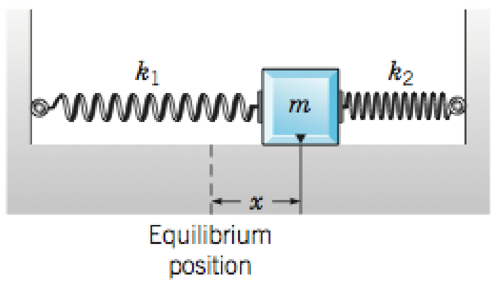
Tutorial Problems #8

MAT 292 – Calculus III – Fall 2014

SOLUTIONS

4.1 - # 18 A body of mass m is attached between two springs with spring constants k_1 and k_2 as shown in the figure. The springs are at their rest length when the system is in the equilibrium state. Assume that the mass slides without friction but the motion is subject to viscous air resistance with coefficient γ . Find the differential equation satisfied by the displacement $x(t)$ of the mass from its equilibrium position.



Solution Recall that the force acting from each springs obeys Hook's Law

$$F_i = -k_i x$$

where k_i is the spring constant and x is the displacement. We also have an air resistance acting by

$$F_{air} = -\gamma|\dot{x}|$$

Thus, by Newton's second law, we have the following ODE.

$$F_{net} = F_2 + F_1 + F_{air} \implies \boxed{m\ddot{x} = (k_2 - k_1)x - \gamma|\dot{x}|}$$

4.2 - # 15 Can an equation $y'' + p(t)y' + q(t)y = 0$, with continuous coefficients, have $y = \sin(t^2)$ as a solution on an interval containing $t = 0$? Explain your answer.

Solution To check if this is possible, we assume p, q are continuous (since it's given), and check how the solution would solve the ODE (assuming it is a solution of course). We clearly have

$$y = \sin t^2, \quad \& \quad y' = 2t\cos t^2, \quad \& \quad y'' = 2\cos t^2 - 4t^2 \sin t^2$$

Plugging this back into the ODE gives the following:

$$2(1 + p(t)t)\cos(t^2) + (q(t) - 4t^2)\sin(t^2) = 0$$

Notice since the above should hold for t around 0, this implies the coefficients must be zero. i.e

$$1 + p(t)t = 0 \quad \& \quad 4t^2 + q(t) = 0$$

This fixes our choice of p and q, namely

$$p(t) = -\frac{1}{t} \quad \& \quad q(t) = 4t^2$$

This means $p(t)$ isn't continuous around $t = 0$, i.e. a contradiction to $p(t)$ being continuous. Therefore we cannot have $y = \sin(t^2)$ as a solution on an interval containing $t = 0$.

4.2 - # 25 Prove Theorem 4.2.4 and Corollary 4.2.5

Theorem[4.2.4]: Let $K[x] = x' - P(t)x$, where the entries of P are continuous functions on an interval I. If x_1 and x_2 are continuously differentiable vector functions on I, and c_1 and c_2 are any constants, then,

$$K[c_1x_1 + c_2x_2] = c_1K[x_1] + c_2K[x_2]$$

Proof. By explicit computation we have

$$\begin{aligned} K[c_1x_1 + c_2x_2] =& (c_1x_1 + c_2x_2)' - P(t)(c_1x_1 + c_2x_2) \Longleftarrow \text{ Def'n of } K \\ =& c_1x_1' + c_2x_2' - c_1P(t)x_1 - c_2P(t)x_2 \Longleftarrow x_1, x_2 \text{ are differentiable} \\ =& c_1(x_1' - P(t)x_1) + c_2(x_2' - P(t)x_2) \Longleftarrow \text{ rearranging} \\ =& c_1K[x_1] + c_2K[x_2] \Longleftarrow \text{ Def'n of } K \end{aligned}$$

□

Corollary[4.2.5]: Let $K[x] = x' - P(t)x$ and suppose the entries of P are continuous functions on an interval I. If x_1 and x_2 are two solutions of $K[x] = 0$, then the linear combination

$$x = c_1x_1(t) + c_2x_2(t)$$

is also a solution for any values of the constants c_1 and c_2.

Proof. Using the above theorem, we have

$$K[x] = K[c_1x_1 + c_2x_2] = c_1K[x_1] + c_2K[x_2]$$

Thus, if x_1 and x_2 are solutions, i.e. $K[x_1] = K[x_2] = 0$, we see

$$K[x] = 0$$

Therefore it is also a solution. □

4.2 - # 36 The differential equation

$$y'' + \delta(xy' + y) = 0$$

arises in the study of the turbulent flow of a uniform stream past a circular cylinder. Verify that $y_1 = \exp(-\delta x^2/2)$ is one solution and then find the general solution in the form of an integral.

Solution We begin by checking the solution.

$$y = e^{-\delta x^2/2}, \quad \& \quad y' = -\delta xy, \quad \& \quad y'' = -\delta y + \delta^2 x^2 y$$

Thus:

$$(-\delta y + \delta^2 x^2 y) + \delta(x(-\delta xy) + y) = 0$$

So it is indeed a solution. Recall that when you have the first solution, the full solution is readily found via

$$y(x) = y_1(x) \int \frac{W[y_1, y_2]}{y_1^2} dx$$

where the Wronskian is interpreted via Abel's formula:

$$W[y_1, y_2] = C \exp\left(-\int p(x)dx\right) = C \exp\left(-\delta \int xdx\right) = C \exp(-\delta x^2/2)$$

where $C \in \mathbb{R}$. Explicitly, we have y in the integral form of

$$\boxed{y(x) = Ce^{-\delta x^2/2} \int e^{\delta x^2/2} dx}$$

Since the inside is a Gaussian, we are unable to write a closed form.

4.3 - # 51 Consider the equation $ay'' + by' + cy = 0$, where a, b, and c are constants with $a > 0$. Find conditions on a, b, and c such that the roots of the characteristic equation are:

(a) Real, different, and negative,

(b) Real with opposite signs

(c) Real, different, and positive.

Solution We begin by computing the characteristic equation, and finding the eigenvalues.

$$P(\lambda) = a\lambda^2 + b\lambda + c = 0 \implies \lambda_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We see that if we want real eigenvalues, we require the discriminant to be positive. i.e.

$$\Delta = b^2 - 4ac > 0$$

If we want the eigenvalues to have the same sign, we require that

$$\Delta < b^2 \implies -4ac < 0 \implies c > 0$$

since $a > 0$. The positive and negative now come down to the sign of b. If $b > 0$, this implies the eigenvalues are negative. If $b < 0$, this implies the eigenvalues are positive. To make sure that they're different, we need $\Delta \neq 0$. If we want eigenvalues with opposite signs, we require

$$\Delta > b^2 \implies -4ac > 0 \implies c < 0$$

Summarizing we have

(a) Real, different, and negative,

$$c > 0 \quad \& \quad b > 0 \quad \& \quad a > 0 \quad \& \quad b^2 - 4ac \neq 0$$

(b) Real with opposite signs

$$c < 0 \quad \& \quad a > 0$$

(c) Real, different, and positive.

$$c > 0 \quad \& \quad b < 0 \quad \& \quad a > 0 \quad \& \quad b^2 - 4ac \neq 0$$