Tutorial Problems #11

MAT 292 – Calculus III – Fall 2015

Solutions

5.4 - # 7 Solve using the Laplace Transform,

$$y'' + \omega^2 y = \cos 2t, \quad y(0) = 1, y'(0) = 0, \quad \omega^2 \neq 4$$

Solution Apply the Laplace transform to the ODE,

$$\mathcal{L}\{y''\} + \omega^2 \mathcal{L}\{y\} = \mathcal{L}\{\cos 2t\}$$

Recall

$$\mathcal{L}\{y''\} = s^2 \mathcal{L}\{y\} - sy(0) - y'(0) \quad \& \quad \mathcal{L}\{\cos bt\} = \frac{s}{s^2 + b^2}$$

Thus we may rewrite the above equation as

$$\mathcal{L}\{y\} = \frac{s}{s^2 + \omega^2} + \frac{s}{(s^2 + \omega^2)(s^2 + 4)}$$

If we perform a partial fraction decomposition on the second term, we see

$$\mathcal{L}\{y\} = \frac{s}{s^2 + \omega^2} + \frac{1}{\omega^2 - 4}\left(\frac{s}{s^2 + 4} - \frac{s}{s^2 + \omega^2}\right)$$

It's easy to apply $\mathcal{L}^{-1}$ since we know exactly what function corresponds to every term, thus

$$y = \cos \omega t + \frac{1}{\omega^2 - 4}\left(\cos 2t - \cos \omega t\right)$$

$\square$

5.5 - # 8 Find $\mathcal{L}\{f\}$ for

$$f(t) = \begin{cases} 0 & t < 1 \\ t^2 - 2t + 2 & t \geqslant 1 \end{cases}$$

Solution Notice that if we define $g(x) = x^2 + 1$, we see that

$$g(t-1) = (t-1)^2 + 1 = t^2 - 2t + 2$$

Thus we may rewrite f as

$$f(t) = g(t-1)H(t-1)$$

Recall that

$$\mathcal{L}\{g(t-c)H(t-c)\} = e^{-cs}\mathcal{L}\{g(t)\} \quad \& \quad \mathcal{L}\{t^2\} = \frac{2}{s^3} \quad \& \quad \mathcal{L}\{1\} = \frac{1}{s}$$

Thus

$$\mathcal{L}\{g(t-1)H(t-1)\} = e^{-s}\mathcal{L}\{g(t)\} = e^{-s}\left(\mathcal{L}\{t^2\} + \mathcal{L}\{1\}\right) = e^{-s}\left(\frac{2}{s^3} + \frac{1}{s}\right)$$

□

Lemma If $f(t) = f(t+T)$ and is piecewise continuous on $[0, T]$, then

$$\mathcal{L}\{f\} = \frac{\int_0^T e^{-st} f(t)dt}{1 - e^{-sT}}$$

Proof Without the loss of generality assume $f = 0$ when $t < 0$. The first period of the function f may be isolated as

$$f_T(t) = f(t)\big(1 - H(t-T)\big) = \begin{cases} f & 0 \leqslant t \leqslant T \\ 0 & else \end{cases}$$

The kth period of the function f may be isolated as $f_T(t-kT)H(t-kT)$. Now consider the Laplace transform of the first n periods,

$$\begin{aligned}\mathcal{L}\{f_{nT}\} &= \int_0^{nT} e^{-st} f(t)dt \\ &= \sum_{k=0}^{n-1} \mathcal{L}\{f_T(t-kT)H(t-kT)\} \\ &= \sum_{k=0}^{n-1} e^{-kTs}\mathcal{L}\{f_T(t)\} \\ &= \mathcal{L}\{f_T(t)\}\sum_{k=0}^{n-1}(e^{-sT})^k \\ &= \mathcal{L}\{f_T(t)\}\frac{1 - e^{-s(n-1)T}}{1 - e^{-sT}}\end{aligned}$$

Taking the limit as $n \to \infty$ recovers f, and we see

$$\mathcal{L}\{f(t)\} = \lim_{n\to\infty} \mathcal{L}\{f_{nT}\} = \lim_{n\to\infty} \mathcal{L}\{f_T(t)\}\frac{1 - e^{-s(n-1)T}}{1 - e^{-sT}} = \frac{\mathcal{L}\{f_T(t)\}}{1 - e^{-sT}}$$

□

5.5 - # 22 Find $\mathcal{L}\{f\}$ where

$$f(t) = \begin{cases} 1 & 0 \leqslant t < 1 \\ -1 & 1 \leqslant t < 2 \end{cases} \qquad f(t+2) = f(t)$$

Solution Use the previous formula:

$$\mathcal{L}\{f\} = \frac{\int_0^T e^{-st} f(t)dt}{1 - e^{-sT}}$$

We see $T = 2$, and

$$\mathcal{L}\{f\} = \frac{\int_0^2 e^{-st} f(t)dt}{1 - e^{-2s}} = \frac{\int_0^1 e^{-st}dt - \int_1^2 e^{-st}dt}{1 - e^{-2s}} = \frac{e^{-2s} - 2e^{-s} + 1}{1 - e^{-2s}}$$

□

5.6 - #21 Solve using the Laplace Transform,

$$y'' + y = g = 1 + \sum_{k=1}^{n} (-1)^k H(t - k\pi), \quad y(0) = 0, \quad y'(0) = 0$$

Solution Apply the Laplace Transform on the ODE, we obtain by linearity

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{1\} + \sum_{k=1}^{n} (-1)^k \mathcal{L}\{H(t - k\pi)\}$$

Using our previous table, we see

$$\mathcal{L}\{y\} = \frac{1}{s(s^2+1)} + \sum_{k=1}^{n} (-1)^k \frac{e^{-k\pi s}}{s(s^2+1)}$$

A partial fractions decomposition gives

$$\mathcal{L}\{y\} = \frac{1}{s} - \frac{s}{s^2+1} + \sum_{k=1}^{n} (-1)^k e^{-k\pi s} \left(\frac{1}{s} - \frac{s}{s^2+1} \right)$$

Using the table again, we see the solution is given by

$$y = 1 - \cos t + \sum_{k=1}^{n} (-1)^k (1 - \cos(t - k\pi)) H(t - k\pi)$$

□