac{\sqrt{63}}{8}$$

Thus...

$$y_{max} = 2 \approx \frac{8k}{\sqrt{63}} e^{-\pi/16} \implies k_1 \approx \frac{\sqrt{63}}{4} e^{\pi/16}$$

(c) Determine how k_1 varies as γ decreases. What is the value of k_1 when $\gamma = 0$?

We see that k_1 decreases as γ decreases. The value of k_1 when $\gamma = 0$ is just the case when

$$a = 0 \quad \& \quad b = 1$$

Thus

$$y_{max} = 2 = k \implies k_1 = 2$$

5.7 - #25c Solve the IVP using the Laplace Trasnform

$$y'' + 2y' + 2y = \delta(t - \pi), \quad y(0) = 0, y'(0) = 0$$

Solution Apply the Laplace Transform to the ODE,

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\delta(t - \pi)\}$$

Recall that

$$\mathcal{L}\{\delta(t - c)\} = e^{-cs} \quad \& \quad \mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$$

Therefore the above equation may be rewritten as

$$\mathcal{L}\{y\} = \frac{e^{-\pi s}}{s^2 + 2s + 2}$$

By completing the square in the denominator, we see

$$\mathcal{L}\{y\} = \frac{e^{-\pi s}}{(s + 1)^2 + 1}$$

Recall that

$$\mathcal{L}\{H(t - c)f(t - c)\} = e^{-cs}\mathcal{L}\{f(t)\} \quad \& \quad \mathcal{L}\{e^{at} \sin bt\} = \frac{b}{(s - a)^2 + b^2}$$

Therefore we see the inverse transform gives

$$y(t) = e^{-(t-\pi)} \sin(t - \pi)H(t - \pi)$$

□

9.5 - # 1 Determine whether separation of variables can be used in the PDE. If so, find the separated ODE's.

$$xu_{xx} + u_t = 0$$

Solution We see the answer is yes here since if $u(x, t) = X(x)T(t)$, we obtain

$$xX''T + XT' = 0 \implies \frac{xX''}{X} + \frac{T'}{T} = 0 \implies \underbrace{\frac{xX''}{X}}_{f(x)} = -\underbrace{\frac{T'}{T}}_{g(t)}$$

since both functions (f and g) do not depend on each other, they must be constant. Call the constant $\lambda \in \mathbb{R}$, thus

$$\boxed{xX'' = \lambda X \quad \& \quad T' = -\lambda T}$$

are the ODE's we seek.

9.5 - # 3 Determine whether separation of variables can be used in the PDE. If so, find the separated ODE's.

$$u_{xx} + u_{xt} + u_t = 0$$

Solution We again see the answer is yes here. If we try $u(x,t) = X(x)T(t)$, we'll obtain

$$\frac{X''}{X} + \frac{X'T'}{XT} + \frac{T'}{T} == \frac{X''}{X} + \frac{T'}{T}\left(\frac{X'}{X} + 1\right) = 0 \implies \frac{X''}{X' + X} = -\frac{T'}{T}$$

thus we see the above both equal some constant $\lambda \in \mathbb{R}$, hence

$$\boxed{X'' = \lambda(X' + X) \quad \& \quad T' = -\lambda T}$$

are the ODE's we seek.

9.5 - # 5 Determine whether separation of variables can be used in the PDE. If so, find the separated ODE's.

$$u_{xx} + (x + y)u_{yy} = 0$$

Solution We won't be able to decouple the system due to the mixed variables. If we try $u(x,y) = X(x)Y(y)$, we see

$$\frac{X''}{X} + (x + y)\frac{Y''}{Y} = 0$$

which can not be decoupled.