Tutorial Problems #3

MAT 292 – Calculus III – Fall 2015

SOLUTIONS

2.2 - # 13 The population of mosquitoes in a certain area increases at a rate proportional to the current population, and in the absence of other factors, the population doubles each week. There are 200,000 mosquitoes in the area initially, and predators (birds, bats, and so forth) eat 20,000 mosquitoes/day. Determine the population of mosquitoes in the area at any time.

Solution Let $P(t)$ be the population of mosquitoes in the area with t in weeks, and k be the rate at which they double. We see in absence of other factors that

$$P' = kP$$

Thus the population grows like

$$P(t) = P_0 e^{kt}$$

where $P_0 = 200,000$ is the initial population. We may find the rate k since we're given that the population doubles each week, thus

$$2 = \frac{P(t+1)}{P(t)} = \frac{e^{k(t+1)}}{e^{kt}} = e^k \implies k = \ln 2 \frac{1}{weeks}$$

Now that we have the rate at which the population grows, we may introduce the predators to the model now by

$$P' = \ln 2 P - 20,000 \frac{mosquitoes}{day} * 7 \frac{day}{weeks} = \ln 2 P - 140,000$$

The solution to this ODE is given by

$$P(t) = e^{t \ln 2} \left(P_0 - \frac{140,000}{\ln 2} \right) + \frac{140,000}{\ln 2} = \left(\frac{200,000 \ln 2 - 140,000}{\ln 2} \right) 2^t + \frac{140,000}{\ln 2}$$

$\square$

Example Consider

$$\begin{cases} y' = -y^2 \\ y(0) = -1 \end{cases}$$

Show a) the solution is unique and exists, b) solve the system, c) What is the domain of the solution? and d) Compare that to the existence and uniqueness theorem for linear DE's.

Solution Well, $y' = f(t,y)$, thus in our case we have

$$f(t,y) = -y^2 \implies \frac{\partial f}{\partial y} = -2y$$

which is continuous. Thus the solution is unique and exists by the Picard-Lindelöf Theorem. The solution to the ODE can be found using separation of variables, i.e.

$$y' = -y^2 \implies \int \frac{dy}{y^2} = \int -dt \implies \frac{1}{y} = t + C \implies \frac{1}{y} = t - 1 \implies y(t) = \frac{1}{t-1}$$

The domain is seen to be $(-\infty, 1)$ since we need to satisfy $0 \in (-\infty, 1)$. Note for linear differential equations $y' + py = g$, we know the answer is given by

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)g(t)dt$$

which just requires the integrals to make sense (i.e. continuity of p and g is enough). □

Picard Iterations Sheet # 2 Consider

$$y' = f(t, y(t)), \quad y(t_0) = y_0$$

Describe how to do the Picard Iteration Method here. Then assume $\phi_n \to \phi$, and show that ϕ is a solution of the IVP.

Solution Notice by the fundamental theorem of Calculus we may write

$$y(t) = y(t_0) + \int_{t_0}^{t} f(s, y(s))ds$$

To approximate the answer to this integral equation, take the approximating sequence

$$\phi_0 = y_0, \quad \phi_{k+1} = y_0 + \int_{t_0}^{t} f(s, \phi_k(s))ds$$

This is why the Picard Iteration works if f is nice enough. If we assume that our approximation converges to some answer ϕ, we see this implies that

$$\phi(t) = y_0 + \int_{t_0}^{t} f(s, \phi(s))ds$$

This solves the IVP by differentiating the integral equation. □

2.3 - # 32 Solve

$$\begin{cases} y' + 2y = g(t) \\ y(0) = 0 \end{cases}, \quad \text{where} \quad g(t) = \begin{cases} 1 & 0 \leqslant t \leqslant 1 \\ 0 & t > 1 \end{cases}$$

Solution We know

$$y(t) = \frac{1}{\mu} \int \mu g dt \quad \text{where} \quad \mu = \exp\left[\int p(t)dt\right] = e^{2t}$$

Thus the solution on $[0,1]$ is given by

$$y(t) = e^{-2t} \int_0^t e^{2s} ds = \frac{1 - e^{-2t}}{2}$$

The solution on $t > 1$ is given by

$$y(t) = Ce^{-2t}$$

Since we'd like a continuous solution, we find C by setting both pieces equal to each other at $t = 1$.

$$Ce^{-2} = \frac{1 - e^{-2}}{2} \implies C = \frac{e^2 - 1}{2}$$

Thus a piecewise solution to the ODE is given by

$$y(t) = \begin{cases} \frac{1-e^{-2t}}{2} & 0 \leqslant t \leqslant 1 \\ \frac{e^{2(1-t)}-e^{-2t}}{2} & t > 1 \end{cases}$$

$\square$