Q. Consider the DE \( y' = f(y) \) where the function \( f(y) \) is differentiable. Assume also that \( f(y_1) = f(y_2) = 0 \) and \( y_1 < y_2 \).

(a) If the equilibrium solution \( y = y_1 \) is stable, then what do we know about \( f(y) \) around the point \( y_1 \)?

(b) Assume that both equilibria \( y = y_1 \) and \( y = y_2 \) are stable. Show that there must be another equilibria point \( y^* \) such that \( y_1 < y^* < y_2 \) and \( y = y^* \) is unstable.

Solution

(a) We know if \( y = y_1 \) is stable, then we have for small positive \( \epsilon \) that

\[
f(y_1 - \epsilon) > 0 \quad \& \quad f(y_1 + \epsilon) < 0
\]

(b) Since \( y = y_1 \) and \( y = y_2 \) are stable, we have for small positive \( \epsilon \) that

\[
f(y_1 + \epsilon) < 0 \quad \& \quad f(y_2 - \epsilon) > 0
\]

We also know that \( f(y) \) is continuous, thus the Intermediate value theorem gives us that there exists \( y^* \in (y_1, y_2) \) such that

\[
f(y^*) = 0 \quad \& \quad f(y^* - \epsilon) < 0 \quad \& \quad f(y^* + \epsilon) > 0
\]

i.e. \( y = y^* \) is an unstable equilibrium.

2.4 - # 24 Consider the equation

\[
dy/dt = ay - y^3 = y(a - y^2)
\]

(a) Again consider the cases \( a < 0, a = 0 \) and \( a > 0 \). In each case, find the critical points, draw the phase line, and determine whether each critical point is asymptotically stable, semistable, or unstable.

Recall that a critical point is simply \( y' = 0 \), thus

\[
y' = 0 \iff y = 0 \quad \text{or} \quad a - y^2 = 0 \implies y = \pm\sqrt{a}
\]

If \( a < 0 \) we have that \( y = 0 \) is the only critical point. If \( a = 0 \), we again have \( y = 0 \). If \( a > 0 \), we have the two roots \( \pm\sqrt{a} \) and \( y = 0 \).
(b) In each case, sketch several solutions of the ODE in the y-plane. We sketch the resulting phase portraits.

(c) Draw the bifurcation diagram for the ODE. Note that $a = 0$ is a pitch fork bifurcation.

2.5 - # 23 Show that if $(N_x - M_y)/M = Q$, where $Q$ is a function of $y$ only, then the differential equation

$$M + Ny' = 0$$

has an integrating factor of the form

$$\mu(y) = \exp \int Q(y)dy$$

**Solution** Suppose that $M + Ny' = 0$ is not exact and consider

$$\underbrace{\mu(y)M}_{M} dx + \underbrace{\mu(y)N}_{N} dy = 0$$

We’ll try to find the condition on $\mu$ to make this exact. How do we do this? Check $M'_y = N'_x$.

$$\tilde{M}_y = \frac{\partial}{\partial y}(\mu(y)M) = \mu'(y)M + \mu(y)M'$$

$$\tilde{N}_x = \frac{\partial}{\partial x}(\mu(y)N) = \mu(y)N'$$

Using these equations, we can form an ODE in $\mu$. Namely

$$0 = \tilde{N}_x - \tilde{M}_y = \mu(y)(N_x - M_y) - \mu'(y)M \iff \frac{\mu'(y)}{\mu(y)} = \frac{N_x - M_y}{M} = Q$$
By solving the above ODE for $\mu$, we obtain

$$\mu(y) = \exp \int Q(y) dy$$

### 2.5 - # 26
Find an integrating factor and solve the given equation

$$y' = e^{2x} + y - 1$$

**Solution**  Rewrite the ODE in differential form

$$\left( e^{2x} + y - 1 \right) dx + ( -1) dy = 0$$

We check the partials.

$$M_y = 1$$

$$N_x = 0$$

Since the equation is not exact, we’ll need an integrating factor. Following the same logic as the previous question, we deduce

$$\mu(x) = \exp \int \left( \frac{M_y - N_x}{N} \right) dx = \exp \left( - \int dx \right) = e^{-x}$$

will work. Let’s check

$$\left( e^{x} + e^{-x}(y - 1) \right) dx + ( -e^{-x}) dy = 0$$

$$M_y = e^{-x}$$

$$N_x = e^{-x}$$

Now the equation is exact! Thus we can just integrate each part respectively.

$$\int Mdx = \int (e^{x} + e^{-x}(y - 1)) dx = e^{x} + e^{-x}(1 - y) + C(y)$$

$$\int Ndy = \int -e^{-x} dy = -ye^{-x} + C(x)$$

By comparing the above equation, we see that a function satisfying the partials is

$$F(x, y) = e^{x} + e^{-x}(1 - y)$$

This implies the general solution is

$$\text{const} = e^{x} + e^{-x}(1 - y)$$
2.4 - # 18  A pond forms as water collects in a conical depression of radius $a$ and depth $h$. Suppose that water flows in at a constant rate $k$ and is lost through evaporation at a rate proportional to the surface area.

(a) Show that the volume $V(t)$ of water in the pond at time $t$ satisfies the differential equation

$$\frac{dV}{dt} = k - \alpha \pi (3a/\pi h)^{2/3} V^{2/3}$$

where $\alpha$ is the coefficient of evaporation.

The model we’d like to use is

$$\frac{dV}{dt} = V_{in} - V_{out}$$

we’re given that $V_{in} = k$, and that $V_{out} = \alpha S A$ (out of the top, i.e. just a circle). We just have to compute the surface area of the cone in terms of it’s Volume. Recall that

$$V_{cone} = \frac{\pi r^2 l}{3} \quad & \quad S A_{circle} = \pi r^2$$

where $r$ is radius and $l$ is the length. By drawing a picture, you’ll find that the ratio between the length and radius is always the same i.e. $l/r = h/a$. Thus we have

$$V_{cone} = \frac{\pi r^2 l}{3} = \frac{\pi r^3 h}{3a} \implies \sqrt[3]{\frac{3\alpha V_{cone}}{\pi h}} = r$$

$$\implies S A = \pi \left(\frac{3\alpha V_{cone}}{\pi h}\right)^{2/3}$$

Therefore, the ODE is

$$\frac{dV}{dt} = k - \alpha \pi (3a/\pi h)^{2/3} V^{2/3}$$

(b) Find the equilibrium depth of water in the pond. Is the equilibrium asymptotically stable?

Recall that equilibrium occurs when $V' = 0$, so we have to find the roots of the ODE. We see

$$\frac{dV}{dt} = k - \alpha \pi (3a/\pi h)^{2/3} V^{2/3} = 0 \iff V = \pm \frac{(k/\alpha \pi)^{3/2} \pi h}{3a}$$

Since the Volume cannot be negative, we discard that root. To find the depth $l$, just substitute back in as in the previous part.

(c) Find a condition that must be satiated if the pond is not to overflow.

For the pond to not overflow, we need $dV/dt = 0$ when the cone is full. Thus

$$V_{cone} = \frac{\pi a^2 h}{3} = \frac{(k/\alpha \pi)^{3/2} \pi h}{3a} \implies k = \alpha \pi a^{4/3}$$