Tutorial Problems #4

MAT 292 – Calculus III – Fall 2014

SOLUTIONS

Q. Write an Autonomous Differential Equation where 0 is a semi-stable critical point. Justify

Solution Take the example from last week,

$$y' = -y^2$$

If we try the line, $y = 0$, we see this is indeed a critical point. Since $y' \leqslant 0$, this is indeed semi-stable.

2.4 - # 23 Consider the equation

$$dy/dt = a - y^2$$

(a) Find all of the critical points for the above ODE. Observe that there are no critical points if $a < 0$, one critical point if $a = 0$, and two critical points if $a > 0$

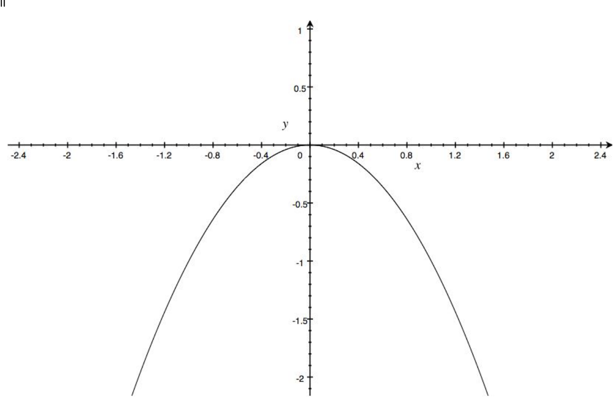
¶Recall that a critical point is simply $y' = 0$, thus

$$y' = 0 \iff a - y^2 = 0 \iff y = \pm\sqrt{a}$$

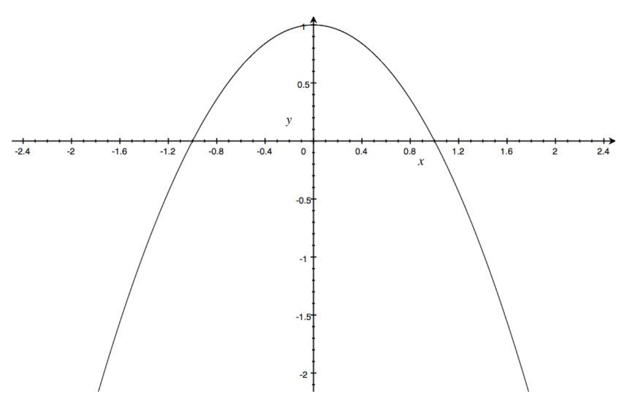
If $a < 0$, there are no real roots. If $a = 0$, we have the single root of $y = 0$. If $a > 0$, we have the two roots $\pm\sqrt{a}$.

(b) Draw the phase line in each case and determine whether each critical point is asymptotically stable, semistable, or unstable.

¶



$a = 0$

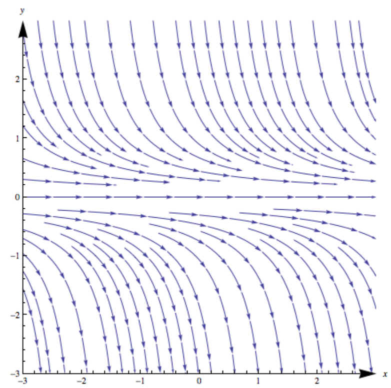


$a > 0$

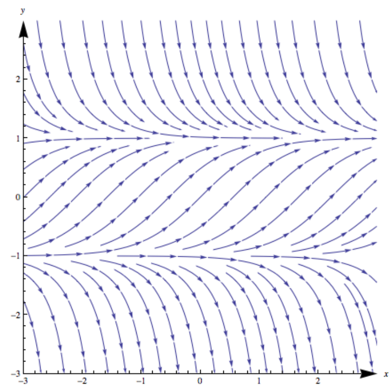
We see that the first is semistable ($a = 0$), the second has that $a > 0$ is stable while $a < 0$ is unstable. The last has no critical point.

(c) In each case, sketch several solution of the ODE.

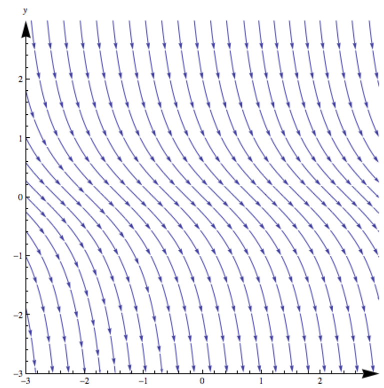
¶Follow the lines in the below portraits.



$a = 0$

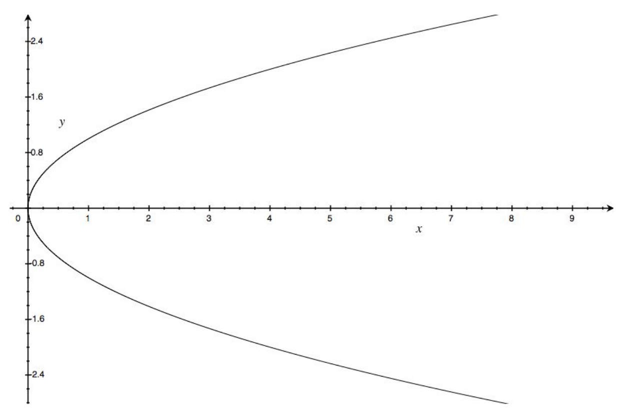


$a > 0$



$a < 0$

(d) If we plot the location of the critical points as a function of a in the ay-plane, we obtain



This is called the bifurcation diagram for the above ODE. The bifurcation at $a = 0$ is called a saddle-node bifurcation.

2.4 - # 24 Consider the equation

$$dy/dt = ay - y^3 = y(a - y^2)$$

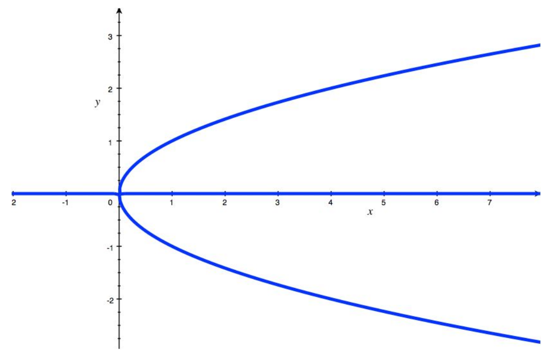
(a) Again consider the cases $a < 0, a = 0$ and $a > 0$. In each case, find the critical points, draw the phase line, and determine whether each critical point is asymptotically stable, semistable, or unstable.

¶The analysis from the previous question is identical with the addition that $y = 0$ is in every case.

(b) In each case, sketch several solutions of the ODE in the y-plane

¶Use the previous part.

(c) Draw the bifurcation diagram for the ODE. Note that $a = 0$ is a pitch fork bifurcation.



2.5 - # 16 Find b such that the equation is exact. Then solve.

$$\underbrace{(ye^{2xy}+x)}_{M}dx + \underbrace{bxe^{2xy}}_{N}dy = 0$$

Solution For the equation to be exact we need the partials to match. i.e. $M_y = N_x$. We compute

$$M_y = \frac{\partial}{\partial y}(ye^{2xy}+x) = e^{2xy} + 2xye^{2xy}$$
$$N_x = \frac{\partial}{\partial x}(bxe^{2xy}) = be^{2xy} + 2bxye^{2xy}$$

We easily see that we need $b = 1$ for the above function to be equal. Thus the equation is exact! Now we solve by comparing the integrand of M and N.

$$\int M dx = \int (ye^{2xy}+x)dx = \frac{e^{2xy}}{2} + \frac{x^2}{2} + C(y)$$
$$\int N dy = \int xe^{2xy}dy = \frac{e^{2xy}}{2} + \tilde{C}(x)$$

By comparing both integrals, we deduce

$$F(x,y) = \frac{e^{2xy}}{2} + \frac{x^2}{2}$$

is a function that satisfies

$$\frac{\partial F}{\partial x} = M \quad \& \quad \frac{\partial F}{\partial y} = N$$

Therefore our solution must be

$$\boxed{const = \frac{e^{2xy}}{2} + \frac{x^2}{2}}$$

2.5 - # 23 Show that if $(N_x - M_y)/M = Q$, where Q is a function of y only, then the differential equation

$$M + Ny' = 0$$

has an integrating factor of the form

$$\mu(y) = \exp \int Q(y)dy$$

Solution Suppose that $M + Ny' = 0$ is not exact and consider

$$\underbrace{\mu(y)M}_{\bar{M}}\,dx + \underbrace{\mu(y)N}_{\bar{N}}\,dy = 0$$

We'll try to find the condition on μ to make this exact. How do we do this? Check $M'_y = N'_x$.

$$\bar{M}_y = \frac{\partial}{\partial y}(\mu(y)M) = \mu'(y)M + \mu(y)M_y$$

$$\bar{N}_x = \frac{\partial}{\partial x}(\mu(y)N) = \mu(y)N_x$$

Using these equations, we can form an ODE in μ. Namely

$$0 = \bar{N}_x - \bar{M}_y = \mu(y)(N_x - M_y) - \mu'(y)M \iff \frac{\mu'(y)}{\mu(y)} = \frac{N_x - M_y}{M} = Q$$

By solving the above ODE for μ, we obtain

$$\boxed{\mu(y) = \exp\int Q(y)dy}$$

2.5 - # 26 Find an integrating factor and solve the given equation

$$y' = e^{2x} + y - 1$$

Solution Rewrite the ODE in differential form

$$\underbrace{(e^{2x} + y - 1)}_{M}\,dx + \underbrace{(-1)}_{N}\,dy = 0$$

We check the partials.

$$M_y = 1$$

$$N_x = 0$$

Since the equation is not exact, we'll need an integrating factor. Following the same logic as the previous question, we deduce

$$\mu(x) = \exp\int\left(\frac{M_y - N_x}{N}\right)dx = \exp\left(-\int dx\right) = e^{-x}$$

will work. Let's check

$$\underbrace{(e^x + e^{-x}(y-1))}_{\bar{M}}\,dx + \underbrace{(-e^{-x})}_{\bar{N}}\,dy = 0$$

$$\bar{M}_y = e^{-x}$$

$$\bar{N}_x = e^{-x}$$

Now the equation is exact! Thus we can just integrate each part respectively.

$$\int \bar{M}dx = \int(e^x + e^{-x}(y-1))dx = e^x + e^{-x}(1-y) + C(y)$$

$$\int \bar{N}dy = \int -e^{-x}dy = -ye^{-x} + \tilde{C}(x)$$

By comparing the above equation, we see that a function satisfying the partials is

$$F(x,y) = e^x + e^{-x}(1-y)$$

This implies the general solution is

$$\boxed{const = e^x + e^{-x}(1-y)}$$

2.4 - # 18 A pond forms as water collects in a conical depression of radius a and depth h. Suppose that water flows in at a constant rate k and is lost through evaporation at a rate proportional to the surface area.

(a) Show that the volume $V(t)$ of water in the pond at time t satisfies the differential equation

$$dV/dt = k - \alpha\pi(3a/\pi h)^{2/3}V^{2/3}$$

where α is the coefficient of evaporation

¶The model we'd like to use is

$$\frac{dV}{dt} = V_{in} - V_{out}$$

we're given that $V_{in} = k$, and that $V_{out} = \alpha SA$ (out of the top, i.e. just a circle). We just have to compute the surface area of the cone in terms of it's Volume. Recall that

$$V_{cone} = \frac{\pi r^2 l}{3} \quad \& \quad SA_{circle} = \pi r^2$$

where r is radius and l is the length. By drawing a picture, you'll find that the ratio between the length and radius is always the same i.e. $l/r = h/a$. Thus we have

$$V_{cone} = \frac{\pi r^2 l}{3} = \frac{\pi r^3 h}{3a} \implies \sqrt[3]{\frac{3aV_{cone}}{\pi h}} = r$$

$$\implies SA = \pi\left(\frac{3aV_{cone}}{\pi h}\right)^{2/3}$$

Therefore, the ODE is

$$\boxed{dV/dt = k - \alpha\pi(3a/\pi h)^{2/3}V^{2/3}}$$

(b) Find the equilibrium depth of water in the pond. Is the equilibrium asymptotically stable?

¶Recall that equilibrium occurs when $V' = 0$, so we have to find the roots of the ODE. We see

$$\frac{dV}{dt} = k - \alpha\pi(3a/\pi h)^{2/3}V^{2/3} = 0 \iff \boxed{V = \pm\frac{(k/\alpha\pi)^{3/2}\pi h}{3a}}$$

Since the Volume cannot be negative, we discard that root. To find the depth l, just substitute back in as in the previous part.

(c) Find a condition that must be satiated if the pond is not to overflow.

¶For the pond to not overflow, we need $dV/dt = 0$ when the cone is full. Thus

$$V_{cone} = \frac{\pi a^2 h}{3} = \frac{(k/\alpha\pi)^{3/2}\pi h}{3a} \implies \boxed{k = \alpha\pi a^{4/3}}$$