Tutorial Problems #6

MAT 292 – Calculus III – Fall 2015

SOLUTIONS

3.2 - #31 Consider two interconnected tanks similar to those in Figure 3.2.9. Tank 1 initially contains 60 gal of water and Q_1^0 oz of salt, and Tank 2 initially contains 100 gal of water and Q_2^0 oz of salt. Water containing q_1 oz/gal of salt flows into Tank 1 at a rate of 3 gal/min. The mixture in Tank 1 flows out at a rate of 4 gal/min, which half flows into Tank 2, while the remainder leaves the system. Water containing q_2 oz/gal of salt also flows into Tank 2 from the outside at the rate of 1 gal/min. The mixture in Tank 2 leaves the tank at a rate of 3 gal/min, of which some flows back into Tank 1 at a rate of 1 gal/min, while the rest leaves the system.

(a) Draw a diagram that depicts the flow process described above. Let $Q_1 t)$ and $Q_2(t)$, respectively, be the amount of salt in each tank at time t. Write down differential equations and initial conditions for Q_1 and Q_2 that model the flow process

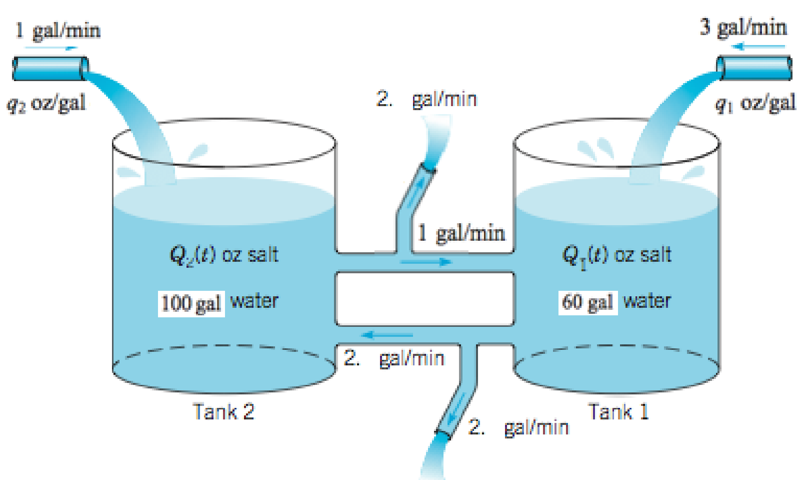
(b) Find the equilibrium values Q_1^E and Q_2^E in terms of the concentrations q_1 and q_2.

(c) Is it possible (by adjusting q_1 and q_2) to obtain $Q_1^E = 60$ and $Q_2^E = 50$ as an equilibrium state?

(d) Describe which equilibrium states are possible for this system for various values of q_1 and q_2

Solution

(a) We see the diagram to be given by



The DE's are easily seen to be

$$Q_1' = -\frac{4}{60}Q_1 + \frac{1}{100}Q_2 + 3q_1 \quad Q_1(0) = Q_1^0$$
$$Q_2' = \frac{2}{60}Q_1 - \frac{3}{100}Q_2 + q_2 \quad Q_2(0) = Q_2^0$$

(b) The equilibrium is easily found setting $Q_1' = Q_2' = 0$, we see

$$\frac{1}{20}\begin{pmatrix} -4/3 & 1/5 \\ 2/3 & -3/5 \end{pmatrix} Q_E = \begin{pmatrix} -3q_1 \\ -q_2 \end{pmatrix} \implies \boxed{Q_E = \begin{pmatrix} 18 & 6 \\ 20 & 40 \end{pmatrix}\begin{pmatrix} 3q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 54q_1 + 6q_2 \\ 60q_1 + 40q_2 \end{pmatrix}}$$

(c) Is it isn't possible to obtain $Q_1^E = 60$ and $Q_2^E = 50$ as an equilibrium state since it'd require a negative q_2. Since

$$\frac{2}{60} * 60 - \frac{3}{100} * 50 + q_2 = 0 \implies q_2 = -\frac{1}{2}$$

(d) The states that we'd allow must satisfy

$$\frac{2}{60}Q_1^E - \frac{3}{100}Q_2^E \leqslant 0 \quad \& \quad -\frac{4}{180}Q_1^E + \frac{1}{300}Q_2^E \leqslant 0$$
$$\implies \boxed{\frac{3}{20}Q_2^E \leqslant Q_1^E \leqslant \frac{9}{10}Q_2^E}$$

□

3.3 - # 17-20 Consider $x' = Ax$. If given the eigenvectors and eigenvalues:

(a) Sketch a phase portrait of the system.

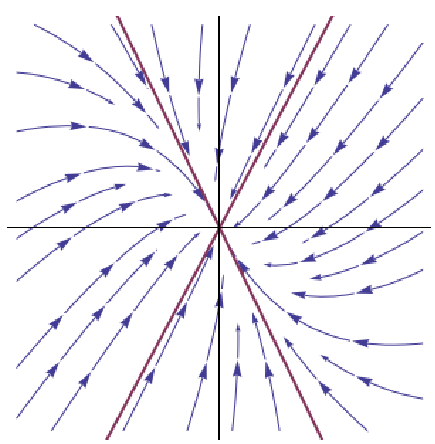
(b) Sketch the trajectory passing through the initial point (2,3)

(c) For the trajectory in part b), sketch the component plots of x_1 versus t and of x_2 versus t on the same set of axes.

#17

$$\lambda_1 = -1, \quad \vec{\lambda}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \& \quad \lambda_2 = -2, \quad \vec{\lambda}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \implies x(t) = C_1\begin{pmatrix} -1 \\ 2 \end{pmatrix}e^{-t} + C_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{-2t}$$

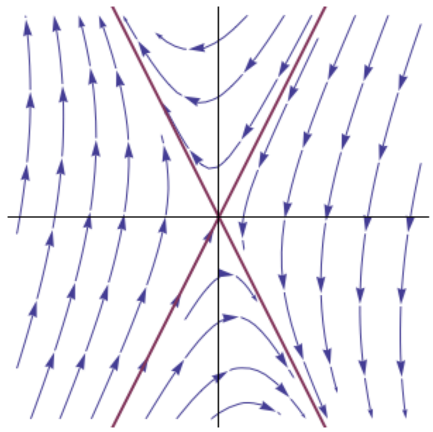
with a portrait like



#18

$$\lambda_1 = 1, \quad \vec{\lambda}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \& \quad \lambda_2 = -2, \quad \vec{\lambda}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \implies x(t) = C_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{t} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}$$

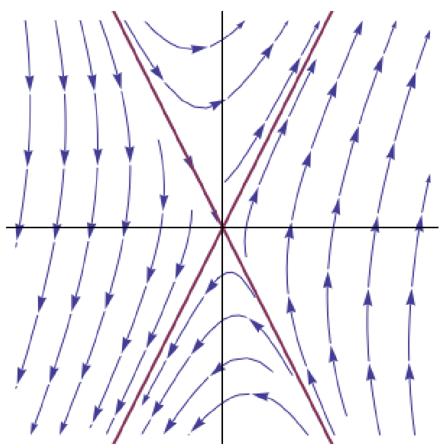
with a portrait like



#19

$$\lambda_1 = -1, \quad \vec{\lambda}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \& \quad \lambda_2 = 2, \quad \vec{\lambda}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \implies x(t) = C_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$$

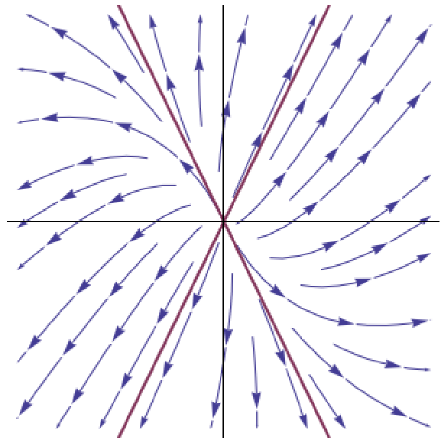
with a portrait like



#20

$$\lambda_1 = 1, \quad \vec{\lambda}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \& \quad \lambda_2 = 2, \quad \vec{\lambda}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \implies x(t) = C_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{t} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$$

with a portrait like



3.3 - #28 Consider

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} -R_1/L & -1/L \\ 1/C & -1/CR_2 \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix}$$

(a) Find a condition on R_1, R_2, C and L that must be satisfied if the eigenvalues of the efficient matrix are to be real and different.

(b) If the condition found in part a) is satisfied, show that both eigenvalues are negative. Then show that $I(t) \to 0$ and $V(t) \to 0$ as $t \to \infty$, regardless of the initial conditions.

Solution

(a) Let's compute the characteristic equation. We see

$$P(\lambda) = \det(1\lambda - A) = \begin{vmatrix} \lambda + R_1/L & 1/L \\ -1/C & \lambda + 1/CR_2 \end{vmatrix} = \lambda^2 + \lambda\left(\frac{CR_1R_2 + L}{CLR_2}\right) + \frac{R_1 + R_2}{CLR_2}$$

Now the eigenvalues are given by the roots, but we know the discriminate needs to be positive and non-zero for the roots to be real and distinct. i.e

$$b^2 - 4ac = \left(\frac{CR_1R_2 + L}{CLR_2}\right)^2 - 4\frac{R_1 + R_2}{CLR_2} = \frac{R_1^2}{L^2} - 2\frac{R_1 L}{CL^2R_2} + \frac{1}{C^2R_2^2} - 4\frac{1}{CL} = \boxed{\left(\frac{R_1}{L} - \frac{1}{CR_2}\right)^2 - \frac{4}{CL} > 0}$$

(b) The actual eigenvalues are given by

$$\lambda_\pm = -\frac{1}{2}\left(\frac{R_1}{L} + \frac{1}{CR_2}\right) \pm \frac{1}{2}\sqrt{\left(\frac{R_1}{L} - \frac{1}{CR_2}\right)^2 - \frac{4}{CL}}$$

Clearly λ_- is negative, and λ_+ is negative since

$$\begin{aligned}
\lambda_+ &= -\frac{1}{2}\left(\frac{R_1}{L} + \frac{1}{CR_2}\right) + \frac{1}{2}\sqrt{\left(\frac{R_1}{L} - \frac{1}{CR_2}\right)^2 - \frac{4}{CL}} \\
&\leqslant -\frac{1}{2}\left(\frac{R_1}{L} + \frac{1}{CR_2}\right) + \frac{1}{2}\sqrt{\left(\frac{R_1}{L} - \frac{1}{CR_2}\right)^2} \\
&= -\frac{1}{2}\left(\frac{R_1}{L} + \frac{1}{CR_2}\right) + \frac{1}{2}\left|\frac{R_1}{L} - \frac{1}{CR_2}\right| \\
&< 0
\end{aligned}$$

We know the solution takes the form

$$\begin{pmatrix} I \\ V \end{pmatrix} = C_1 \vec{\lambda}_+ e^{\lambda_+ t} + C_2 \vec{\lambda}_- e^{\lambda_- t}$$

Therefore the solution goes to zero as $t \to \infty$ since the eigenvalues are negative regardless of the initial data. $\square$

3.4 - #7 Solve the following system, draw direction field and a phase portrait. Describe the behaviour of solutions as $t \to \infty$

$$x' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

Solution Let's compute the characteristic equation to find the eigenvalues.

$$P(\lambda) = \det(1\lambda - A) = \begin{vmatrix} \lambda+1 & 4 \\ -1 & \lambda+1 \end{vmatrix} = \lambda^2 + 2\lambda + 5 = 0 \implies \lambda_\pm = -1 \pm 2i$$

Now that we've found the eigenvalues, let's compute the eigenvectors.

$$\ker(1\lambda_+ - A) = \ker \begin{pmatrix} 2i & 4 \\ -1 & 2i \end{pmatrix} = \operatorname{span} \begin{pmatrix} 2i \\ 1 \end{pmatrix} \implies \vec{\lambda}_+ = \begin{pmatrix} 2i \\ 1 \end{pmatrix}$$

The conjugate gives us the other vector.

$$\vec{\lambda}_- = \begin{pmatrix} -2i \\ 1 \end{pmatrix}$$

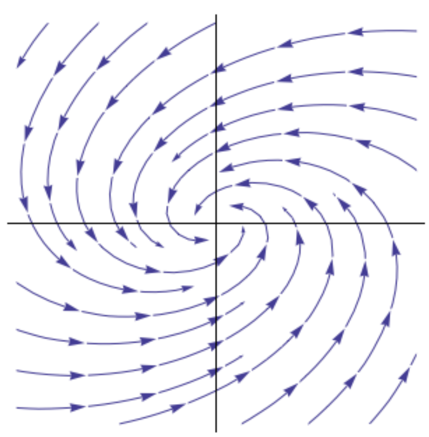
Thus the complex valued solution is given by

$$x(t) = e^{-t} \left[C_1 \vec{\lambda}_- e^{-2it} + C_2 \vec{\lambda}_+ e^{2it} \right]$$

Using Euler's identity and assuming $C_2 = \overline{C}_1$ we obtain

$$x(t) = e^{-t} \left[A \begin{pmatrix} -2\sin(2t) \\ \cos(2t) \end{pmatrix} + B \begin{pmatrix} 2\cos(2t) \\ \sin(2t) \end{pmatrix} \right]$$

The phase portrait is given by



$\square$

3.4 - #22 The electric circuit shown below is described by the system of differential equations

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 0 & 1/L \\ -1/C & -1/RC \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix}$$

where I is the current through the inductor and V is the voltage across the capacitor.

(a) Show that the eigenvalues of the coefficient matrix are real and different if $L > 4R^2C$. Show that they are complex conjugates if $L < 4R^2C$.

(b) Suppose that $R = 1\Omega$, $C = 1/2$ F, and $L = 1$ H. Find the general solution of the system in this case.

(c) Find $I(t)$ and $V(t)$ if $I(0) = 2$ Amp and $V(0) = 1$ Volt.

Solution

(a) This follows from our previous calculation setting $R_1 = 0$ and $R_2 = R$, we see the eigenvalues are real and distinct if

$$\frac{1}{C^2R^2} - \frac{4}{CL} > 0 \implies L > 4R^2C$$

and complex conjugates if the discriminate is negative, i.e. $L < 4R^2C$

(b) Suppose that $R = 1\Omega$, $C = 1/2$ F, and $L = 1$ H, we see the system is

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix}$$

It's easy to see that

$$P(\lambda) = \det(A - 1\lambda) = \begin{vmatrix} -\lambda & 1 \\ -2 & -2-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 2$$

Thus the eigenvalues are

$$\lambda_{\pm} = -1 \pm i$$

The eigenvectors are found by checking the kernel.

$$\lambda_+ \implies \ker\begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix} = \text{span}\begin{pmatrix} 1+i \\ -2 \end{pmatrix} \implies \vec{\lambda}_+ = \begin{pmatrix} 1+i \\ -2 \end{pmatrix}$$

We may take the complex conjugate to retrieve the second

$$\vec{\lambda}_- = \overline{\lambda_+} = \begin{pmatrix} 1-i \\ -2 \end{pmatrix}$$

Thus the complex valued solution is given by

$$\begin{pmatrix} I \\ V \end{pmatrix} = e^{-t}\left[C_1\vec{\lambda}_+e^{it} + C_2\vec{\lambda}_-e^{-it}\right] \quad C_1, C_2 \in \mathbb{C}$$

To make this real valued, fix $C_2 = \overline{C}_1$. Then using Euler's identity we obtain

$$\begin{pmatrix} I \\ V \end{pmatrix} = e^{-t}\left[A\begin{pmatrix} \sin(t) \\ \cos(t) - \sin(t) \end{pmatrix} + B\begin{pmatrix} \sin(t) + \cos(t) \\ -2\sin(t) \end{pmatrix}\right], \quad A, B \in \mathbb{R}$$

(c) Here we have $I(0) = 2$ and $V(0) = 1$, so we solve for A and B:

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \left[A \begin{pmatrix} 0 \\ 1 \end{pmatrix} + B \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \implies A = 1 \quad \& \quad B = 2$$

Thus the solution is given by

$$\begin{pmatrix} I \\ V \end{pmatrix} = e^{-t} \begin{pmatrix} 3\sin(t) + 2\cos(t) \\ \cos(t) - 5\sin(t) \end{pmatrix}$$