Tutorial Problems #1

MAT 267 – Advanced Ordinary Differential Equations – Fall 2014

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Solutions

pg.56-# 19 Solve

$$\begin{cases} (1-x)dy = x(y+1)dx \\ y(0) = 0 \end{cases}$$

Solution The equation is separable, thus

$$\frac{dy}{y+1} = \frac{xdx}{1-x} \implies \int \frac{dy}{y+1} = \int \frac{xdx}{1-x} \implies \ln|y+1| = \ln\left|\frac{1}{1-x}\right| + x + C \implies y(x) = \frac{\tilde{C}e^x}{1-x} - 1$$

is the general solution. The initial condition fixes the constant.

$$y(0) = 0 \implies 0 = \tilde{C} - 1 \implies \tilde{C} = 1$$

Thus the solution to the IVP is

$$\boxed{y(x) = \frac{e^x}{1-x} - 1}$$

□

pg.69 - #10 Solve

$$(3x - 2y + 4)dx - (2x + 7y - 1)dy = 0$$

Solution We know it is possible to convert this to a homogeneous equation, we just need to find the change of variables. We'll use the differential method, i.e. we want u and v such that

$$3x - 2y + 4 = u \quad \& \quad -2x - 7y + 1 = v$$

This implies that

$$\begin{cases} du = 3dx - 2dy \\ dv = -2dx - 7dy \end{cases} \iff \begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -2 & -7 \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

By inverting the matrix we see that

$$\frac{1}{25} \begin{pmatrix} 7 & -2 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} dx \\ dy \end{pmatrix}$$

Thus the ODE becomes the following in coordinates u and v,

$$\begin{aligned} udx + vdy =& u\left(\frac{7}{25}du - \frac{2}{25}\right) + v\left(-\frac{2}{25}du - \frac{3}{25}dv\right) \\ =& \frac{1}{25}[(7u-2v)du + (-2u-3v)dv] \end{aligned}$$

Now the equation is homogeneous, which we know is possible to solve using $u = tv$, $du = tdv + vdt$. Substitute the change of variables again. (note we've dropped the 1/25 since the RHS is 0)

$$(7tv - 2v)(tdv + vdt) + (-2tv - 3v)dv = (7t^2 - 4t - 3)vdv + (7t-2)v^2dt = 0$$

This equation is separable, and we see the solution is (using $\omega = 7t^2 - 4t - 3, d\omega = 2(7t-2)dt$)

$$\int \frac{dv}{v} = \int \frac{(2-7t)dt}{7t^2 - 4t - 3} = -\frac{1}{2}\int \frac{d\omega}{\omega} \implies \ln|v^2\omega| = C$$

Now we just back substitute everything to revert to the original coordinates.

$$\omega v^2 = \tilde{C} \implies 7(vt)^2 - 4tv^2 - 3v^2 = \tilde{C} \implies 7u^2 - 4uv - 3v^2 = \tilde{C}$$

Hence the implicit general solution is

$$\boxed{7y^2 + (2-4x)y + 3x^2 + 8x = Const}$$

pg.79 - #16 Solve

$$\begin{cases} \sin x \cos y dx + \cos x \sin y dy = 0 \\ y(\pi/4) = \pi/4 \end{cases}$$

Solution By the symmetry of the equation, we check if it is exact. i.e

$$M_y = N_x \quad \text{where} \quad \underbrace{\sin x \cos y}_{=M} dx + \underbrace{\cos x \sin y}_{=N} dy = 0$$

Clearly

$$M_y = -\sin x \sin y = N_x$$

Thus the equation is exact, the solution is therefore given by a level set of linearity independent factors of the integrated functions. I write this as

$$\begin{aligned} F(x,y) &= \int M dx \oplus \int N dy \\ &= \int \sin x \cos y dx \oplus \int \cos x \sin y dy \\ &= -\cos x \cos y \oplus -\cos x \cos y \\ &= -\cos x \cos y \end{aligned}$$

Thus the general solution is

$$\cos x \cos y = C$$

The initial data implies that

$$C = \cos(\pi/4)\cos(\pi/4) = \frac{1}{2} \implies \boxed{1 = 2\cos x \cos y}$$

is the implicit solution to the IVP. □

pg.91 - #10 Solve

$$e^x(x+1)dx + (ye^y - xe^x)dy = 0$$

Solution The equation has a certain symmetry about it, so we check if it is exact. Letting $M = e^x(x+1)$ and $N = ye^y - xe^x$ we see that

$$M_y = 0 \quad \& \quad N_x = -e^x(x+1) = -M$$

More specifically, we notice that

$$\frac{N_x - M_y}{M} = -1$$

So we my multiply the equation by an integrating factor to make it exact, namely

$$\mu(y) = \exp\left(\int \frac{N_x - M_y}{M} dy\right) = e^{-y}$$

The ODE with the integrating factor becomes

$$\underbrace{e^{x-y}(x+1)}_{=\tilde{M}}\, dx + \underbrace{(y - xe^{x-y})}_{=\tilde{N}}\, dy = 0$$

By construction this is exact! So we integrate the components and add the linearly independent factors.

$$F(x,y) = \int \tilde{M} dx \oplus \int \tilde{N} dy = \int e^{x-y}(x+1)dx \oplus \int (y - xe^{x-y})dy = xe^{x-y} \oplus xe^{x-y} + \frac{y^2}{2} = xe^{x-y} + \frac{y^2}{2}$$

Thus the general solution the ODE is

$$\boxed{xe^{x-y} + \frac{y^2}{2} = C}$$

□

pg.97 - # 9 Solve

$$\tan\theta \frac{dr}{d\theta} - r = \tan^2\theta$$

Solution Recall for first order linear equations $(y'(x) + p(x)y(x) = q(x),)$ the solution is given by

$$y(x) = \frac{1}{\mu(x)} \int \mu(x)q(x)dx \quad \text{where} \quad \mu(x) = \exp\left(\int p(x)dx\right)$$

In this case we have

$$\frac{dr}{d\theta} - \underbrace{\cot\theta}_{=p}\, r = \underbrace{\tan\theta}_{=q}, \quad \text{with integrating factor} \quad \mu = \exp\left(-\int \cot\theta d\theta\right) = \exp\left(\ln\left|\frac{1}{\sin\theta}\right|\right) = \frac{1}{\sin\theta}$$

Thus the general solution is given by

$$\boxed{y(\theta) = \sin\theta \int \frac{d\theta}{\cos\theta} = \sin\theta(\ln|\sec\theta + \tan\theta| + C)}$$

□