Tutorial Problems #7

MAT 292 – Calculus III – Fall 2015

Solutions

Problems 3.5.13 and 3.5.15 Given a system

$$\mathbf{x}' = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mathbf{x}$$

we want to know the qualitative behaviour of solutions $\mathbf{x}(t)$ for any values of the matrix coefficients a_{ij}. Recall that these are determined by the eigenvalues of the matrix $\mathbf{A} = a_{ij}$. To this end we solve the equation

$$\det \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} = 0.$$

Expanding the determinant, we have a quadratic equation for the eigenvalues λ:

$$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

Using the quadratic formula, we find two eigenvalues λ_+ and λ_- given by

$$\begin{aligned} \lambda_\pm &= \frac{1}{2}(a_{11} + a_{22}) \pm \frac{1}{2}\sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})} \\ &= \frac{1}{2}(a_{11} + a_{22}) \left[1 \pm \sqrt{1 - 4\frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22})^2}} \right] \end{aligned}$$

Recall that the trace $\mathrm{tr}\mathbf{A} = a_{11} + a_{22}$ and determinant $\det \mathbf{A} = a_{11}a_{22} - a_{12}a_{21}$ so that

$$\lambda_\pm = \frac{1}{2}\mathrm{tr}\mathbf{A} \left(1 \pm \sqrt{1 - 4\frac{\det \mathbf{A}}{\mathrm{tr}\mathbf{A}^2}} \right)$$

We now ask the following questions:

1. **When do all solutions tend to zero as $t \to \infty$?**

   Well recall that for solutions to tend to zero, it must be the case that $\Re\lambda < 0$ that is the real part of the eigenvalue is negative.

   (a) $\mathrm{tr}\mathbf{A} \geqslant 0$

   In this case it is easy to see that at least one solution will not tend to zero. Indeed, if $\mathrm{tr}\mathbf{A} = 0$, we have a center (see case) and if $\mathrm{tr}\mathbf{A} > 0$ then either

$$1 - 4\frac{\det \mathbf{A}}{\mathrm{tr}\mathbf{A}^2} < 0 \implies \text{Unstable Spiral}$$

or

$$1 - 4\frac{\det \mathbf{A}}{\mathrm{tr}\mathbf{A}^2} > 0 \implies \lambda_+ > 0$$

(b) $\mathrm{tr}\mathbf{A} < 0$

In this case we just need to ensure $1 - 4\frac{\det \mathbf{A}}{\mathrm{tr}\mathbf{A}^2} < 0$. That is because

$$1 - 4\frac{\det \mathbf{A}}{\mathrm{tr}\mathbf{A}^2} > 1 \implies \lambda_- > 0$$

However

$$1 - 4\frac{\det \mathbf{A}}{\mathrm{tr}\mathbf{A}^2} < 0 \iff \det \mathbf{A} > 0$$

We conclude that all solutions will tend to zero if and only if $\mathrm{tr}\mathbf{A} < 0$ and $\det \mathbf{A} > 0$.

2. **When do we have a node?**

Recall that we have node when either both eigenvalues are positive or both are negative.

Observe that if $\det \mathbf{A} > 0$ and $1 - 4\frac{\det \mathbf{A}}{\mathrm{tr}\mathbf{A}^2} \geqslant 0$ then

$$1 > \sqrt{1 - 4\frac{\det \mathbf{A}}{\mathrm{tr}\mathbf{A}^2}} \geqslant 0$$

and

$$1 \pm \sqrt{1 - 4\frac{\det \mathbf{A}}{\mathrm{tr}\mathbf{A}^2}} > 0$$

Therefore if $\mathrm{tr}\mathbf{A} > 0$ then both λ_+ and λ_- are positive. Else if $\mathrm{tr}\mathbf{A} < 0$ then both λ_+ and λ_- are negative. Therefore we have a stable or unstable node depending on the sign of $\mathrm{tr}\mathbf{A}$.

3. **When do we have a saddle point?**

Recall that we have a saddle point when one eigenvalue is positive and the other is negative.

Suppose that $\det \mathbf{A} < 0$. Then $1 - 4\frac{\det \mathbf{A}}{\mathrm{tr}\mathbf{A}^2} > 1$ and we have that

$$1 + \sqrt{1 - 4\frac{\det \mathbf{A}}{\mathrm{tr}\mathbf{A}^2}} > 0$$
$$1 - \sqrt{1 - 4\frac{\det \mathbf{A}}{\mathrm{tr}\mathbf{A}^2}} < 0.$$

So if $\mathrm{tr}\mathbf{A} > 0$ then $\lambda_+ > 0$ and $\lambda_- < 0$. Meanwhile if $\mathrm{tr}\mathbf{A} < 0$ then $\lambda_+ < 0$ and $\lambda_- > 0$. In either case we have a saddle point.

4. **When do we have a spiral?**

Recall that we have a spiral when the eigenvalues are complex with a non-zero real part. To have a complex eigenvalue we need that $1 - 4\frac{\det \mathbf{A}}{\mathrm{tr}\mathbf{A}^2} < 0$ while to have a nonzero real part we need $\mathrm{tr}\mathbf{A} \neq 0$. In this case we have

$$\lambda_\pm = \frac{1}{2}\mathrm{tr}\mathbf{A}\left[1 \pm i\sqrt{-\left(1 - 4\frac{\det \mathbf{A}}{\mathrm{tr}\mathbf{A}^2}\right)}\right]$$

5. **When do we have a center?**

   Recall that we have center when both the eigenvalues are purely imaginary. Therefore, we need that $\mathrm{tr}\mathbf{A} = 0$. Since the expression $1 - 4\frac{\det \mathbf{A}}{\mathrm{tr}\mathbf{A}^2}$ makes no sense when $\mathrm{tr}\mathbf{A} = 0$, we go back to the original definition of $\lambda_\pm$ and set $\mathrm{tr}\mathbf{A} = 0$:

$$\begin{aligned}\lambda_\pm &= \frac{1}{2}\left(\mathrm{tr}\mathbf{A} \pm \sqrt{\mathrm{tr}\mathbf{A}^2 - 4\det \mathbf{A}}\right)\\ &= \pm\frac{1}{2}\sqrt{\det \mathbf{A}}\end{aligned}$$

   Clearly we will have a center when $\det \mathbf{A} < 0$.

6.2.10.(a) W is given by

$$W(t) = C\exp\left(\int_{t_0}^{t} \mathrm{tr}\big(\mathbf{P}(s)\big)ds\right).$$

If W is zero or not depends on the initial condition thus agreeing with Theorem 6.2.5 and Theorem 6.2.1 which asserts uniqueness of the solution when $\mathbf{P}(t)$ is continuous (as it is assumed here).

6.5.15 Let us find the fun matrix for the system of equations

$$\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}\mathbf{x}$$

Remember that the structure of these types of differential equations is governed by the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}.$$

So let's find the eigenvalues: we solve

$$\det\begin{pmatrix} -1-\lambda & -4 \\ 1 & -1-\lambda \end{pmatrix} = 0$$

$$(\lambda+1)^2 + 4 = 0$$

$$\lambda = -1 \pm 2i$$

and find the corresponding eigenvectors. For $\lambda_1 = -1 + 2i$ we find $\xi_{\mathbf{1}} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ satisfying

$$\mathbf{A}\xi_{\mathbf{1}} = (-1+2i)\,\xi_{\mathbf{1}}$$

Both components of this vector equation read

$$\begin{aligned}-\xi_1 + 4\xi_2 &= (-1+2i)\,\xi_1\\ \xi_2 &= \frac{i}{2}\xi_1\end{aligned}$$

and therefore $\xi_{\mathbf{1}} = \begin{pmatrix} 2 \\ i \end{pmatrix}$ is a good choice for the first eigenvector. You may check that the second eigenvector $\xi_{\mathbf{2}}$ satisfying

$$\mathbf{A}\xi_{\mathbf{2}} = (-1-2i)\,\xi_{\mathbf{2}}$$

is just the complex conjugate $\xi_2 = \begin{pmatrix} 2 \\ -i \end{pmatrix}$. We find that the general solution reads

$$\mathbf{x}(t) = e^{-t}\left[c_1 \begin{pmatrix} 2 \\ i \end{pmatrix} e^{2it} + c_2 \begin{pmatrix} 2 \\ -i \end{pmatrix} e^{-2it}\right].$$

Here c_1 and c_2 are complex numbers set by the initial conditions $\mathbf{x}(0)$. Now since we want to solve the system for three different initial conditions, it would be nice to not have to solve for these coefficients every time. To this end we solve for c_1 and c_2 if $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We find

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ i \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -i \end{pmatrix}$$

$$\begin{cases} 2\,(c_1 + c_2) & = 1 \\ i\,(c_1 - c_2) & = 0 \end{cases}$$

$$c_1 = c_2 = \frac{1}{4}.$$

Therefore the solution corresponding to this initial condition is

$$\begin{aligned} \mathbf{x_1}(t) &= \begin{pmatrix} \frac{1}{2}\left(e^{2it} + e^{-2it}\right) \\ \frac{i}{4}\left(e^{2it} - e^{-2it}\right) \end{pmatrix} \\ &= \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} \end{aligned}$$

where we have used the definition of sin and cos coming from Euler's equation in the last line. Similarly, we solve for c_1 and c_2 corresponding to the initial condition $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We find

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ i \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -i \end{pmatrix}$$

$$\begin{cases} 2(c_1 + c_2) = 0 \\ i(c_1 - c_2) = 0 \end{cases}$$

$$c_1 = -\frac{i}{2}$$

$$c_2 = +\frac{i}{2}$$

Therefore the solution corresponding to this initial condition is

$$\mathbf{x_2}(t) = \begin{pmatrix} -i\left(e^{2it} - e^{-2it}\right) \\ \frac{1}{2}\left(e^{2it} + e^{-2it}\right) \end{pmatrix}$$
$$= \begin{pmatrix} 2\sin(2t) \\ \cos(2t) \end{pmatrix}$$

From these two special solutions we construct the fun matrix (also known as $e^{\mathbf{A}t}$):

$$e^{\mathbf{A}t} = \mathbf{\Phi}(t) = \begin{pmatrix} \cos(2t) & 2\sin(2t) \\ -\frac{1}{2}\sin(2t) & \cos(2t) \end{pmatrix}$$

And now finding the solution corresponding to any initial value is just a matter of matrix multiplication:

- If $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ then

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 3\cos(2t) + 2\sin(2t) \\ -\frac{3}{2}\sin(2t) + \cos(2t) \end{pmatrix}$$

- If $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ then

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\begin{pmatrix} 2 \\ 2 \end{pmatrix}$$
$$= \begin{pmatrix} 2\cos(2t) + 4\sin(2t) \\ -\sin(2t) + 2\cos(2t) \end{pmatrix}$$

- If $\mathbf{x}(0) = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ then

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\begin{pmatrix} -2 \\ 5 \end{pmatrix}$$
$$= \begin{pmatrix} -2\cos(2t) + 10\sin(2t) \\ 3\sin(2t) + 5\cos(2t) \end{pmatrix}$$

The point to take away from all this is that if you have to solve the same linear system for a large number of initial conditions it is well worth it to find the fundamental matrix. The problem is simplified to simple matrix multiplication.