Exam

MAT 244 – ODE – Winter 2015

SOLUTIONS

 # 1 Solve the following IVP(5 points):

$$xy' - 3y = 3x^5 \quad y(1) = 7.5$$

Solution Rewrite the equation

$$y' - \frac{3}{x}y = 3x^4$$

In this form we may use the standard integrating factor:

$$\mu(x) = \exp\left(\int p(x)dx\right) = \exp\left(-\int \frac{3}{x}dx\right) = \exp(-3\ln x) = \frac{1}{x^3}$$

We know the general solution is given by

$$y(x) = \frac{1}{\mu(x)}\int g(x)\mu(x)dx = x^3\int 3x^4\frac{1}{x^3}dx = x^3\int 3xdx = x^3\left(\frac{3}{2}x^2 + C\right)$$

Plugging in the initial data gives

$$7.5 = \frac{3}{2} + C \implies C = 6 \implies \boxed{y(x) = \frac{3}{2}x^5 + 6x^3}$$

 # 2 Find the general solution of the equation $xy^2(xy' + y) = 1$ (hint: find an integrating factor)(5 pt). Then solve the IVP $y(1) = 3$ for this equation.(1 pt)

Solution Lets check if the equation is exact, we rewrite into standard form

$$xy^3 - 1 + x^2y^2y' = 0 \iff \underbrace{(xy^3 - 1)}_{M}dx + \underbrace{x^2y^2}_{N}dy = 0$$

We try to check the exactness condition but see

$$M_y = 3xy^2 \quad \& \quad N_x = 2xy^2$$

Notice that

$$\frac{M_y - N_x}{N} = \frac{1}{x}$$

is just a function of x, so it will work as an integrating factor. We have

$$\mu(x) = \exp\left(\int \frac{M_y - N_x}{N} dx\right) = \exp\left(\int \frac{dx}{x}\right) = x$$

makes the equation exact. Thus we may integrate the ODE after multiplying by the above factor to obtain

$$\underbrace{(x^2y^3 - x)}_{\tilde{M}} dx + \underbrace{x^3y^2}_{\tilde{N}} dy = 0 \implies F(x,y) = \int \tilde{M} dx \oplus \int \tilde{N} dy$$

$$= \int (x^2y^3 - x) dx \oplus \int x^3y^2 dy$$

$$= \left(\frac{x^3y^3}{3} - \frac{x^2}{2}\right) \oplus \frac{x^3y^3}{3}$$

$$= \frac{x^3y^3}{3} - \frac{x^2}{2}$$

Thus the general solution to the ODE is

$$C = \frac{x^3y^3}{3} - \frac{x^2}{2} \implies \boxed{y(x) = \sqrt[3]{\frac{3}{2x} + \frac{C}{x^3}}}$$

The solution to the IVP is found by calculating C, we see

$$C = 25.5 \implies \boxed{y(x) = \sqrt[3]{\frac{3}{2x} + \frac{51}{2x^3}}}$$

 # 3 Find the general solution of the equation $x^3y'' - 2xy = 6\ln x$ (hint: the homogeneous part of this equation can be reduced to an Euler equation)(6 points)

Solution Rewrite the equation

$$x^2y'' - 2y = 6\frac{\ln x}{x}$$

Notice the homogeneous part is now Euler, so try $y(x) = x^\lambda$ as the homogeneous solution, we obtain

$$x^\lambda(\lambda(\lambda - 1) - 2) = 0 \implies (\lambda - 2)(\lambda + 1) = 0 \implies \lambda = -1, 2$$

Thus the homogenous solution is

$$y(x) = A \underbrace{x^2}_{y_1} + B \underbrace{\frac{1}{x}}_{y_2} \qquad A, B \in \mathbb{R}$$

We now may find the non homogeneous solution using the variation of parameters formula, so we calculate the Wronskian:

$$W[y_1, y_2](x) = y_1y_2' - y_2y_1' = -x^2\frac{1}{x^2} - \frac{2}{x}x = -3$$

Thus

$$y(x) = A(x)x^2 + B(x)\frac{1}{x}$$

is the solution, where (don't forget to divided the ODE by x^2!) (and $u = 1/x$)

$$A(x) = -\int \frac{y_2 g}{W} = 2\int \frac{\ln x}{x^4} dx = 2\int u^2 \ln u du = \frac{u^3 \ln(u)}{3} - \int \frac{u^2}{3} du = -\frac{2\ln(x)}{3x^3} - \frac{2}{9x^3} + B$$

and (where $u = \ln x$)

$$B(x) = \int \frac{y_1 g}{W} = -2\int \frac{\ln x}{x}dx = -2\int udu = -u^2 + B = -(\ln x)^2 + B$$

Putting it all together we obtain

$$\boxed{y(x) = Ax^2 + \frac{B}{x} - \frac{(\ln x)^2}{x} - \frac{2\ln x}{3x}}$$

 # 4 Find the general solution of $y''' - 2y'' + 4y' = e^t \sin(\sqrt{3}t)$.

Solution First find the homogeneous solution, try $e^{\lambda t}$, we see

$$e^{\lambda t}(\lambda^3 - 2\lambda^2 + 4\lambda) = 0 \implies \lambda(\lambda^2 - 2\lambda + 4) = 0 \implies \lambda = 0, 1 \pm i\sqrt{3}$$

Thus the homogeneous solution is

$$y(t) = A + B\underbrace{e^t \sin(\sqrt{3}t)}_{y_1} + C\underbrace{e^t \cos(\sqrt{3}t)}_{y_2} \quad A, B, C \in \mathbb{R}$$

To find the non homogeneous solution, we'll use the method of undetermined coefficients and guess

$$y_p(t) = Dte^t \sin(\sqrt{3}t) + Ete^t \cos(\sqrt{3}t) = Dty_1 + Ety_2$$

The derivatives are

$$\begin{aligned} y_p' &= Dy_1 + Ey_2 + Dty_1' + Ety_2' \\ y_p'' &= 2Dy_1' + 2Ey_2' + Dty_1'' + Ety_2'' \\ y_p''' &= 3Dy_1'' + 3Ey_2'' + Dty_1''' + Ety_2'' \end{aligned}$$

Thus we see

$$\begin{aligned} y_p''' - 2y_p'' + 4y_p' &= D(4y_1 - 4y_1' + 3y_1'') + E(4y_2 - 4y_2' + 3y_2'') \\ &= e^t\big(D(-6\sin(\sqrt{3}t) + 2\sqrt{3}\cos(\sqrt{3}t)) + E(-6\cos(\sqrt{3}t) - 2\sqrt{3}\sin(\sqrt{3}t))\big) \\ &= (-6D - 2\sqrt{3}E)e^t \sin(\sqrt{3}t) + (2\sqrt{3}D - 6E)e^t \cos(\sqrt{3}t) \end{aligned}$$

comparing this with the RHS of the ODE, we obtain

$$-6D - 2\sqrt{3}E = 1 \quad \& \quad 2\sqrt{3}D - 6E = 0 \implies E = \frac{1}{8\sqrt{3}} \quad \& \quad D = \frac{1}{8}$$

Thus the general solution to the ODE is

$$\boxed{y(t) = A + Be^t \sin(\sqrt{3}t) + Ce^t \cos(\sqrt{3}t) + \frac{3te^t \sin(\sqrt{3}t) + \sqrt{3}te^t \cos(\sqrt{3}t)}{24}}$$

 # 5 a)Find the general solution (3 points)

$$\begin{cases} x' = 2x - y \\ y' = x + 4y \end{cases}$$

0.5 Point - Computing the characteristic equation to find the eigenvalues

$$P(\lambda) = \det(A - I\lambda) = \begin{vmatrix} 2-\lambda & -1 \\ 1 & 4-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 9 = (\lambda-3)^2 \implies \boxed{\lambda = 3}$$

2 Point - Computing the eigenvector and generalized eigenvector

$$\lambda = 3 \implies \ker(A - 3I) = \ker \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} = \text{span}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \implies \boxed{\vec{\lambda} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

$$(A - 3I)\vec{\lambda}_g = \vec{\lambda} \implies \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \vec{\lambda}_g = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \implies \boxed{\vec{\lambda}_g = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

0.5 Point -Writing the general solution

$$\mathbf{x}(t) = A\vec{\lambda}e^{\lambda t} + Be^{\lambda t}(t\vec{\lambda} + \vec{\lambda}_g) \implies \boxed{\mathbf{x}(t) = A\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{3t} + B\begin{pmatrix} 1-t \\ t \end{pmatrix} e^{3t} \quad A, B \in \mathbb{R}}$$

b) Input initial data (2 points)

$$x(0) = 1 \quad \& \quad y(0) = 3$$

2 point Solve for A and B from the previous part

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} = A\begin{pmatrix} -1 \\ 1 \end{pmatrix} + B\begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies \boxed{B = 4 \quad \& \quad A = 3}$$

$$\implies \boxed{\mathbf{x}(t) = \begin{pmatrix} 1-4t \\ 3+4t \end{pmatrix} e^{3t}}$$

c) If $W(t)$ has the property $W(1) = 2$ for this system, what is $W(3)$ (2 points)

$$W(t) = C\det(x^{(1)}x^{(2)}) = Ce^{6t}\det\begin{pmatrix} -1 & 1-t \\ 1 & t \end{pmatrix} = -Ce^{6t}$$

$$W(1) = 2 \implies C = -2e^{-6} \implies \boxed{W(3) = 2e^{12}}$$

 # 6 Find the general solution to (6 points)

$$\dot{x} = \begin{pmatrix} -1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} x$$

2 Point - Computing the characteristic equation to find the eigenvalues

$$P(\lambda) = \det(A - I\lambda) = \begin{vmatrix} -1-\lambda & 3 & 2 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = -(1+\lambda)\Big((2-\lambda)(2-\lambda)-1\Big) = -(1+\lambda)(\lambda-3)(\lambda-1) \implies \boxed{\lambda = \pm 1, 3}$$

3 Points - Computing the eigenvectors (1 point for each)

$$\lambda = -1 \implies \ker \begin{pmatrix} 0 & 3 & 2 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix} = \text{span} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \implies \boxed{\vec{\lambda}_{-1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

$$\lambda = 1 \implies \ker \begin{pmatrix} -2 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \text{span} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \implies \boxed{\vec{\lambda}_{1} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}}$$

$$\lambda = 3 \implies \ker \begin{pmatrix} -4 & 3 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} = \text{span} \begin{pmatrix} 5 \\ 4 \\ 4 \end{pmatrix} \implies \boxed{\vec{\lambda}_{1} = \begin{pmatrix} 5 \\ 4 \\ 4 \end{pmatrix}}$$

1 Point - Writing the general solution

$$\boxed{x(t) = A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-t} + B \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} e^{t} + C \begin{pmatrix} 5 \\ 4 \\ 4 \end{pmatrix} e^{3t} \quad A, B, C \in \mathbb{R}}$$

 # 7 Consider the nonlinear system

$$\begin{cases} x' = (1-y)(y+x) \\ y' = (2+y)(x-y) \end{cases}$$

Describe the locations of all critical points (2 pts). Classify their types (i.e. specify whether they are nodes, saddles, etc) and stability (3 pts). Sketch the phase portraits near the critical points(2 pts). Sketch the phase portrait of the whole system(1 pt).

Solution We compute the solutions to $x' = y' = 0$ for critical points:

$$x' = 0 = (1-y)(y+x) \implies \begin{cases} y = 1 \\ y = -x \end{cases} \quad \& \quad y' = 0 = (2+y)(x-y) \implies \begin{cases} y = -2 \\ y = x \end{cases}$$

Thus the only critical points are

$$(x, y) = (0, 0), (1, 1), (2, -2)$$

To linearize the system, we know $\dot{z} = Jz$ where $z = x - x_0$, thus compute the Jacobian.

$$J(x, y) = \begin{pmatrix} 1-y & -2y+x+1 \\ 2+y & -2y+x-2 \end{pmatrix}$$

We have 3 points to check, so let's start with $x_0 = (0, 0)$. We see

$$J(0, 0) = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$$

The eigenvalues are found by checking the characteristic equation,

$$P(\lambda) = \det(J(0,0) - 1\lambda) = \begin{vmatrix} 1-\lambda & 1 \\ 2 & -2-\lambda \end{vmatrix} = \lambda^2 + \lambda - 4 = 0 \implies \lambda_{+} = \frac{-1 \pm \sqrt{17}}{2}$$

The eigenvectors are found by checking the kernel,

$$\lambda_+ \implies \ker\begin{pmatrix} \frac{3-\sqrt{17}}{2} & 1 \\ 2 & \frac{-3-\sqrt{17}}{2} \end{pmatrix} = \text{span}\begin{pmatrix} 3+\sqrt{17} \\ 4 \end{pmatrix} \implies \lambda_+ = \begin{pmatrix} 3+\sqrt{17} \\ 4 \end{pmatrix} \approx \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

$$\lambda_- \implies \ker\begin{pmatrix} \frac{3+\sqrt{17}}{2} & 1 \\ 2 & \frac{-3+\sqrt{17}}{2} \end{pmatrix} = \text{span}\begin{pmatrix} 3-\sqrt{17} \\ 4 \end{pmatrix} \implies \lambda_+ = \begin{pmatrix} 3-\sqrt{17} \\ 4 \end{pmatrix} \approx \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

Thus we see this is an unstable saddle. Next up we have $x_0 = (1,1)$,

$$J(1,1) = \begin{pmatrix} 0 & 0 \\ 3 & -3 \end{pmatrix}$$

The eigenvalues are found by checking the characteristic equation,

$$P(\lambda) = \det(J(1,1) - 1\lambda) = \begin{vmatrix} -\lambda & 0 \\ 3 & -3-\lambda \end{vmatrix} = \lambda^2 + 3\lambda = 0 \implies \lambda_1 = 0, \quad \lambda_2 = -3$$

The eigenvectors are found by checking the kernel,

$$\lambda_1 \implies \ker\begin{pmatrix} 0 & 0 \\ 3 & -3 \end{pmatrix} = \text{span}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \implies \lambda_+ = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda_2 \implies \ker\begin{pmatrix} 3 & 0 \\ 3 & 0 \end{pmatrix} = \text{span}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \implies \lambda_+ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus we see this is an asymptotically stable degenerate node (transiting between a node and a saddle). Lastly we check $x_0 = (2,-2)$.

$$J(2,-2) = \begin{pmatrix} 3 & 7 \\ 1 & 4 \end{pmatrix}$$

The eigenvalues are found by checking the characteristic equation,

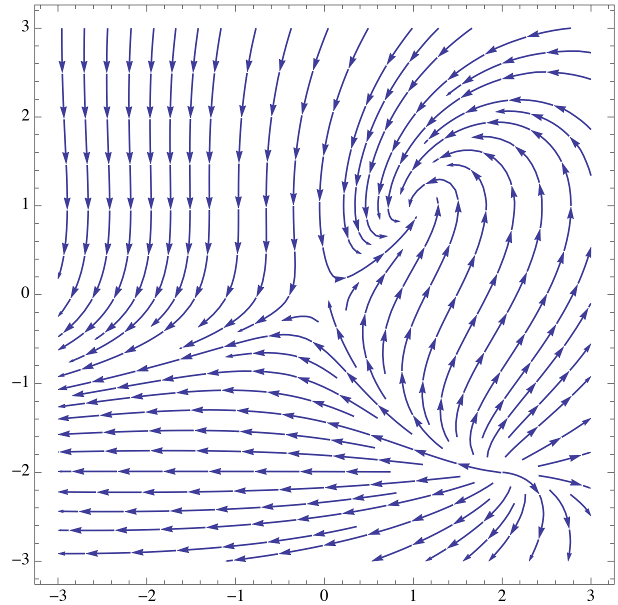
$$P(\lambda) = \det(J(2,-2) - 1\lambda) = \begin{vmatrix} 3-\lambda & 7 \\ 1 & 4-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 5 = 0 \implies \lambda_\pm = \frac{7 \pm \sqrt{29}}{2}$$

The eigenvectors are found by checking the kernel,

$$\lambda_+ \implies \ker\begin{pmatrix} \frac{-1-\sqrt{29}}{2} & 7 \\ 1 & \frac{1-\sqrt{29}}{2} \end{pmatrix} = \text{span}\begin{pmatrix} -1+\sqrt{29} \\ 2 \end{pmatrix} \implies \lambda_+ = \begin{pmatrix} -1+\sqrt{29} \\ 2 \end{pmatrix} \approx \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\lambda_- \implies \ker\begin{pmatrix} \frac{-1+\sqrt{29}}{2} & 7 \\ 1 & \frac{1+\sqrt{29}}{2} \end{pmatrix} = \text{span}\begin{pmatrix} -1-\sqrt{29} \\ 2 \end{pmatrix} \implies \lambda_+ = \begin{pmatrix} -1-\sqrt{29} \\ 2 \end{pmatrix} \approx \begin{pmatrix} -6 \\ 2 \end{pmatrix}$$

Thus this is an unstable node. Since we have all the local information, we know the system looks something like



 # 8 Consider the equation $y'' - 5xy = 0$, find the recurrence relation for the series solution of this equation at $x_0 = 0$, then find the first four terms of the solution with $y(0) = 2, y'(0) = 3$. What is the radius of convergence of the corresponding series solution? (6 pts)

Solution Try

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

as a solution, we obtain the following from substituting the solution

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - 5\sum_{n=0}^{\infty} a_n x^{n+1} = 2a_2 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - 5a_{n-1}\right]x^n = 0$$

This of course implies that the recurrence relation is

$$(n+2)(n+1)a_{n+2} - 5a_{n-1} = 0 \implies \boxed{a_{n+2} = \frac{5a_{n-1}}{(n+2)(n+1)}}$$

and we also see that $a_2 = 0$ from the equation. We compute the first four terms.

$$a_3 = \frac{5a_0}{6} = \frac{5}{3} \quad \& \quad a_4 = \frac{5a_1}{12} = \frac{5}{4}$$

Thus the solution with the first four terms is

$$\boxed{y(x) = 2 + 3x + \frac{5x^3}{3} + \frac{5x^4}{4} + \dots}$$

The radius of convergence is infinity since the coefficients of the ODE are well behaved.