Tutorial Problems #8

MAT 267 – Advanced Ordinary Differential Equations – Winter 2016

Christopher J. Adkins

Solutions

Gaussian Elimination Not much to say here, basically you may use gaussian elimination to simplify systems of equations...

pg.421 - # 7 Solve

$$\frac{dx}{dt} = 3x + 2e^{3t} \quad \& \quad \frac{dx}{dt} + \frac{dy}{dt} - 3y = \sin 2t$$

Solution Let's use elimination (you could solve x, first order linear, then solve the y equation). Rewrite the system in easy notation.

$$(D-3)x = 2e^{3t} \quad [1] \quad \& \quad Dx + (D-3)y = \sin 2t \quad [2]$$

Notice that

$$(D-3)[2] - D[1] \implies (D-3)^2 y = (D-3)\sin 2t + 6e^{3t} = 2\cos(2t) - 3\sin(2t) - 6e^{3t} \quad [2^*]$$

Now that both equations have been uncoupled, we solve using the techniques we know. By the method of undetermined coefficients, we know that

$$x(t) = (At + c_1)e^{3t}$$

will solve [1], we just need to find the $A \in \mathbb{R}$ that works. We see

$$e^{3t}(3(At + c_1) + A - 3At - 3c_1) = Ae^{3t} = 2e^{3t} \implies A = 2$$

Now we solve [2*], using the same method we know

$$y(t) = (c_2 t + c_3)e^{3t} + At^2 e^{3t} + B\cos(2t) + C\sin(2t)$$

will solve [2*], the coefficients turn out to be

$$A = -3 \quad \& \quad B = -\frac{2}{13} \quad \& \quad C = -\frac{3}{13}$$

It seems we've added an extra constant to the expression, but it can easily be solved using the homogeneous part of [2]. i.e.

$$x' + y' - y = e^{3t}(3c_1 + c_2 + 3c_3 - 3c_3) = 0 \implies c_2 = -3c_1$$

pg.421 - # 22 Solve

$$\begin{cases} (D-1)x & + & 0 & + & 0 & = & 0 \\ -x & + & (D-3)y & + & 0 & = & 0 \\ -x & + & y & + & (D-2)z & = & 0 \end{cases}$$

Solution Notice that we see that $x(t) = c_1 e^t$ from [1]. Next we see that

$$(D-1)[2] - [1] \implies (D-3)(D-1)y = 0 \quad [*] \quad \implies y = c_2 e^{3t} + c_3 e^t$$

Lastly,

$$[3] - [2] \implies (D+2)y + (D-2)z = 0 \quad [\#]$$

$$(D-3)(D-1)[\#] - [*] \implies (D-3)(D-1)(D-2)z = 0 \implies z = c_4 e^t + c_5 e^{2t} + c_6 e^{3t}$$

Now let's peg down the constants. By equation [2], we see

$$(D-3)y - x = 0 \implies (c_3 - 3c_3 - c_1)e^t = 0 \implies c_3 = -\frac{c_1}{2}$$

By equation [3] we see

$$(D-2)z + y - x = 0 \implies (c_4 - 2c_4 - \frac{c_1}{2} + c_1)e^t + (c_6 - 2c_6 - c_2)e^{3t} = 0 \implies c_4 = -\frac{3c_1}{2} \quad \& \quad c_6 = -c_2$$

Thus the general solution to the system is

$$\boxed{x(t) = c_1 e^t, \quad y(t) = c_2 e^{3t} - \frac{c_1}{2} e^t, \quad z(t) = -\frac{3c_1}{2} e^t - c_2 e^{3t} + c_5 e^{2t} \quad c_1, c_2, c_5 \in \mathbb{R}}$$

$\square$

2×2 **Phase Portraits i.e. Drawing a Picture of Possible Solutions to a System.** As we saw last time, a very convenient way to classify a system is via the eigenvalues of the associated matrix. The terminology is

$$\boxed{\operatorname{sgn}(\lambda_+) = \operatorname{sgn}(\lambda_-) \quad \& \quad \lambda_+ \neq \lambda_- \implies \text{node,} \begin{cases} \lambda_+, \lambda_- > 0 \implies & \text{unstable} \\ \lambda_+, \lambda_- < 0 \implies & \text{asymptotically stable} \end{cases}}$$

$$\boxed{\lambda_+ = \lambda_- \implies \text{improper node,} \begin{cases} \lambda_+, \lambda_- > 0 \implies & \text{unstable} \\ \lambda_+, \lambda_- < 0 \implies & \text{asymptotically stable} \end{cases}}$$

$$\boxed{\operatorname{sgn}(\lambda_+) \neq \operatorname{sgn}(\lambda_-) \implies \text{saddle, always unstable}}$$

$$\boxed{\lambda_\pm \in \mathbb{C} \quad \& \quad \lambda_+ \neq -\lambda_- \text{(i.e. non-zero real part)} \implies \text{spiral} \begin{cases} \lambda_+ + \lambda_- > 0 \implies & \text{unstable} \\ \lambda_+ + \lambda_- < 0 \implies & \text{asymptotically stable} \end{cases}}$$

$$\boxed{\lambda_\pm \in \mathbb{C} \quad \& \quad \lambda_+ = -\lambda_- \text{(i.e. complex valued, with zero real part)} \implies \text{centre, always stable}}$$

Just as we saw the eigenvectors were important for solving the system, they make drawing the system very simple.

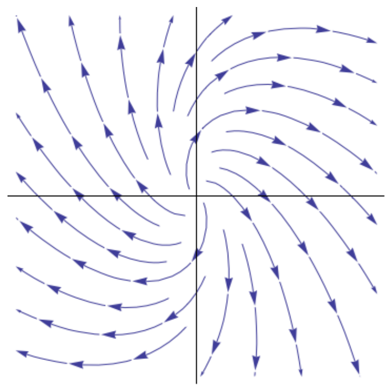
Exercise from Class Notes Draw the phase portrait for

$$\dot{x} = \begin{pmatrix} 5 & 2 \\ -1 & 3 \end{pmatrix} x$$

Solution First let's find the eigenvalues. We see via the characteristic equation that

$$P(\lambda) = \det(A - 1\lambda) = \begin{vmatrix} 5-\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} = (5-\lambda)(3-\lambda) + 2 = \lambda^2 - 8\lambda + 17 = 0 \implies \lambda_{\pm} = 4 \pm i$$

Thus the system is an unstable spiral. We see that the top right entry of the matrix A is 2, which is positive so the system spins clockwise. Thus we see have that



Exercise from Class Notes Draw the phase portrait for

$$\frac{dy}{dx} = \frac{x - 3y}{3x - 9y}$$

Solution Notice that this is the same as the

$$\dot{x} = \begin{pmatrix} 1 & -3 \\ 3 & -9 \end{pmatrix} x$$

And the eigenvalues of the matrix are given by

$$P(\lambda) = \begin{vmatrix} 1-\lambda & -3 \\ 3 & -9-\lambda \end{vmatrix} = \lambda^2 + 8\lambda \implies \lambda = 0, -8$$

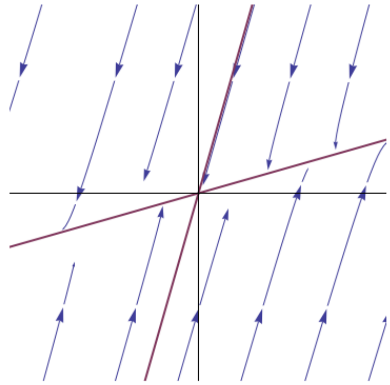
This is a degenerate type of solution, transitioning between a saddle and a node. Regardless, the same procedure applies. Let's find the eigenvalues. We see

$$\lambda = -8 \implies \ker \begin{pmatrix} 9 & -3 \\ 3 & -1 \end{pmatrix} = \text{span} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \implies \vec{\lambda}_{-8} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

is the eigenvector for $\lambda = -8$, then

$$\lambda = 0 \implies \ker \begin{pmatrix} 1 & -3 \\ 3 & -9 \end{pmatrix} = \text{span} \begin{pmatrix} 3 \\ 1 \end{pmatrix} \implies \vec{\lambda}_0 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Thus we see the phase portrait is given by



Quiz When the origin is a centre, i.e. the discriminant is negative and $a + d = 0$, show that

$$\frac{dy}{dx} = \frac{ay + bx}{cy + dx}$$

is an exact equation and integrate it to find the formula for the elliptic integral curves.

Solution In differential form we have

$$\underbrace{-(ay + bx)}_{=M} dx + \underbrace{(cy + dx)}_{=N} dy = 0$$

We see that

$$M_y = -a \quad \& \quad N_x = d \quad \text{then} \quad a + d = 0 \implies M_y = -a = d = N_x$$

so the equation is exact. We have

$$F(x, y) = \int (-ay - bx)dx \oplus \int (cy + dx)dy = -\frac{b}{2}x^2 - axy \oplus \frac{c}{2}y^2 + dxy = \frac{cy^2 + 2dxy - bx^2}{2} = const$$

Since the discriminant is negative, we must have that c and b are opposite signs, which means the above is an equation for an ellipse. $\square$