MAT292 - Calculus III - Fall 2014

Term Test 2 - November 6, 2014

Time allotted: 90 minutes. Aids permitted: None.

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Instructions

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection, turn off all cellular phones, and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- This test contains 20 pages (including this title page). Make sure you have all of them.
- You can use pages 19–20 for rough work or to complete a question (**Mark clearly**).
  DO NOT DETACH PAGES 19–20.

GOOD LUCK!

PART I No explanation is necessary.

For questions 1–4, consider the following systems of differential equations: **(4 marks)**

Letter	System Matrix	Eigenvalues and Eigenvectors	
a	$\mathbf{A} = \begin{pmatrix} -2 & -4 \\ -\frac{1}{2} & -1 \end{pmatrix}$	$\lambda_1 = -3, \vec{\xi}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$	$\lambda_2 = 0, \vec{\xi}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$
b	$\mathbf{A} = \begin{pmatrix} -1 & 4 \\ \frac{1}{2} & -2 \end{pmatrix}$	$\lambda_1 = 0, \vec{\xi}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$	$\lambda_2 = -3, \vec{\xi}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$
c	$\mathbf{A} = \begin{pmatrix} -2 & 5 \\ -1 & 2 \end{pmatrix}$	$\lambda_1 = -i, \vec{\xi}_1 = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$	$\lambda_2 = i, \vec{\xi}_2 = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$
d	$\mathbf{A} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$	$\lambda_1 = i, \vec{\xi}_1 = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$	$\lambda_2 = -i, \vec{\xi}_2 = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$
e	$\mathbf{A} = \begin{pmatrix} -\frac{13}{8} & \frac{3}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix}$	$\lambda_1 = -\frac{7}{4}, \vec{\xi}_1 = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$	$\lambda_2 = -\frac{1}{8}, \vec{\xi}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
f	$\mathbf{A} = \begin{pmatrix} -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{13}{8} \end{pmatrix}$	$\lambda_1 = -\frac{1}{8}, \vec{\xi}_1 = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$	$\lambda_2 = -\frac{7}{4}, \vec{\xi}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
g	$\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\lambda_1 = -1, \vec{\xi}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\lambda_2 = -1, \vec{\xi}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
h	$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	$\lambda_1 = 2, \vec{\xi}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\lambda_2 = 2, \vec{\xi}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

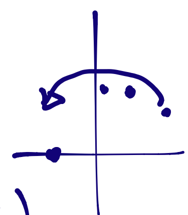
$\boxed{d}$ $y_1 = \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix}$

$y_1(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$y_1(\pi/4) = \left(\frac{\sqrt{2}}{2}, \sqrt{2}/2\right)$

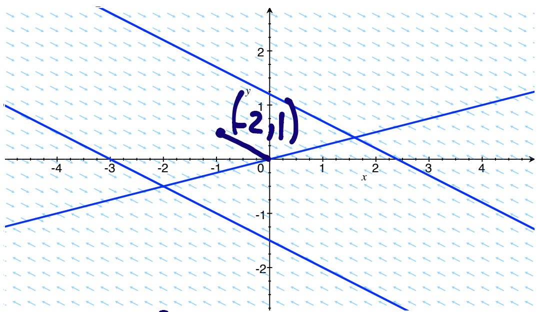
$y_1(\pi/3) = \left(1 - \sqrt{3}/2, 1/2\right)$

$y_1(\pi/2) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$



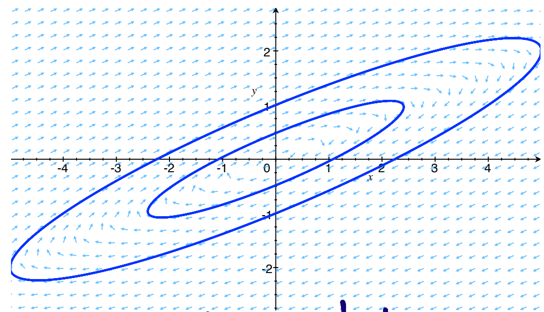
Counter clockwise

Next to each phase plane diagram, write the letter of the corresponding system of differential equations.



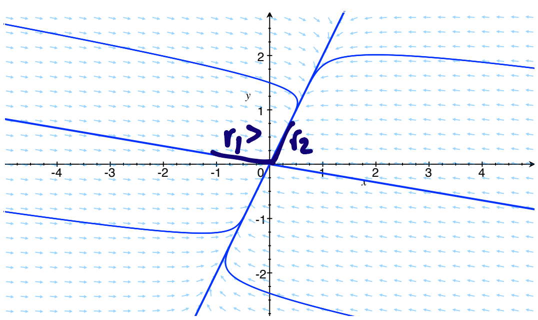
$r=0$

1. This is system b



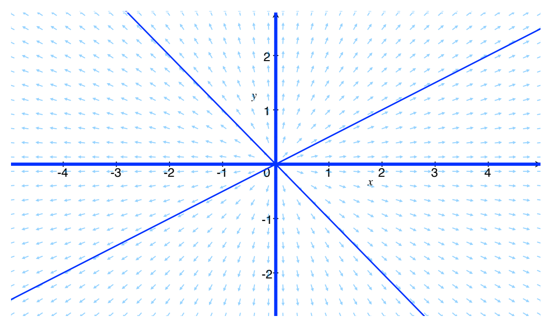
$r=\pm\beta i$ clockwise

2. This is system c



$r_1, r_2 < 0$

3. This is system e



$r_1 = r_2 > 0$

4. This is system h

5. Write a differential equation whose complementary solution is **(2 marks)**

$$y_c(t) = c_1 e^{-2t} + c_2 t e^{-2t} + c_3 t^2 e^{-2t} + c_4$$

(Char eq. $(r+2)^3 r = 0$

$\quad = (r^3+6r^2+12r+8)r$)

$y^{(4)} + 6y^{(3)} + 12y'' + 8y' = 0$

$$(r^4+2r^2+1)r^2 = (r^2+1)^2 r^2$$

$$y_c = c_1 + c_2 t + c_3 \cos t + c_4 \sin t + c_5 t\cos t + c_6 t \sin t$$

$$y_p = (A\cos t + B\sin t)t^2 + (Ct^2 + Dt + E)t^2$$

6. Consider the ODE $y^{(6)} + 2y^{(4)} + y^{(2)} = \cos(t) + t^2$. When using the Method of Undetermined Coefficients, we assume that the terms in the *particular solution* that are *not in the complementary solution* are (**select all that apply**): **(2 marks)**

(a) $A\cos t$	**(d)** $D\sin t$	**(g)** G	**(j)** Jt^3	**(m)** Me^t	**(p)** Pe^{-t}
(b) $Bt\cos t$	**(e)** $Et\sin t$	**(h)** Ht	**(k)** Kt^4	**(n)** Nte^t	**(q)** Qte^{-t}
(c) $Ct^2\cos t$	**(f)** $Ft^2\sin t$	**(i)** It^2	**(l)** Lt^5	**(o)** Ot^2e^t	**(r)** Rt^2e^{-t}

For questions 7 and 8, consider the ODE: **(2 marks)**

$$ay'' + by' + cy = 0,$$

with $b^2 - 4ac < 0$. roots: $r = \underbrace{-\frac{b}{2a}}_{\text{real}} + \underbrace{\sqrt{b^2-4ac}}_{\text{imag}}$

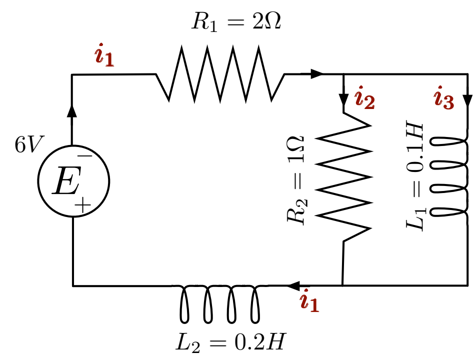
7. The solutions decay while oscillating if $-\frac{b}{2a} < 0 \iff b, a$ same sign.

8. The solutions grow while oscillating if $-\frac{b}{2a} > 0 \iff b, a$ opposite signs

PART II **Justify** your answers.

9. Consider the following parallel circuit. **(10 marks)**

Using Kirchhoff's First Law, we deduce that $i_1 = i_2 + i_3$, so we consider only the currents i_1 and i_2.
Using Kirchhoff's Second Law, we can show that this parallel circuit is modelled by



$$\begin{cases} \dfrac{di_1}{dt} = -10i_1 - 5i_2 + 30 \\ \dfrac{di_2}{dt} = -10i_1 - 15i_2 + 30 \end{cases}$$

(a) Consider a vector $\vec{x} = \vec{i} + \vec{b}$, with $\vec{i} = \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}$.

Find $\vec{b}$ so that the system of differential equations for $\vec{x}$ is homogeneous.

$$\frac{d\vec{x}}{dt} = \frac{d\vec{i}}{dt} = \begin{pmatrix} -10 & -5 \\ -10 & -15 \end{pmatrix}(\vec{x} - \vec{b}) + \begin{pmatrix} 30 \\ 30 \end{pmatrix}$$

$$= \begin{pmatrix} -10 & -5 \\ -10 & -15 \end{pmatrix}\vec{x} + \underbrace{\begin{pmatrix} 10b_1 + 5b_2 + 30 \\ 10b_1 + 15b_2 + 30 \end{pmatrix}}_{\text{Choose } b_1 \text{ and } b_2 \text{ st. } = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$$

$$\Rightarrow \begin{cases} 10b_1 + 5b_2 = -30 \\ 10b_1 + 15b_2 = -30 \end{cases} \Rightarrow 10b_2 = 0 \Rightarrow b_2 = 0$$

$$\Rightarrow b_1 = -3.$$

So $\boxed{\vec{b} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}}$.

(b) The new system is

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} -10 & -5 \\ -10 & -15 \end{pmatrix} \vec{x}.$$

Find the general solution $\vec{x}$.

Eigenvals. $\begin{vmatrix} -10-r & -5 \\ -10 & -15-r \end{vmatrix} = 0 \iff (10+r)(15+r) - 50 = 0$

$\iff r^2 + 25r + 100 = 0$

$\iff r = \frac{-25 \pm \sqrt{25^2 - 4 \cdot 10^2}}{2} = \frac{-25 \pm 5\sqrt{5^2-4^2}}{2}$

$r = \frac{-25 \pm 15}{2} \iff \begin{cases} r_1 = -20 \\ r_2 = -5 \end{cases}$

$\boxed{r_1 = -20}$

$\begin{pmatrix} 10 & -5 \\ -10 & 5 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \xi_2 = 2\xi_1$

$\vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\boxed{r_2 = -5}$

$\begin{pmatrix} -5 & -5 \\ -10 & -10 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \xi_2 = -\xi_1$

$\vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

General solution

$\vec{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-20t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-5t}$

(c) Given the initial conditions $\vec{i}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, what is the solution $\vec{i}$ of the original system?

$$\Rightarrow \vec{\dot{i}} = \vec{x} - \vec{b} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-20t} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-5t} + \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\dot{i}(0) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} C_1 + C_2 = -3 \\ 2C_1 - C_2 = 0 \end{cases} \Rightarrow \begin{cases} 3C_1 = -3 \\ C_2 = 2C_1 \end{cases} \Rightarrow \begin{cases} C_1 = -1 \\ C_2 = -2 \end{cases}$$

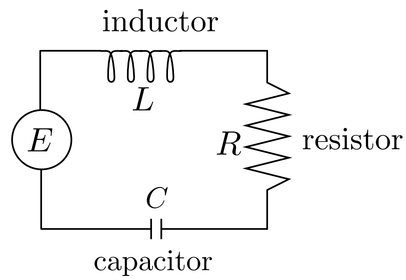
Answer: $\vec{\dot{i}} = \begin{pmatrix} 3 - e^{-20t} - 2e^{-5t} \\ -2e^{-20t} + 2e^{-5t} \end{pmatrix}$

(d) What is i_3?

$$\dot{i}_3 = \dot{i}_1 - \dot{i}_2 = 3 - e^{-20t} - 2e^{-5t} + 2e^{-20t} - 2e^{-5t}$$

$$\dot{i}_3 = 3 + e^{-20t} - 4e^{-5t}$$

10. Consider the following electrical circuit. **(10 marks)**



The charge on the capacitor $q(t)$ is modelled by

$$Lq'' + Rq' + \frac{1}{C}q = E(t),$$

(a) Give a condition on the constants L, R, C that guarantees that the solution oscillates. Justify your answer.

Solution oscillates if it has sin and/or cos. We need to obtain complex roots of the characteristic equation.

Characteristic Equation: $Lr^2 + Rr + \frac{1}{C} = 0 \iff r = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L}$

This means that r_1 and r_2 are complex iff $R^2 - \frac{4L}{C} < 0$

$\iff \boxed{R^2 C < 4L}$

(b) Let $L = 1$, $R = 0$, and $C = \frac{1}{4}$, and $E(t) = \sin(2t)$. Also assume that the capacitor starts with no charge and the circuit starts with no current. Find the solution of this initial-value problem. (**Hint.** Recall that current $i(t) = q'(t)$)

$$y'' + 4y = \sin(2t)$$

Char. eq. : $r^2 + 4 = 0 \iff r = \pm 2i$

So $\boxed{y_c(t) = C_1 \cos(2t) + C_2 \sin(2t)}$

Method of Undetermined Coefficients :

$$y_p = At\cos(2t) + Bt\sin(2t)$$

$$y_p' = A\cos(2t) - 2At\sin(2t) + B\sin(2t) + 2Bt\cos(2t)$$

$$y_p'' = -2A\sin(2t) - 2A\sin(2t) - 4At\cos(2t) + 2B\cos(2t) + 2B\cos(2t) - 4Bt\sin(2t)$$

So $y_p'' + 4y$

$$= -4A\sin(2t) - \cancel{4At\cos(2t)} + 4B\cos(2t) - \cancel{4Bt\sin(2t)} + \cancel{4At\cos(2t)} + \cancel{4Bt\sin(2t)} = \sin(2t)$$

$$\iff \begin{cases} -4A = 1 \\ 4B = 0 \end{cases} \Rightarrow \begin{cases} A = -1/4 \\ B = 0 \end{cases}$$

$$y_p = \frac{-t}{4}\cos(2t).$$

General Solution is $y = C_1\cos(2t) + C_2\sin(2t) - \frac{t}{4}\cos(2t)$

$$\begin{cases} y(0) = C_1 = 0 \\ y'(0) = 2C_2 - \frac{1}{4} = 0 \Rightarrow C_2 = 1/8 \end{cases}$$

Solution. $\boxed{q = \frac{1}{8}\sin(2t) - \frac{t}{4}\cos(2t)}$

(c) How does the solution to **(b)** behave (grow / decay / oscillate) as t becomes larger and larger? Justify your answer.

(**Hint.** You don't need to have solved **(b)** to answer this question)

It grows while oscillating.
The circuit is resonating!

11. Consider the ODE **(10 marks)**

$$y'' - (3+2t)y' + (6t-2)y = 0. \qquad (\star)$$

(a) Show that $y_1(t) = e^{t^2}$ is a solution of this differential equation.

$y_1' = 2te^{t^2}$ and $y_1'' = (2+4t^2)e^{t^2}$

So $y_1'' - (3+2t)y_1' + (6t-2)y_1 = (\cancel{2} + \cancel{4t^2})y_1 - (\cancel{6t} + \cancel{4t^2})y_1 + (\cancel{6t} - \cancel{2})y_1 = 0$

(b) Using reduction of order, consider a second solution of the form

$$y_2(t) = u(t)y_1(t).$$

Deduce a differential equation for $u(t)$.

$$y_2' = u'y_1 + uy_1'$$

$$y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

$$\Rightarrow \quad y_2'' - (3+2t)y_2' + (6t-2)y_2$$

$$= u''y_1 + 2u'y_1' + uy_1'' - (3+2t)(u'y_1 + uy_1') + (6t-2)uy_1$$

$$= u\underbrace{\left(y_1'' - (3+2t)y_1' + (6t-2)y_1\right)}_{=0} + u''y_1 + 2u'\underset{\substack{\| \\ 2ty_1}}{y_1'} - (3+2t)u'y_1 = 0$$

$$\Leftrightarrow \quad u'' + 4tu' - (3+2t)u' = 0$$

$$\Leftrightarrow \quad \boxed{u'' + (2t-3)u' = 0}$$

(c) Find $u(t)$.

(**Hint.** You can leave u in the form of an integral)

$v = u'$ satisfies $v' + (2t-3)v = 0$

$\Leftrightarrow \quad v = C e^{3t-t^2}$

$\Leftrightarrow \quad \boxed{u = C \int e^{3t-t^2}\, dt}$

(d) Write the second solution $y_2(t)$ of $(\star)$ using a definite integral between 0 and t. Show that y_1 and y_2 form a fundamental set of solutions.

$$\text{So} \quad y_2 = u y_1 = e^{t^2} \int_0^t e^{3s-s^2}\, ds$$

$$y_2' = 2te^{t^2} \int_0^t e^{3s-s^2}\, ds + e^{3t}$$

$$\text{Then} \quad W[y_1, y_2] = y_1 y_2' - y_1' y_2 = y_1 \left(y_2' - 2t y_2 \right)$$

$$= e^{t^2} \left[\cancel{2te^{t^2} \int_0^t e^{3s-s^2}\, ds} + e^{3t} - \cancel{2te^{t^2} \int_0^t e^{3s-s^2}\, ds} \right]$$

$$= e^{3t+t^2} \neq 0 \; !$$

(e) What is the general solution of the differential equation $(\star)$?

$$\boxed{y = C_1 e^{t^2} + C_2 e^{t^2} \int_0^t e^{3s-s^2}\, ds}$$

12. Consider the system of differential equations: **(10 marks)**

$$\vec{x}' = \begin{pmatrix} -2 & -4 \\ -\frac{1}{2} & -1 \end{pmatrix} \vec{x}.$$

(a) Find two linearly independent solutions $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$.

Eigenvalues: $\begin{vmatrix} -2-r & -4 \\ -1/2 & -1-r \end{vmatrix} = 0 \iff (2+r)(1+r) - 2 = 0$

$\iff r^2 + 3r = 0 \iff r(r+3) = 0$

$\boxed{r=0}$ $\begin{pmatrix} -2 & -4 \\ -1/2 & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \xi_1 = -2\xi_2$

$\vec{\xi}^{(1)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\boxed{r=-3}$ $\begin{pmatrix} 1 & -4 \\ -1/2 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \xi_1 = 4\xi_2$

$\vec{\xi}^{(2)} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$\boxed{\vec{x}^{(1)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}}$ and $\boxed{\vec{x}^{(2)} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{-3t}}$

(b) Consider the eigenvectors found in **(a)**. Construct a matrix $\mathbf{T}$ by putting each eigenvector as a column.

Find the matrix $\mathbf{T}^{-1}$.

$\left(\textbf{Hint.}\quad \text{For the forgetful ones, } \mathbf{A}^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\right)$

$$T = \begin{pmatrix} -2 & 4 \\ 1 & 1 \end{pmatrix} \Rightarrow T^{-1} = \frac{1}{-6}\begin{pmatrix} 1 & -4 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} -1/6 & 2/3 \\ 1/6 & 1/3 \end{pmatrix}$$

Continued...

(c) Consider a new variable $\vec{x} = \mathbf{T}\,\vec{y}$. Which system of differential equations does it satisfy?

$$\vec{x}\,' = \begin{pmatrix} -2 & -4 \\ -1/2 & -1 \end{pmatrix}\vec{x} \quad \Leftrightarrow \quad T\vec{y}\,' = \begin{pmatrix} -2 & -4 \\ -1/2 & -1 \end{pmatrix} T\vec{y}$$

$$\Leftrightarrow \quad \vec{y}\,' = \boxed{T^{-1}\begin{pmatrix} -2 & -4 \\ -1/2 & -1 \end{pmatrix} T}\,\vec{y}$$

$$\begin{pmatrix} -1/6 & 2/3 \\ 1/6 & 1/3 \end{pmatrix}\begin{pmatrix} -2 & -4 \\ -1/2 & -1 \end{pmatrix}\begin{pmatrix} -2 & 4 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1/6 & 2/3 \\ 1/6 & 1/3 \end{pmatrix}\begin{pmatrix} 0 & -12 \\ 0 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix}$$

So $\vec{y}$ satisfies:

$$\vec{y}\,' = \begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix}\vec{y}$$

(d) Find $\vec{y}$.

$$\vec{y} = \begin{pmatrix} c_1 \\ c_2 e^{-3t} \end{pmatrix}$$

(e) What is the special fundamental matrix $\mathbf{\Phi}$ for the system of differential equations in **(c)**?

$$\Phi = e^{At} \quad \text{with} \quad A = \begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix}$$

$$\Rightarrow \boxed{\Phi = \begin{pmatrix} 1 & 0 \\ 0 & e^{-3t} \end{pmatrix}}$$

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