MAT292 - Tutorial 2 - C.J. Adkins

$\boxed{(1.2\text{-}\#33)}$ Show if a & λ are positive constants, and $b \in \mathbb{R}$, then all solutions of

$$y' + ay = be^{-\lambda t}$$

has the property $y \to 0$ as $t \to \infty$.

¶ Since the equation is first order linear, we know $y = \frac{1}{\mu}\int \mu g$ is the solution, where $g = be^{-\lambda t}$ & $\mu = \exp(\int a) = e^{at}$, thus

$$y(t) = \frac{1}{e^{at}}\int be^{(a-\lambda)t}\,dt = \begin{cases} \frac{b}{a-\lambda}e^{-\lambda t} + Ce^{-at} & \text{if } a \neq \lambda \\ bte^{-at} + Ce^{-at} & \text{if } a = \lambda \end{cases}$$

can check with L'H

Since $\lambda, a > 0 \Rightarrow y \to 0$ as $t \to \infty$ (i.e $\lim_{t\to\infty} e^{-at} = \lim_{t\to\infty} te^{-at} = 0$)

$\boxed{(2.1\text{-}\#30)}$ Consider

$$\frac{dy}{dx} = \frac{y - 4x}{x - y}$$

a) Notice that we may rewrite the equation in the form:

$$\frac{dy}{dx} = \frac{y-4x}{x-y} = \frac{\frac{1}{x}(y-4x)}{\frac{1}{x}(x-y)} = \frac{\frac{y}{x} - 4}{1 - \frac{y}{x}}$$

b) let $v(x) = \frac{y(x)}{x}$, i.e $y = v(x)x \Rightarrow \frac{dy}{dx} = \frac{dv}{dx}x + v(x)$. Therefore the O.D.E becomes

c) $$\frac{dv}{dx}x + v = \frac{v-4}{1-v} \iff v'x = \frac{v - 4 - v(1-v)}{1-v} = \frac{v^2-4}{1-v}$$

d) Now the O.D.E is seperable! lets solve!

$$\int \frac{1-v}{v^2-4}dv = \int \frac{dx}{x} \iff \underbrace{\int \frac{dv}{v^2-4}}_{2} - \underbrace{\int \frac{v\,dv}{v^2-4}}_{1} = \ln|x| + C$$

1 u-sub, $u = v^2 - 4$, $du = 2v\,dv$

$$\therefore \int \frac{v\,dv}{v^2-4} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|v^2-4|$$

2 by partial fractions,

$$\frac{1}{v^2-4} = \frac{1}{(v+2)(v-2)} = \frac{A}{v+2} + \frac{B}{v-2} \iff A(v-2) + B(v+2) = 1 \Rightarrow A = -\frac{1}{4},\ B = \frac{1}{4}$$

$$\therefore \int \frac{dv}{v^2-4} = \frac{1}{4}\int \frac{1}{v-2} - \frac{1}{v+2}\,dv = \frac{1}{4}\ln(v-2) - \frac{1}{4}\ln(v+2) = \frac{1}{4}\ln\left(\frac{2-v}{v+2}\right)$$

Now we put it all together!

Our solution is...

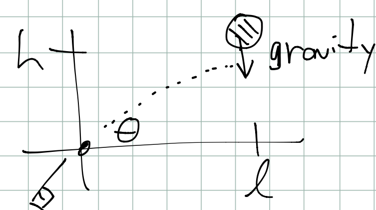
$$\frac{1}{4}\log\left(\frac{2-v}{v+2}\right) - \frac{1}{2}\ln(v^2-4) = \ln(x) + C$$

$$\Rightarrow \sqrt[4]{\frac{2-v}{v+2}}\sqrt{\frac{1}{(v-2)(v+2)}} = Cx \iff \frac{C}{x} = \sqrt[4]{(v+2)(v-2)}\sqrt{v+2}$$

In the orginal function y, we have

$$\frac{C}{x} = \left(\frac{y}{x}+2\right)^{3/4}\left(\frac{y}{x}-2\right)^{1/4}$$

$\boxed{\text{Example not from text.}}$ Find θ so the shot hits the ball



1) path of ball. We have that gravity $\Rightarrow y(t) = h - \frac{gt^2}{2}$ (from rest)

$x(t) = \ell$

$$\therefore \gamma_{Ball}(t) = \left(\ell, h - \frac{gt^2}{2}\right)$$

2) path of bullet. lets say it shoots out with velocity v_0.

$\Rightarrow$ x velocity$_i$ $= v_0 \cos\theta$ ... so $y(t) = v_0 t \sin\theta - \frac{gt^2}{2}$

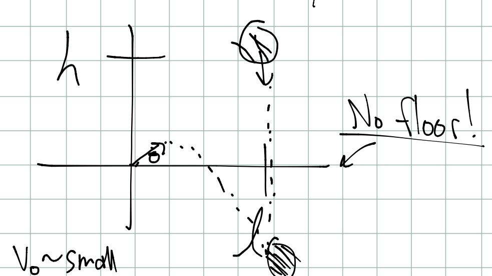
y velocit$_i$ $= v_0 \sin\theta$ $\quad x(t) = v_0 t \cos\theta$

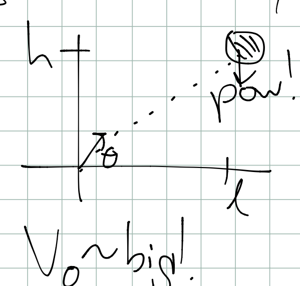
Thus $\gamma_{bullet}(t) = \left(v_0 t\cos\theta, v_0 t\sin\theta - \frac{gt^2}{2}\right)$

We want $\gamma_{bullet}(t_0) = \gamma_{Ball}(t_0) \Rightarrow v_0 t_0 \cos\theta = \ell$ & $h - \cancel{\frac{gt^2}{2}} = v_0 t_0 \sin\theta - \cancel{\frac{gt^2}{2}} \iff h = v_0 t_0 \sin\theta$

$$\Rightarrow \frac{h}{\ell} = \tan\theta \Rightarrow \arctan\left(\frac{h}{\ell}\right) = \theta$$ (i.e point directly at the ball!)

Why do we not see velocity? Well...we forgot the floor...actually!





Note the critical case is $\sqrt{\frac{2h}{g}} = t_{crit}$ (if $t_0 > t_{crit}$ we've gone through the floor)

(2.2-#16) Newton's Law of cooling

$$u' = -k(u - T_0)$$

if coffee obeys the above, $u(0) = 200°F$, 1 minute later it's $190°F$ in a room at 70 find when the coffee is $150°F$

→ Solve the O.D.E!... from before $u(t) = e^{-kt}(u_0 - T_0) + T_0 = e^{-kt}(200-70) + 70$
$= 130\exp(-kt) + 70$

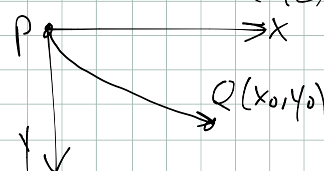
Find k! we know 1 minute in $u(1) = 190$... so

$$190 = 130e^{-k} + 70 \iff \ln\left(\frac{13}{12}\right) = k$$

Now we can find when $u(t) = 150$

$$150 = 130e^{-kt} + 70 \iff \ln\left(\frac{13}{8}\right) = kt \iff t = \frac{\ln(13/8)}{\ln\left(\frac{13}{12}\right)}$$

(2.2-#31) (Brachistochrone Problem)



Minimum curve (in time) is

$$\left(1 + \left(\frac{dy}{dx}\right)^2\right) y = k^2$$

a) solve for $\frac{dy}{dx}$, $\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = \frac{k^2}{y} \iff \frac{dy}{dx} = \pm\sqrt{\frac{k^2}{y} - 1}$ (take + root since + velocity)

b) let $y = k^2\sin^2 t \Rightarrow \frac{dy}{dx} = k^2 2\sin t\cos t\frac{dt}{dx}$, plug this in

$$\Rightarrow k^2 2\sin t\cos t\frac{dt}{dx} = \sqrt{\frac{k^2}{k^2\sin^2 t} - 1} = \frac{\cos t}{\sin t} \iff 2k^2\sin^2 t\frac{dt}{dx} = 1$$

c) let $\theta = 2t$, then $k^2\sin^2\frac{\theta}{2}\frac{d\theta}{dx} = 1$ ← Separable eq, solve

(const is 0 by $t=0 \Rightarrow \begin{matrix} x=0 \\ y=0 \end{matrix}$)

$$k^2\int\sin^2\frac{\theta}{2}\,d\theta = \int dx \iff k^2\int\underbrace{\frac{1-\cos\theta}{2}}_{\text{half angle}}\,d\theta = x \Rightarrow k^2(\theta - \sin\theta)/2 = x$$

since $y = k^2\sin^2\left(\frac{\theta}{2}\right) = k^2\left(\frac{1-\cos\theta}{2}\right)$

we see $\begin{cases} k^2(\theta - \sin\theta)/2 = x \\ k^2(1-\cos\theta)/2 = y \end{cases}$ are parametric equations of the solution. This is called a cycloid.

d) Find k if $x_0 = 1$ & $y_0 = 2$. i.e there is θ_0 s.t k^2 works.

$$\begin{matrix} k^2(\theta_0 - \sin\theta_0) = 2 \\ k^2(1-\cos\theta_0) = 4 \end{matrix} \Rightarrow \frac{1-\cos\theta_0}{\theta_0 - \sin\theta_0} = 2 \xrightarrow{\text{By Computer}} \theta_0 \approx \sqrt{2} \Rightarrow k \approx 2.193$$