Tutorial Problems #9

MAT 292 – Calculus III – Fall 2015

SOLUTIONS

Section 4.5, # 12) Solve

$$y'' - y' - 2y = \cosh(2t).$$

Before we can solve the non-homogeneous problem we need to know the general solution of the corresponding homogeneous problem. In this case, we solve

$$w'' - w' - 2w = 0.$$

Looking for solutions in the form $w = e^{\lambda t}$ we find the characteristic equation for the homogeneous problem:

$$\lambda^2 - \lambda - 2 = 0 \, (\lambda - 2)(\lambda + 1) \qquad = 0$$

Therefore $w = c_1 e^{2t} + c_2 e^{-t}$ is the general solution to the homogeneous problem. We now write

$$\cosh(2t) = \frac{1}{2}\left(e^{2t} + e^{-2t}\right)$$

to see that the inhomogeneity falls into the framework of the much simpler method of undetermined coefficients. Here, to find a particular solution to the equation

$$Y_1'' - Y_1' - 2Y_1 = \frac{1}{2}e^{-2t}$$

we can simply guess $Y_1 = Ae^{-2t}$ and solve for the coefficient A. Since $Y_1' = -2Ae^{-2t}$ and $Y_1'' = 4ae^{-2t}$, we use the equation to find that

$$\begin{aligned} (4A + 2A - 2A)\, e^{-2t} &= \frac{1}{2}e^{-2t} \\ 4A &= \frac{1}{2} \\ A &= \frac{1}{8}. \end{aligned}$$

Things are a little bit more subtle for the other term in the inhomogeneity. To find a particular solution to

$$Y_2'' - Y_2' - 2Y_2 = \frac{1}{2}e^{2t}$$

we cannot make the guess $Y_2 = Ae^{2t}$. This is because the function e^{2t} is a solution of the homogeneous problem and so any choice of A will give zero on the rhs. Instead we make the guess $Y_2 = Bte^{2t}$ (these are the types of solutions we saw with repeating eigenvalues for example). Computing

$$\begin{aligned} Y_2' &= (B + 2Bt)\, e^{2t} \\ Y_2'' &= (4B + 4Bt)\, e^{2t} \end{aligned}$$

we use the equation to solve for B to get

$$\begin{aligned} Y_2'' - Y_2' - 2Y_2 &= (4B + 4Bt - B - 2Bt - 2Bt)\, e^{2t} \\ &= 3Be^{2t} = \frac{1}{2}e^{2t}. \end{aligned}$$

Therefore $Y_2 = \frac{1}{6}te^{2t}$ is a particular solution to this equation. Using the linearity of the equation, we have a particular solution to the original problem:

$$\begin{aligned} Y &= Y_1 + Y_2 \\ &= \frac{1}{8}e^{-2t} + \frac{1}{6}e^{2t} \end{aligned}$$

Finally the general solution to the inhomogeneous differential equation can be found by adding the general solution of the homogeneous one:

$$y = c_1 e^{2t} + c_2 e^{-t} + \frac{1}{8}e^{-2t} + \frac{1}{6}te^{2t}$$

where the constants c_1 and c_2 are fixed by initial conditions.

Section 4.5, # 33 Find the general solution of

$$y'' + y = \begin{cases} t, & 0 \leqslant t \leqslant \pi \\ \pi e^{\pi - t}, & t > \pi. \end{cases}$$

As always we first find the general solution to the homogeneous problem:

$$w'' + w = 0 \implies w = c_1 \sin t + c_2 \cos t$$

Since the inhomogeneity is a piecewise defined function, the right approach to take is to solve the problem separately on the two subdomains $t \in [0, \pi]$ and $t > \pi$ and then ensure that the solution is continuous and differentiable at the point $t = \pi$.

Region 1: $0 \leqslant t \leqslant \pi$:

In this region we find the general solution $y_<$ to the problem

$$y''_< + y <= t$$

We guess a particular solution in the form $Y_< = At + B$ and use the equation to trivially find $A = 1$ and $B = 0$. Therefore the general solution in this region is given by

$$y <= c_1^< \sin t + c_2^< \cos t + t$$

Region 2: $t > \pi$

In this region, we find the general solution $y_>$ to the problem

$$y_>'' + y_> = \pi e^{\pi - t}.$$

We guess a particular solution in the form $Y_> = Ae^{-t}$. Then $Y_>'' = Ae^{-t}$ and we have that

$$Y_>'' + Y_> = (2A)e^{-t} = \pi e^{\pi - t}$$
$$A = \frac{1}{2}\pi e^{\pi}.$$

Therefore the general solution in this region is given by

$$y_> = c_1^> \sin t + c_2^> \cos t + \frac{1}{2}\pi e^{\pi - t}$$

Patching $y_<$ and $y_>$

We have found the general solution is

$$y(t) = \begin{cases} c_1^< \sin t + c_2^< \cos t + t, & 0 \leqslant t \leqslant \pi \\ c_2^> \sin t + c_2^> \cos t + \frac{1}{2}\pi e^{t-\pi}, & t > \pi \end{cases}$$

with 4 undetermined constants. We need to ensure that at $t = \pi$, the solution is both continuous and differentiable. Evaluating at $t = \pi$:

$$y_<(\pi) = -c_2^< + \pi$$
$$y_>(\pi) = -c_2^> + \frac{1}{2}\pi$$

implies that we must set $c_2^> = c_2^< - \frac{\pi}{2}$.
Similarly, evaluating the derivatives at $t = \pi$:

$$y_<'(\pi) = -c_1^< + 1$$
$$y_>'(\pi) - -c_1^> - \frac{1}{2}\pi$$

implies $c_1^> = c_1^< - 1 - \frac{1}{2}\pi$. Therefore the general solution to the original equation is given by

$$y(t) = \begin{cases} c_1 \sin t + c_2 \cos t + t, & 0 \leqslant t \leqslant \pi \\ \left(c_1 - \left(1 + \frac{\pi}{2}\right)\right) \sin t + \left(c_2 - \frac{\pi}{2}\right) \cos t + \frac{1}{2}\pi e^{\pi - t}, & t > \pi \end{cases}$$

Section 4.7, # 39 This problem is an alternative to the horrible derivation of the variation of parameters formula. Suppose you want to solve

$$y'' + py' + qy = F$$

and you know that the corresponding homogeneous problem

$$w'' + pw' + qw = 0$$

has the general solution $w(t) = c_1 y_1(t) + c_2 y_2(t)$. We make a change of variables to reduce the non-homogeneous second order equation to a first order equation that we can solve easily. To this end, set

$$y(t) = v(t) y_1(t)$$

and find the equation satisfied by $v(t)$. We compute

$$\begin{aligned} y &= vy_1 \\ y' &= v'y_1 + vy_1' \\ y'' &= v''y_1 + 2v'y_1' + vy_1''. \end{aligned}$$

Plugging these into the equation we have

$$v''y_1 + 2v'y_1' + \underbrace{vy_1''} + p\left(v'y_1 + \underbrace{vy_1'}\right) + \underbrace{qvy_1} = F$$

Note that the underlined terms add up to zero:

$$\begin{aligned} vy_1'' + pvy_1' + qvy_1 &= v\left(y_1'' + py_1' + qy_1\right) \\ &= 0 \end{aligned}$$

because y_1 is a solution to the homogeneous problem. We use the method of integrating factors to solve the resulting first order equation for v'. That is, after removing the underlined terms, we have

$$\begin{aligned} v''y_1 + v'\left(2y_1' + py_1\right) &= F \\ v'' + v'\left(2\frac{y_1'}{y_1} + p\right) &= \frac{F}{y_1} \\ \left(e^{\int\left(2\frac{y_1'}{y_1}+p\right)}v'\right)' &= e^{\int\left(2\frac{y_1'}{y_1}+p\right)}\frac{F}{y_1}. \end{aligned}$$

This looks ugly but simplifies a lot when you notice that

$$\begin{aligned} \int 2\frac{y_1'}{y_1} &= 2\ln y_1 \\ &= \ln y_1^2 \end{aligned}$$

so that

$$e^{\int 2\frac{y_1'}{y_1}} = y_1^2.$$

We thus have

$$\left(e^{\int p}y_1^2 v'\right)' = e^{\int p}y_1 F.$$

Even further, recall Abel's theorem (something very easy to derive): the Wronskian $W(t)$ of any two solutions to the homogeneous problem can be written

$$W = Ce^{-\int p}$$

where the constant $C \neq 0$ if the two solutions are independent (that we assume is the case). Therefore, cancelling the C from both sides of the equation, we have

$$\left(\frac{y_1^2}{W}v'\right)' = \frac{y_1 F}{W}$$

Integrating both sides of this equation, we have that

$$v' = \frac{W}{y_1^2}\int\frac{Fy_1}{W}.$$

Great, now there is one more simplification to be made. Recall that the Wronskian was originally defined as a determinant:

$$\begin{aligned} W(t) &= \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \\ &= y_1 y_2' - y_2 y_1'. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{W}{y_1^2} &= \frac{y_1 y_2' - y_2 y_1'}{y_1^2} \\ &= \left(\frac{y_2}{y_1}\right)' \end{aligned}$$

where all we did was reverse the quotient rule for derivatives to get the last line. Therefore we can find v by integrating by parts

$$\begin{aligned} v &= \int \left(\frac{y_2}{y_1}\right)' \int \frac{F y_1}{W} \\ &= \frac{y_2}{y_1} \int \frac{F y_1}{W} - \int \frac{F y_1}{W} \frac{y_2}{y_1} \\ &= \frac{y_2}{y_1} \int \frac{F y_1}{W} - \int \frac{F y_2}{W}. \end{aligned}$$

Finally, we recall that $y = v y_1$ to obtain a particular solution to the original non-homogeneous equation:

$$y(t) = -y_1 \int \frac{F y_2}{W} + y_2 \int \frac{F y_1}{W}$$

which is exactly the same as the variation of parameters equation derived in the chapter.
This method is a bit more computational but has the advantage that it works for any form of the inhomogeneity. (The easier method of the undetermined coefficients only works easily when $F = \sin, \cos, e^{\pm at}$, when F is a polynomial or F is a sum or product of these functions)