Tutorial Problems #10

MAT 292 – Calculus III – Fall 2015

SOLUTIONS

5.1 - # 15 Find the Laplace Transform of

$$f(t) = \begin{cases} 0, & 0 \leqslant t \leqslant 1 \\ 1, & 1 < t \leqslant 2 \\ 0, & 2 < t \end{cases}$$

Solution By direct computation we have

$$\mathcal{L}\{f(t)\} = \int_1^2 e^{-st}dt = \frac{-1}{s}e^{-st}\Big|_1^2 = \frac{e^{-s} - e^{-2s}}{s}$$

$\square$

5.1 - # 35 Prove Corollary[5.1.7]: If $f(t)$ is piecewise continuous on $[0, A]$ for any positive A, and f is of exponential order, that is, there exists real constants $M \geqslant 0$,$K > 0$, and a such that $|f(t)| \leqslant Ke^{at}$ when $t \geqslant M$. Then

$$|F(s)| \leqslant \frac{L}{s}$$

for some constant L as $s \to \infty$. i.e.

$$\lim_{s\to\infty} F(s) = 0$$

Proof We know that $F(s)$ exists for $s > a$ via theorem 5.1.6. So let's take $M \geqslant 0$, and split the integral in the definition of the Laplace transform,

$$|F(s)| = \left|\int_0^\infty f(t)e^{-st}dt\right| = \left|\int_0^M f(t)e^{-st}dt + \int_M^\infty f(t)e^{-st}dt\right| \leqslant \int_0^M |f(t)|e^{-st}dt + \int_M^\infty |f(t)|e^{-st}dt$$

Since $f(t)$ is bounded on $[0, M]$, and of exponential order, we have the following bound

$$|F(s)| \leqslant \max_{t\in[0,M]} |f(t)| \int_0^M e^{-st}dt + K\int_M^\infty e^{at}e^{-st}dt = \max_{t\in[0,M]} |f(t)| \frac{1 - e^{-sM}}{s} + \frac{K}{s-a}e^{-(s-a)M}$$

Now we note that

$$1 - e^{-sM} \leqslant 1 \quad \& \quad \frac{K}{s-a}e^{-(s-a)M} = \frac{1}{s}\left(\frac{K}{1 - a/s}e^{-(s-a)M}\right) \leqslant \frac{K_1}{s}$$

since $a/s < 1$. Thus

$$|F(s)| \leqslant \frac{\max_{t\in[0,M]} |f(t)| + K_1}{s} = \frac{L}{s}$$

$\square$

5.2- # 11 Let $F(s) = \mathcal{L}\{f(t)\}$, where $f(t)$ is piecewise continuous and of exponential order on $[0, \infty)$. Show that

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s)$$

Solution Recall that we have

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

via integration by parts since $f(t)$ is piecewise continuous and of exponential order. Define

$$g(t) = \int_0^t f(\tau)d\tau \implies g'(t) = f(t)$$

Hence, by plugging this into the above formula we have

$$\mathcal{L}\{f(t)\} = s\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} \implies \mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$

□

5.2 - #26 A tank originally contains 100 gal of fresh water. Then water containing 1/2 lb of salt per gal is poured into the tank at a rate of 2 gal/min, and the mixture is allowed to leave at the same rate. After 10 min the salt water solution flowing into the tank suddenly switches to fresh water flowing in at a rate of 2 gal/min, while the solution continues to leave the tank at the same rate. Find the Laplace transform of the amount of salt $y(t)$ in the tank.

Solution We see the amount of salt in the tank is begin at $t = 0$ min, so $y(0) = 0$. We see the change in salt is given by

$$y' = \begin{cases} \frac{1}{2} * 2 - 2y & t \in [0, 10] \\ -2y & t > 10 \end{cases}$$

since volume is constant. We see that y is given by

$$y(t) = \begin{cases} \frac{1-e^{-2t}}{2} & t \in [0, 10] \\ \frac{e^{20}-1}{2}e^{-2t} & t > 10 \end{cases}$$

Now we compute the Laplace Transform

$$\begin{aligned} \mathcal{L}\{y\} &= \int_0^{10} e^{-st}\frac{1-e^{-2t}}{2}dt + \int_{10}^{\infty} \frac{e^{20}-1}{2}e^{-2t}e^{-st}dt \\ &= \frac{1-e^{-10s}}{2s} + \frac{e^{-10(s+2)}-1}{2(s+2)} + (e^{20}-1)\frac{e^{-10(s+2)}}{2(s+2)} \\ &= \frac{1-e^{-10s}}{2s} - \frac{1-e^{-10s}}{2(s+2)} \\ &= \frac{1-e^{-10t}}{s(s+2)} \end{aligned}$$

Note you could have perform the Laplace transform directly on the ODE to obtain

$$\mathcal{L}\{y'\} = \int_0^{10} e^{-st}dt - 2\mathcal{L}\{y\} \implies \mathcal{L}\{y\}(s+2) = \frac{1-e^{-10t}}{s} \implies \boxed{\mathcal{L}\{y\} = \frac{1-e^{-10t}}{s(s+2)}}$$

□

5.3 - #27 A damped oscillator with mass m, damping constant γ, and spring constant k, is subjected to an external force $F(t) = F_0 t$ over the time interval $0 \leqslant t \leqslant T$. The external force is then removed at time T. Find the Laplace transform of the displacement $y(t)$ of the mass, assuming that the oscillator is initially in the equilibrium state.

Solution We see the system is given by Hook's Law with damping, i.e.

$$my'' = \begin{cases} -ky - \gamma y' + F_0 t & t \in [0, T] \\ -ky - \gamma y' & t > T \end{cases}$$

The initial data is $y(0) = 0, y'(0) = 0$ since the system is in equilibrium. Let's apply the Laplace transform directly this time, we see

$$\begin{aligned} m\mathcal{L}\{y''\} =& -k\mathcal{L}\{y\} - \gamma\mathcal{L}\{y'\} + F_0 \int_0^T te^{-st} \\ \implies ms^2\mathcal{L}\{y\} =& -k\mathcal{L}\{y\} - \gamma s\mathcal{L}\{y\} + F_0 \frac{1 - e^{-sT}(sT+1)}{s^2} \\ \implies \mathcal{L}\{y\} =& F_0 \frac{1 - e^{-sT}(sT+1)}{s^2(ms^2 + \gamma s + k)} \end{aligned}$$

To find the solution y from this, we'll use injectivity and linearity of the Laplace transform. Performing a partial fraction decomposition, we see

$$\begin{aligned} \frac{1}{s^2(ms^2 + \gamma s + k)} =& \frac{\gamma^2 + \gamma ms - km}{k^2(ms^2 + \gamma s + k)} - \frac{\gamma}{k^2 s} + \frac{1}{ks^2} \\ =& \frac{1}{k^2}\left[\frac{\gamma(s + \gamma/2m)}{(s + \gamma/2m)^2 - \gamma^2/4m + k/m} + \frac{\gamma/2m^2 - k}{(s + \gamma/2m)^2 - \gamma^2/4m^2 + k/m} - \frac{\gamma}{s} + \frac{k}{s^2}\right] \end{aligned}$$

To simplify the above, let's introduce some notation. Let $\eta = -\gamma/2m$, and $\xi = \sqrt{k/m - \gamma^2/4m}$. Then we see the Laplace Transform may be written as

$$\begin{aligned} \mathcal{L}\{y\} = \frac{F_0}{k^2}\Big[\frac{\gamma(s-\eta)}{(s-\eta)^2 + \xi^2} + \frac{\gamma/2m^2 - k}{\xi}\frac{\xi}{(s-\eta)^2 + \xi^2} - \frac{\gamma}{s} + \frac{k}{s^2}\Big] - \\ - \frac{F_0}{k^2} e^{-sT}(sT+1)\left[\frac{\gamma(s-\eta)}{(s-\eta)^2 + \xi^2} + \frac{\gamma/2m^2 - k}{\xi}\frac{\xi}{(s-\eta)^2 + \xi^2} - \frac{\gamma}{s} + \frac{k}{s^2}\right] \end{aligned}$$

As we can see, injectivity implies the solution y contains the Heaviside function, $e^{\eta t}\cos(\xi t)$, $e^{\eta t}\sin(\xi t)$, constant terms, and linear terms. It is very messy to continue in full generalities, so we leave it to the reader to push through and match the terms.

□