Tutorial Problems #5

MAT 292 – Calculus III – Fall 2015

SOLUTIONS

1. Consider the IVP

$$\begin{cases} x''(t) = f(t, x, x') \\ x(t_0) = x_0 \\ x'(t_0) = v_0 \end{cases}$$

Let's define a "new" function $v(t) = x'(t)$ and transform the problem into a system of first order equations:

$$\begin{aligned} x'(t) &= v(t) \\ v'(t) &= f(t, x(t), v(t)). \end{aligned}$$

Suppose that you want to solve this numerically on some interval $[t_0, T]$ with $T > t_0$. To use a numerical method, we must discretize the time interval into some finite number (say N) of pieces:

$$t_0 < t_1 < ... < t_{N-1} < t_N = T.$$

To make the method simple, we take all the little subintervals $[t_n, t_{n+1}]$ of the same length. That is we set $t_{n+1} - t_n = h > 0$ and the length h is the same for every index n.
Now let's use Euler's method to approximate the solution to the equation for v at t_1. The equation for v is:

$$v'(t) = f(t, x, v) \quad v(t_0) = v_0$$

and Euler's method gives us, for $v_1 \equiv v(t_1)$,

$$v_1 = v_0 + hf(t_0, x_0, v_0).$$

Now the equation for x reads

$$x'(t) = v(t) \quad x(t_0) = x_0$$

and we use Heun's method on this equation to improve on the method of Euler. For $x_1 = x(t_1)$ we get

$$\begin{aligned} x_1 &= x_0 + \frac{h}{2}(v_0 + v_1) \\ &= x_0 + \frac{h}{2}(2v_0 + hf(t_0, x_0, v_0)) \\ &= x_0 + hv_0 + \frac{1}{2}h^2 f(t_0, x_0, v_0). \end{aligned}$$

We can easily repeat this procedure for any index n between 1 and N. Assuming you have the approximate solution x_n and v_n at some time t_n, at the next step we approximate:

$$\begin{aligned} v_{n+1} &= v_n + hf(t_n, x_n, v_n) \\ x_{n+1} &= x_n + \frac{h}{2}\left(v_n + v_{n+1}\right) \\ &= x_n + \frac{h}{2}\left(2v_n + hf(t_n, x_n, v_n)\right) \\ &= x_n + hv_n + \frac{1}{2}h^2 f(t_n, x_n, v_n) \end{aligned}$$

We used Euler's method on the equation for the helper function $v(t)$ and an improved method on the equation describing $x(t)$. The result is far more accurate than using Euler's method throughout because of the corrective term proportional to h^2. But why didn't we just use Heun's method on both equations?

2. a) Show that both Euler's and backward Euler's methods give the exact solution if the solution is linear.

If the linear function $y(x) = ax + b$ is the solution of some IVP, then it isn't very hard to determine exactly what the IVP is:

$$\begin{cases} y'(x) & = a \\ y(0) = b. \end{cases}$$

In this case, the derivative of y depends on neither of the variables y or x and there is no difference between the forward and backward methods. So let's use them: suppose I know that $y_0 = y(x_0) = ax_0 + b$ for some x_0. Then for any x_1, we compute $y_1 = y(x_1)$ by

$$\begin{aligned} y_1 &= y_0 + (x_1 - x_0)a \\ &= ax_0 + b + (x_1 - x_0)a \\ &= ax_1 + b \end{aligned}$$

which is the exact solution at x_1.

2. b) Show also that these two methods make exactly the same error, but on opposite sides when the solution is $y(x) = x^2$.

If $y(x) = x^2$ it is also not too hard to find the relevant IVP:

$$\begin{cases} y'(x) = 2x \\ y(0) = 0 \end{cases}$$

Suppose we know that $y(x_0) = x_0^2$ for some x_0 and we want to find $y_1 = y(x_1)$. Letting $h = x_{n+1} - x_n$, the forward Euler's method gives us:

$$\begin{aligned} y_1 &= x_0^2 + h\,(2x_0) \\ &= x_0^2 + 2hx_0 \\ &= (x_0 + h)^2 - h^2 \\ &= x_1^2 - h^2 \end{aligned}$$

So we are making an error of $-h^2$ with the true solution.
Similarly, let's use the backwards Euler's method on the same problem:

$$\begin{aligned} y_1 &= x_0^2 + h\,(2x_1) \\ &= x_0^2 + 2hx_1 \\ &= x_0^2 + 2h\,(x_0 + h) \\ &= x_0^2 + 2hx_0 + 2h^2 \\ &= (x + 0 + h)^2 + h^2 \\ &= x_1^2 + h^2 \end{aligned}$$

and we are making an error of $+h^2$ with the true solution. So both methods give the same error with a different sign. If you repeated this over many steps, the errors would add up and your result would be rather imprecise. By the way, $y(x) = x^2$ is also a solution to the following IVP:

$$\begin{cases} y'(x) = 2\sqrt{y(x)} \\ y(0) = 0. \end{cases}$$

Why does Euler's method give $y(x) = 0$ for every x?

3. Show that the Improved Euler's (or Heun's) Method gives a perfect result if the solution is $y = x^p$ for $p = 0, 1, 2$.

Let's get rid of the easy cases: when $p = 0$, $y(x) = 1$ and satisfies the trivial IVP

$$\begin{cases} y' = 0 \\ y(0) = 1 \end{cases}$$

Both Euler's and Heun's methods in this case can be summed up by the algorithm: take the initial condition and do nothing.
When $p = 1$ and $y(x) = x$ then Euler's and Heun's method can be shown to be exactly the same. Since $f(x, y) = 1$ it doesn't matter which points t and y we evaluate f at. Explicitly: suppose you know that $y(x_0) = x_0$ for some x_0 and you want to find $y_1 = y(x_1)$ at some other point x_1. With $h = x_1 - x_0$, Heun's method reads

$$y_1 = x_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, \tilde{y}_1)\right)$$

where $\tilde{y}_1 = x_0 + hf(x_0, y_0)$ is the result of the Euler's method. Since f is a constant function we get

$$\begin{aligned} y_1 &= x_0 + \frac{h}{2}(1+1) \\ &= x_0 + h \\ &= x_1 \end{aligned}$$

which is the true solution.

The more interesting case is when $y(x) = x^2$. We saw that Euler's method produces a fairly large error in this IVP:

$$\begin{cases} y' = 2x \\ y(0) = 0. \end{cases}$$

Now suppose you know that $y(x_0) = x_0^2$ for some x_0 and you want to find $y_1 = y(x_1)$ at some other point x_1. Setting $h = x_1 - x_0$, the Euler's method gives you

$$\tilde{y}_1 = x_0^2 + 2hx_0.$$

The Heun's method however gives (with $f(x, y) = 2x$)

$$\begin{aligned} y_1 &= x_0^2 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1)\right) \\ &= x_0^2 + \frac{h}{2}(2x_0 + 2x_1) \\ &= x_0^2 + h(x_0 + x_1) \\ &= x_0^2 + (x_1 - x_0)(x_1 + x_0) \\ &= x_0^2 + x_1^2 - x_0^2 \\ &= x_1^2 \end{aligned}$$

that is the exact solution in this case.