MAT292 - Tutorial 1 - C.J. Adkins

The Questions the Prof suggest:

Solve: $y' = 1+2y$

Well, notice the equation is seperable, hence

$$\frac{dy}{dx} = 1+2y \iff \int \frac{dy}{1+2y} = \int dx \therefore \frac{1}{2}\ln(1+2y) = x + \check{C} \iff 1+2y = \tilde{C}e^{2x}$$

The solution is $\boxed{y(x) = C\exp(2x) - \frac{1}{2}}$

Solve: $y' = -(1+2y)$

following the previous solution, we see the solution is:

$$\boxed{y(x) = C\exp(-2x) - \frac{1}{2}}$$

(1.1-#34) A certain drug is administered to some guy. Fluid contains $5\,\frac{mg}{mL}$ of the drug, and the guy receives it at a rate of $100\,mL/h$. The drug goes in cells or leaving the body at a rate proportional to the amount present with a rate constant of $0.4/h$

story

initial data

O.D.E

a) Write the O.D.E (assuming nice distribution of the drug)

Let Q be the amount of the drug (mg) in the system. Thus

$$Q' = Q_{in} - \alpha Q \quad , Q_{in} = 5\,\frac{mg}{mL} \cdot 100\,\frac{mL}{h} = 500\,\frac{mg}{h} \quad , \alpha = 0.4/h$$

$$\therefore \boxed{Q' = 500 - 0.4Q}$$

b) After a long time, how much of the drug is there?

Well... we'd expect an equilibrium, so $Q' = 0 \Rightarrow \boxed{Q = \frac{5}{2} \cdot 500 = 1250\,mg}$

(1.2-#32) Show that all solutions of $2y' + ty = 2$ have a limit as $t \to \infty$

Well... you can solve for y, so lets do that. Use the integrating factor method, i.e if $y' + py = g$, then take $\mu = \exp\left(\int p\,dt\right)$, so $y = \frac{1}{\mu}\int g\mu\,dt$. In our case:

$$y' + \frac{t}{2}y = 1, \text{ so } \mu = \exp\left(\int \frac{t}{2}dt\right) = e^{t^2/4} \Rightarrow y(t) = e^{-\frac{t^2}{4}}\int e^{\frac{t^2}{4}}dt$$

Now lets check:

$$\lim_{t\to\infty} \frac{\int e^{t^2/4}dt}{e^{t^2/4}} \overset{L'H}{=} \lim_{t\to\infty} \frac{e^{t^2/4}}{\frac{t}{2}e^{t^2/4}} = \lim_{t\to\infty} \frac{2}{t} = 0$$, thus all solutions have a limit.

(1.1-#29) Discrete time approx for

$$u' = -k(u-T_0), \quad u(0) = u_0$$

using $u'(t_j) \cong \frac{u(t_j+\Delta t) - u(t_j)}{\Delta t} = \frac{u(t_{j+1}) - u(t_j)}{\Delta t}$

where $\Delta t = \frac{t}{n}$ & $t_j = j\Delta t$

a) turn the O.D.E into a discrete eq. (let $u(t_j) = u_j$)

$$u' = -k(u-T_0) \Rightarrow \frac{u_{j+1} - u_j}{\Delta t} = -k(u_j - T_0) \Leftrightarrow u_{j+1} = (1-k\Delta t)u_j + kT_0\Delta t$$

b) find a form for u_j in terms of u_0 (i.e initial data).

Proceed by induction.

$$u_1 = (1-k\Delta t)u_0 + kT_0\Delta t, \quad u_2 = (1-k\Delta t)u_1 + kT_0\Delta t = (1-k\Delta t)^2 u_0 + (1-k\Delta t)kT_0\Delta t + kT_0\Delta t$$ ✓

That shows the base case, so suppose

$$u_n = (1-k\Delta t)^n u_0 + kT_0\Delta t \sum_{j=0}^{n-1} (1-k\Delta t)^j$$

i.e the nth case holds, and prove the n+1 th case.

$$u_{n+1} = (1-k\Delta t)u_n + kT_0\Delta t = \underbrace{(1-k\Delta t)(1-k\Delta t)^n u_0}_{(1-k\Delta t)^{n+1}} + \underbrace{(1-k\Delta t)\, kT_0\Delta t \sum_{j=0}^{n-1} (1-k\Delta t)^j + kT_0\Delta t}_{kT_0\Delta t \sum_{j=0}^{n} (1-k\Delta t)^j}$$

that shows the formula is true. Thus, using the formula for geometric series, we get

$$\boxed{u_n = (1-k\Delta t)^n u_0 + T_0\left(1 - (1-k\Delta t)^n\right)}$$

$$\sum_{n=0}^{N} a^n = \frac{1-a^n}{1-a}$$

c) Show that $\lim_{n\to\infty}\left(1-\frac{kt}{n}\right)^n = e^{-kt}$

This depends on the definition of the exp, lets assume exp(x) is the solution to $u' = u$. Thus, we need to show

$u = \lim_{n\to\infty}\left(1-\frac{kt}{n}\right)^n$ solves $u' = -ku$, well... notice

$$\frac{d}{dt}\left(1-\frac{kt}{n}\right)^n = -k\left(1-\frac{kt}{n}\right)^{n-1}, \text{ then note } u = \lim_{n\to\infty}\left(1-\frac{kt}{n}\right)^n = \lim_{n\to\infty}\left(1-\frac{kt}{n}\right)^{n-1}$$

$$\therefore \quad u' = -ku \Rightarrow \lim_{n\to\infty}\left(1-\frac{kt}{n}\right)^n = e^{-kt}$$

Thus,

$$\lim_{n\to\infty} u_n = e^{-kt}(u_0 - T_0) + T_0$$ ✓ ✓

✓

✓

This shows the discrete problem is exact under the limit.

Other Questions may be from tutorial 1&2 notes from MAT244 (Summer 2014).