**Tutorial Problems #5**

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**Reduction of Order Via Differential Operators**  
Let \( D = \frac{d}{dx} \) be our differential operator. Then any \( n \)-th order linear non-homogeneous equation may be written as

\[
L(D)[y(x)] = f(x) \quad \text{where} \quad L(D) = D^n + a_{n-1}D^{n-1} + \ldots + a_0 \left( D^n = \frac{d^n}{dx^n} \right)
\]

with \( a_i \in \mathbb{R} \). Factor \( L \) into a product of its roots (which may be complex and we’ll deal with later), i.e.

\[
L(D) = (D - \lambda_1) \ldots (D - \lambda_n)
\]

Notice this factorization is not possible if \( a_i \) are functions since the differential operator isn’t commutative (\( D_1D_2 \neq D_2D_1 \)). Thus, if we let \( y_n = (D - \lambda_n)y \) and \( y_i = (D - \lambda_i)y_{i+1} \) we effectively reduce the \( n \)-th order equations into \( n \) first order equations (which we know how to handle).

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**pg. 267 - # 28**  
Solve (using reduction of order)

\[
y'' + y' = x^2 + 2x
\]

**Solution**  
We see that if \( L = D^2 + D \), then

\[
L(D)[y(x)] = x^2 + 2x
\]

is the ODE we’re looking to solve. Notice we may use the above method to deduce

\[
L(D) = D(D+1) \implies u' = x^2 + 2x \quad \text{where} \quad (D + 1)y = u
\]

The above ODE in \( u \) is separable, thus

\[
\frac{d}{dx} u(x) = \int x^2 + 2xdx = \frac{x^3}{3} + x^2 + C_1 \quad C_1 \in \mathbb{R}
\]

We now know

\[
y' + y = \frac{x^3}{3} + x^2 + C_1
\]

which is a first order linear ODE, we know this may be solved using an integrating factor. We know

\[
y(x) = \frac{1}{\mu(x)} \int \mu(x)g(x)dx \quad \text{where} \quad \mu(x) = \exp \left( \int dx \right) = e^x
\]
Thus the general solution to the ODE is

$$y(x) = e^{-x} \int e^x \left( \frac{x^3}{3} + x^2 + C_1 \right) \, dx = \frac{x^3}{3} + C_2 e^{-x} + C_1, \quad C_2 \in \mathbb{R}$$

The Inverse of a Differential Operator  Let’s talk about $D^{-1}$ now. Formally we need an operator with the property if $Dx = y$, then $x = D^{-1}y$. Intuitively, you should think the integral operator is a natural left inverse for $D$ since

$$\frac{d}{dx} \int f(x) \, dx = f(x)$$

by the fundamental theorem of calculus. Now what about factors of $(D - \lambda)$ we had...using a formal series expansion(notably a geometric series), we may algebraically write

$$(D - \lambda)^{-1} = -\frac{1}{\lambda(1 - D/\lambda)} = -\frac{1}{\lambda} \left[ 1 + \frac{D}{\lambda} + \frac{D^2}{\lambda^2} + \frac{D^3}{\lambda^3} + \ldots \right]$$

Convergence of this series is a slight issue at the moment...but for any solution that terminates after a finite number of derivatives we know convergence is guaranteed. Let’s revisit the example we just saw.

pg. 267 - # 28  Solve (using Inverse Operators)

$$y'' + y' = x^2 + 2x$$

Solution  As we saw before we have

$$D(D+1)y = x^2 + 2x \implies y_p(x) = \frac{1}{D(D+1)}(x^2 + 2x)$$

Notice we’ll only be able to pick up the particular solution to the ODE with this method (not the general) since $L$ is not injective in general(i.e. $L[y_{hom}] = 0$). Expanding the inverse into formal series shows

$$y_p(x) = \frac{1}{D} \left( 1 + D + D^2 \right) (x^2 + 2x) = \left[ \frac{1}{D} - 1 + D \right] (x^2 + 2x)$$

Thus

$$y_p(x) = \int (x^2 + 2x) \, dx - (x^2 + 2x) + \frac{d}{dx} (x^2 + 2x) = \frac{x^3}{3} + 2$$

You may recover the general solution using your knowledge of homogeneous equation, but seeing the eigenvalues of $\lambda = 0$ and $\lambda = -1$, thus 1 and $e^{-x}$ solve the homogeneous problem.

A Special Case, $(D - \lambda)^{-1}$ Applied To $e^{ax}$  Notice in the case of exponential, we may factor out $e^{ax}$ in the formal expansion since

$$\frac{d^n}{dx^n} e^{ax} = a^n e^{ax}$$

Thus

$$\frac{1}{D - \lambda} e^{ax} = \frac{e^{ax}}{\lambda} \left[ 1 + \frac{a}{\lambda} + \frac{a^2}{\lambda^2} + \ldots \right] = \frac{e^{ax}}{a - \lambda}$$
where we side-stepped the notion of convergence once again, but clearly this is an inverse since

\[(D - \lambda) \left( \frac{1}{D - \lambda} e^{ax} \right) = (D - \lambda) \frac{e^{ax}}{a - \lambda} = e^{ax} \]

Since this will work with any $a \in \mathbb{C}$ and the inverse raised to integer powers, we’ve therefore found a way to handle exponentials. The only issue that may occur is if $a = \lambda$ since the expansion isn’t defined (in other words, $a$ is an eigenvalue). This can easily be fixed using the exponential shift theorem,

\[L(D)[e^{ax}y] = e^{ax}L(D + a)[y] \]

when $L(x)$ is a polynomial (the proof goes by induction, and also applies to the inverse). Thus if $\lambda$ is a root of $L$, i.e. $L(D) = (D - \lambda)^k g(D)$ and $g(\lambda) \neq 0$, we see that

\[
\frac{1}{(D - \lambda)^k g(D)} e^{\lambda x} = e^{\lambda x} \frac{1}{D^k g(D + \lambda)} 1 = e^{\lambda x} \frac{x^k}{k!g(\lambda)}
\]

Note there is a similar version from the Laplace Transform which is defined as

\[\mathcal{L}[y(t)](s) \equiv \int_0^\infty e^{-st} y(t) dt\]

which is another useful tool for solving ODE’s. It takes the form $\mathcal{L}[e^{at} f(t)] = \mathcal{L}[f(t-a)]$.

pg. 282 - # 32  Solve

\[y''' + y' = \cos x\]

Solution  Well, in terms of $D$ we have that

\[D(D - i)(D + i)y = \cos x\]

Now since we’ve just dealt with exponentials so far, note $e^{ix} = \cos x + i \sin x$, so lets solve

\[D(D - i)(D + i)y = e^{ix}\]

and take the real part. Letting $g(D) = D(D + i)$ like the above, we see

\[y_p(x) = \frac{1}{(D - i)g(D)} e^{ix} = \frac{e^{ix}}{g(i)} = -\frac{e^{ix} x}{2} = -\frac{x \cos x}{2} - i \frac{x \sin x}{2}\]

Since we just want the real part of the solution, we see the particular solution to the ODE is

\[y_p(x) = -\frac{x \cos x}{2}\]

Noting that the eigenvalues of the equation are $\lambda = 0, \pm i$, we have that

\[y(x) = C_1 + C_2 \cos x + C_3 \sin x - \frac{x \cos x}{2}\]

is the general solution.

Partial Fraction Decomposition with Differential Operators  As you’ve probably seen before with polynomials, you may decompose

\[\frac{1}{(D + \lambda_1)(D + \lambda_2)} = \frac{c_1}{D + \lambda_1} + \frac{c_2}{D + \lambda_2}\]

Let’s see how this would apply to the previous example.
pg. 282 - # 32  Solve (using partial fractions)

\[ y''' + y' = \cos x \]

**Solution** Using what we saw before, let’s try to decompose into pieces:

\[
\frac{1}{D(D-i)(D+i)} = \frac{c_1}{D} + \frac{c_2}{D-i} + \frac{c_3}{D+i}
\]

This implies we need

\[
(D-i)(D+i)c_1 + D(D+i)c_2 + D(D-i)c_3 = 1 \implies \begin{cases} c_1 + c_2 + c_3 = 0 \\ c_2 - c_3 = 0 \\ c_1 = 1 \end{cases} \implies c_2 = -\frac{1}{2}, c_3 = -\frac{1}{2}
\]

Thus

\[
\frac{1}{D(D-i)(D+i)} = \frac{1}{D} - \frac{1}{2(D-i)} - \frac{1}{2(D+i)}
\]

Now if we apply this to \( e^{ix} \), we see

\[
\frac{1}{D(D-i)(D+i)} e^{ix} = \frac{e^{ix}}{i} - \frac{xe^{ix}}{2} - \frac{e^{ix}}{4i} + C = \frac{3}{4} e^{ix} - \frac{xe^{ix}}{2} + C
\]

If we take the real part of this solution we see the following particular solution

\[
y_p(x) = \frac{3}{4} \sin x - \frac{x \cos x}{2} + C
\]

**Quiz**  Find a partial solution using any inverse operator method for

\[ y'' + 3y' + 2y = 2(e^{-2x} + x^2) \]

**Solution** We see that

\[ L(D) = (D+2)(D+1) \]

Thus we want to solve

\[
y_p(x) = \frac{1}{(D+2)(D+1)}(2e^{-2x} + 2x^2)
\]

For the exponential, we may use what we’ve previous talked about to find that we have \( L = (D+2)g(D) \), hence

\[
y_{pe}(x) = \frac{2xe^{-2x}}{g(-2)} = -2xe^{-2x}
\]

For the polynomial, we have that

\[
y_{pp}(x) = \frac{1}{(D+2)(D+1)} 2x^2 = \frac{1}{2} \left( 1 - \frac{D}{2} + \frac{D^2}{4} \right) (1 - D + D^2) 2x^2 = \frac{1}{2} \left( 1 - \frac{3D}{2} + \frac{7D^2}{4} \right) 2x^2
\]

Thus we see

\[
y_{pp}(x) = x^2 - 3x + \frac{3}{2}
\]

So

\[
y_p(x) = -2xe^{-2x} + x^2 - 3x + \frac{7}{2}
\]

is a particular solution.