Tutorial Problems #3

MAT 267 – Advanced Ordinary Differential Equations – Spring 2016

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Solutions

pg.109 - # 2 - Petrov Solve

$$\begin{cases} xy_1' = 2y_1 - y_2 \\ xy_2' = 2y_1 - y_2 \end{cases}$$

(a) Show if $x_0 \neq 0$, the solution exists and is unique on the real axis and if $x_0 = 0$, the solution exists only if $2y_1 - y_2 = 0$ and is not unique.

(b) Show the Wronskian of the linearity independent solutions is Cx with $C \neq 0$,

Solution We'll first solve the system. Notice that

$$xy_1' = xy_2' \implies y_1' = y_2' \quad \text{when} \quad x \neq 0$$

Thus $y_1 = y_2 + C_1$ with some constant $C_1 \in \mathbb{R}$. Using this, we see the system reduces to

$$xy_1' = 2y_1 - y_1 - C_1 = y_1 - C_1$$

This equation is separable, thus

$$\int \frac{dy_1}{y_1 - C_1} = \int \frac{dx}{x} \implies \ln|y_1 - C_1| = \ln|x| + C \implies y_1 = C_1 + C_2 x$$

Now that we have y_1 it's easy to see that

$$y_2 = 2C_1 + C_2 x$$

You may write this in vector notation as

$$\boxed{y(x) = C_1 y^{(1)} + C_2 y^{(2)} = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 x \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

We compute the Wronskian by definition:

$$W(x) = \det(y^{(1)} y^{(2)}) = \begin{vmatrix} 1 & x \\ 2 & x \end{vmatrix} = Cx \quad \text{where} \quad C \neq 0$$

Notice that $x_0 \neq 0$ implies the Wronskian is non-zero as long as x remains on x_0's side of zero, hence the two solutions we found are linearly independent and unique. If $x_0 = 0$, then $W(x) = 0$ since it is either always non-zero or zero, we know the solutions cannot be linearly independent, i.e. $y_1 = ay_2$ with some $a \in \mathbb{R}$. But this means that we need $xy_1' = xy_2' = axy_1'$ which implies that $2y_1 - y_2 = 0$ for all x. So

$$y_1 = C \quad \& \quad y_2 = 2C \quad \text{where} \quad C \in \mathbb{R}$$

$\square$

n-th order ODE's as first order systems Notice that we have

$$y^{(n)} + p_{n-1}y^{(n-1)} + \ldots + p_0 y^{(0)} = 0 \iff \dot{x} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & & 1 \\ -p_0 & -p_1 & \ldots & \ldots & -p_{n-1} \end{pmatrix} x \quad \text{where} \quad x = \begin{pmatrix} y \\ y' \\ \vdots \\ \vdots \\ y^{(n-1)} \end{pmatrix}$$

HW-1 Give an example of an equation $y' = f(x, y)$ with continuous f on $\mathbb{R}^2$ and a sequence of Euler Curves $y_{\epsilon_k}(x)$ with $\epsilon_k \to 0$, defined on some interval $[x_0 - \delta, x_0 + \delta]$ passing through a point (x_0, y_0), such that the sequence $y_{\epsilon_k}(x)$ does not converge on $[x_0 - \delta, x_0 + \delta]$

Solution Consider $y' = \sqrt{|y|}$, it is clearly condition on $\mathbb{R}^2$. Then take the sequence of alternating max and min of $f(x, y)$ built around the point $(x_0, 0)$ for any $x_0 \geqslant 0$. As we've seen the minimizing limit inferior gives

$$y(x) = 0$$

in the limit, but the upper limit is given by

$$y(x) = \begin{cases} \frac{1}{4}(x - x_0)^2 & x \geqslant x_0 \\ 0 & x < x_0 \end{cases}$$

Thus

$$\max_{x \in [x_0 - \delta, x_0 + \delta]} |\limsup y_{\epsilon_k} - \liminf y_{\epsilon_k}| = \delta^2/4$$

$\square$

HW-2 Suppose that a function $f(x, y)$ satisfies the condition of Osgood's theorem,

$$|f(x, y_1) - f(x, y_2)| \leqslant \varphi(|y_1 - y_2|)$$

on the domain D which is convex, and suppose that $\varphi(0) = 0$ and $\varphi'(0) = 0$. Prove that f doesn't depend on y.

Solution By the Mean Value Theorem (have a convex domain), we have

$$|f(x, y_1) - f(x, y_2)| = \left| \frac{\partial f}{\partial y}(x, \xi(x)) \right| |y_1 - y_2| \leqslant \varphi(|y_1 - y_2|)$$

Then we have that by taking the limit as $y_1 \to y_2 = y$

$$\left| \frac{\partial f}{\partial y}(x, y(x)) \right| \leqslant \lim_{y_1 \to y_2} \frac{\varphi(|y_1 - y_2|)}{|y_1 - y_2|} = \lim_{h \to 0} \frac{\varphi(h) - \varphi(0)}{h} = \varphi'(0) = 0$$

which implies that f doesn't depend on y. $\square$

Picard Iterations for first order systems Suppose that

$$\begin{cases} \dot{x} = F(t, x(t)) \\ x(t_0) = x_0 \end{cases} \qquad x \in \mathbb{R}^n, \quad F(t,x) : \mathbb{R} \times C[\mathbb{R}]^n \to \mathbb{R}^n$$

Then we still have the fundamental theorem of calculus element wise to conclude Picard iterations of the form

$$\phi_0 = x_0 \quad \& \quad \phi_{k+1} = x_0 + \int_{t_0}^{t} F(s, x(s))ds$$

where the integral is element wise. Thus the previous existence and uniqueness proof follows if $F(t,x)$ has Lipschitz functions.

pg.726 - # 9 Find the first few Picard iterates for

$$\frac{dx}{dt} = y^2, \frac{dy}{dt} = x + z, \frac{dz}{dt} = z - y, \quad x(0) = 1, y(0) = 0, z(0) = 1$$

Solution Note that we may rewrite the above as

$$\dot{x} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} x + \begin{pmatrix} y^2 \\ 0 \\ 0 \end{pmatrix} \quad x(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Using the above formula for Picard iterations we see

$$\phi_0 = x_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\phi_1 = x_0 + \int_0^t F(s, \phi_0)ds = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} ds = \begin{pmatrix} 1 \\ 2t \\ 1+t \end{pmatrix}$$

$$\phi_2 = x_0 + \int_0^t F(s, \phi_1)ds = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} 4s^2 \\ 2+s \\ 1-s \end{pmatrix} ds = \begin{pmatrix} 1 + 4t^3/3 \\ 2t + t^2/2 \\ t - t^2/2 \end{pmatrix}$$

□

A Helpful Formula to Remember Liouville's Formula. Let X be the fundamental solution to $\dot{X} = AX$ with $X(x_0) = X_0$, then you have

$$\det X(x) = \det X_0 \exp\left(\int_{x_0}^{x} \operatorname{tr}(A(s))ds\right)$$

Abel's Formula for the Wronskian of n-th order ODE is now an easy corollary. If $y_1, \ldots, y_n$ solve

$$y^{(n)} + p_{n-1}y^{(n-1)} + \ldots + p_0 y^{(0)} = 0$$

The Wronskian for the solutions is given by

$$W[y_1, \ldots, y_n](x) = C \exp\left(\int p_{n-1}(x)dx\right)$$

Quiz Prove that if $\phi(0) = 0$ and $\phi'(0)$ exists (and $\phi(x) > 0$), then

$$\int_0^\epsilon \frac{du}{\phi(u)} = \infty \quad \text{for any} \quad \epsilon > 0$$

Solution Since $\phi'(0)$ exists, fix $\epsilon' > 0$, then for some $\delta > 0$, $x \in (0, \delta)$ implies

$$\left| \frac{\phi(x)}{x} - \phi'(0) \right| < \epsilon' \implies 0 < \phi(x) < x(\phi'(0) + \epsilon') = cx$$

Thus we have

$$\frac{1}{cx} \leqslant \frac{1}{\phi(x)}$$

This implies that

$$\infty = \int_0^\delta \frac{dx}{cx} \leqslant \int_0^\delta \frac{dx}{\phi(x)} \implies \int_0^\epsilon \frac{du}{\phi(u)} = \infty$$

$\square$

Alternate Solution Since $\phi'(0)$ exists, we see that

$$\phi'(0) = \lim_{h \to 0} \frac{\phi(h) - \phi(0)}{h} = \lim_{h \to 0} \frac{\phi(h)}{h} = c \in \mathbb{R}$$

Thus we have that ϕ has leading order

$$\phi(x) \approx cx^n \quad , n \geqslant 1 \quad c \in \mathbb{R} \setminus \{0\}$$

around 0. Thus

$$x^n < x \quad \forall n \in [1, \infty) \quad \text{when} \quad x \in [0, 1) \implies \phi(x) < ax \quad a \in \mathbb{R} \setminus \{0\}, x \in [0, \delta)$$

for some $\delta < 1$. By limit comparison we know

$$\frac{1}{x} \sim \frac{1}{\phi(x)}$$

so

$$\int_0^\epsilon \frac{dx}{x} = \infty \implies \int_0^\epsilon \frac{du}{\phi(u)} = \infty$$

$\square$