Tutorial Problems #5

MAT 267 – Advanced Ordinary Differential Equations – Winter 2016

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Solutions

Reduction of Order Via Differential Operators Let $D = \frac{d}{dx}$ be our differential operator. Then any n-th order linear non-homogeneous equation may be written as

$$L(D)[y(x)] = f(x) \quad \text{where} \quad L(D) = D^n + a_{n-1}D^{n-1} + \ldots + a_0 \qquad \left(D^n = \frac{d^n}{dx^n}\right)$$

with $a_i \in \mathbb{R}$. Factor L into a product of it's roots (which may be complex and we'll deal with later), i.e.

$$L(D) = (D - \lambda_1)\ldots(D - \lambda_n)$$

Notice this factorization is not possible if a_i are functions since the differential operator isn't commutative ($D_1 D_2 = D_2 D_1$). Thus, if we let $y_n = (D - \lambda_n)y$ and $y_i = (D - \lambda_i)y_{i+1}$ we effectively reduce the n-th order equations into n first order equations (which we know how to handle)

pg. 267 - # 28 Solve (using reduction of order)

$$y'' + y' = x^2 + 2x$$

Solution We see that if $L = D^2 + D$, then

$$L(D)[y(x)] = x^2 + 2x$$

is the ODE we're looking to solve. Notice we may use the above method to deduce

$$L(D) = D(D+1) \implies u' = x^2 + 2x \quad \text{where} \quad (D+1)y = u$$

The above ODE in u is separable, thus

$$u(x) = \int x^2 + 2x dx = \frac{x^3}{3} + x^2 + C_1 \quad C_1 \in \mathbb{R}$$

We now know

$$y' + y = \frac{x^3}{3} + x^2 + C_1$$

which is a first order linear ODE, we know this may be solved using an integrating factor. We know

$$y(x) = \frac{1}{\mu(x)} \int \mu(x) g(x) dx \quad \text{where} \quad \mu(x) = \exp\left(\int dx\right) = e^x$$

Thus the general solution to the ODE is

$$\boxed{y(x) = e^{-x}\int e^{x}\left(\frac{x^3}{3}+x^2+C_1\right)dx = \frac{x^3}{3}+C_2e^{-x}+C_1 \quad C_2\in\mathbb{R}}$$

$\square$

The Inverse of a Differential Operator Let's talk about D^{-1} now. Formally we need an operator with the property if $Dx = y$, then $x = D^{-1}y$. Intuitively, you should think the integral operator is a natural left inverse for D since

$$\frac{d}{dx}\int f(x)dx = f(x)$$

by the fundamental theorem of calculus. Now what about factors of $(D-\lambda)$ we had...using a formal series expansion(notably a geometric series), we may algebraically write

$$(D-\lambda)^{-1} = -\frac{1}{\lambda(1-D/\lambda)} = -\frac{1}{\lambda}\left[1+\frac{D}{\lambda}+\frac{D^2}{\lambda^2}+\frac{D^3}{\lambda^3}\dots\right]$$

Convergence of this series is a slight issue at the moment...but for any solution that terminates after a finite number of derivatives we know convergence is guaranteed. Let's revisit the example we just saw.

pg. 267 - # 28 Solve (using Inverse Operators)

$$y'' + y' = x^2 + 2x$$

Solution As we saw before we have

$$D(D+1)y = x^2+2x \implies y_p(x) = \frac{1}{D(D+1)}(x^2+2x)$$

Notice we'll only be able to pick up the particular solution to the ODE with this method (not the general) since L is not injective in general(i.e. $L[y_{hom}] = 0$). Expanding the inverse into formal series shows

$$y_p(x) = \frac{1}{D}\left(1+D+D^2\right)(x^2+2x) = \left[\frac{1}{D}-1+D\right](x^2+2x)$$

Thus

$$\boxed{y_p(x) = \int (x^2+2x)dx - (x^2+2x) + \frac{d}{dx}(x^2+2x) = \frac{x^3}{3}+2}$$

You may recover the general solution using your knowledge of homogeneous equation, but seeing the eigenvalues of $\lambda = 0$ and $\lambda = -1$, thus 1 and e^{-x} solve the homogeneous problem. $\square$

A Special Case, $(D-\lambda)^{-1}$ Applied To e^{ax} Notice in the case of exponential, we may factor out e^{ax} in the formal expansion since

$$\frac{d^n}{dx^n}e^{ax} = a^n e^{ax}$$

Thus

$$\frac{1}{D-\lambda}e^{ax} = -\frac{e^{ax}}{\lambda}\left[1+\frac{a}{\lambda}+\frac{a^2}{\lambda^2}+\dots\right] = \frac{e^{ax}}{a-\lambda}$$

where we side-stepped the notion of convergence once again, but clearly this is an inverse since

$$(D-\lambda)\left(\frac{1}{D-\lambda}e^{ax}\right) = (D-\lambda)\frac{e^{ax}}{a-\lambda} = e^{ax}$$

Since this will work with any $a \in \mathbb{C}$ and the inverse raised to integer powers, we've therefore found a way to handle exponentials. The only issue that may occur is if $a = \lambda$ since the expansion isn't defined (in other words, a is an eigenvalue). This can easily be fixed using the exponential shift theorem,

$$L(D)[e^{ax}y] = e^{ax}L(D+a)[y]$$

when $L(x)$ is a polynomial (the proof goes by induction, and also applies to the inverse). Thus if λ is a root of L, i.e. $L(D) = (D-\lambda)^k g(D)$ and $g(\lambda) \neq 0$, we see that

$$\frac{1}{(D-\lambda)^k g(D)}e^{\lambda x} = e^{\lambda x}\frac{1}{D^k g(D+\lambda)}1 = e^{\lambda x}\frac{x^k}{k!g(\lambda)}$$

Note there is a similar version from the Laplace Transform which is defined as

$$\mathcal{L}[y(t)](s) \equiv \int_0^\infty e^{st}y(t)dt$$

which is another useful tool for solving ODE's. It takes the form $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t-a)]$.

pg. 282 - # 32 Solve

$$y''' + y' = \cos x$$

Solution Well, in terms of D we have that

$$D(D-i)(D+i)y = \cos x$$

Now since we've just dealt with exponentials so far, note $e^{ix} = \cos x + i\sin x$, so lets solve

$$D(D-i)(D+i)y = e^{ix}$$

and take the real part. Letting $g(D) = D(D+i)$ like the above, we see

$$y_p(x) = \frac{1}{(D-i)g(D)}e^{ix} = e^{ix}\frac{x}{g(i)} = -e^{ix}\frac{x}{2} = -\frac{x\cos x}{2} - i\frac{x\sin x}{2}$$

Since we just want the real part of the solution, we see the particular solution to the ODE is

$$y_p(x) = -\frac{x\cos x}{2}$$

Noting that the eigenvalues of the equation are $\lambda = 0, \pm i$, we have that

$$\boxed{y(x) = C_1 + C_2\cos x + C_3\sin x - \frac{x\cos x}{2}}$$

is the general solution. □

Partial Fraction Decomposition with Differential Operators As you've probably seen before with polynomials, you may decompose

$$\frac{1}{(D+\lambda_1)(D+\lambda_2)} = \frac{c_1}{D+\lambda_1} + \frac{c_2}{D+\lambda_2}$$

Let's see how this would apply to the previous example.

pg. 282 - # 32 Solve (using partial fractions)

$$y''' + y' = \cos x$$

Solution Using what we saw before, let's try to decompose into pieces:

$$\frac{1}{D(D-i)(D+i)} = \frac{c_1}{D} + \frac{c_2}{D-i} + \frac{c_3}{D+i}$$

This implies we need

$$(D-i)(D+i)c_1 + D(D+i)c_2 + D(D-i)c_3 = 1 \implies \begin{cases} c_1 + c_2 + c_3 = 0 \\ c_2 - c_3 = 0 \\ c_1 = 1 \end{cases} \implies c_2 = -\frac{1}{2}, c_3 = -\frac{1}{2}$$

Thus

$$\frac{1}{D(D-i)(D+i)} = \frac{1}{D} - \frac{1}{2(D-i)} - \frac{1}{2(D+i)}$$

Now if we apply this to e^{ix}, we see

$$\frac{1}{D(D-i)(D+i)}e^{ix} = \frac{e^{ix}}{i} - \frac{xe^{ix}}{2} - \frac{e^{ix}}{4i} + C = \frac{3}{4i}e^{ix} - \frac{xe^{ix}}{2} + C$$

If we take the real part of this solution we see the following particular solution

$$\boxed{y_p(x) = \frac{3}{4}\sin x - \frac{x\cos x}{2} + C}$$

□

Quiz Find a partial solution using any inverse operator method for

$$y'' + 3y' + 2y = 2(e^{-2x} + x^2)$$

Solution We see that

$$L(D) = (D+2)(D+1)$$

Thus we want to solve

$$y_p(x) = \frac{1}{(D+2)(D+1)}(2e^{-2x} + 2x^2)$$

For the exponential, we may use what we've previous talked about to find that we have $L = (D+2)g(D)$, hence

$$y_{p_e}(x) = \frac{2xe^{-2x}}{g(-2)} = -2xe^{-2x}$$

For the polynomial, we have that

$$y_{p_p}(x) = \frac{1}{(D+2)(D+1)}2x^2 = \frac{1}{2}\left(1 - \frac{D}{2} + \frac{D^2}{4}\right)\left(1 - D + D^2\right)2x^2 = \frac{1}{2}\left(1 - \frac{3D}{2} + \frac{7D^2}{4}\right)2x^2$$

Thus we see

$$y_{p_p}(x) = x^2 - 3x + \frac{3}{2}$$

So

$$\boxed{y_p(x) = -2xe^{-2x} + x^2 - 3x + \frac{7}{2}}$$

is a particular solution. □