Tutorial #2,#3, & #4

MAT 244 – ODE – Summer 2013

1st Order Equations

1st Order O.D.E In general we write a first order equation in the form of

$$\frac{dy}{dt} = f(t,y)$$

where f is some function of possibly t and y. We have essentially 4 cases that we'll cover in this corse for 1st order ODE. We'll start by covering how to solve them and deal with existence/justification of our work later.

(a) Separable Equations These are definitely the nicest form of 1st order ODE. They take the form

$$M(t) = N(y)\frac{dy}{dt}$$

where M and N are function of the indicated single variable. Suggestively, we may rewrite this in "differential form" as

$$M(t)dt = N(y)dy$$

and integrate with respect to each differential, i.e. we turn the differential equation into an integral equation.

$$\boxed{M(t) = N(y)\frac{dy}{dt} \iff \int M(t)dt = \int N(y)dy}$$

This forms are equivalent, and the integral formulation allows a nice method to solve the equation explicitly as long as we can evaluate the integral. **Remark: It is sometimes impossible to get y in terms of just t, i.e. what is $y(t)$? We may leave the solution as an implicit solution though.** Think $x^2 + y^2 = 1$ is an implicit solution for a circle, when we rearranging for y obtain $y = \pm\sqrt{1-x^2}$ which loses half the circle.

(b) 1st Order Linear In this case we have a general form of

$$y'(t) + p(t)y(t) = g(t)$$

A method of solving this is to recall product rule from calculus. It looks an awful lot like

$$\frac{d}{dt}(\mu(t)y(t)) = \mu(t)y'(t) + \mu' y(t)$$

Suppose we had, the above form, we could make it look like the 1st Order linear general form if

$$\frac{d}{dt}(\mu(t)y(t)) = \mu(t)g(t) \iff y'(t) + \frac{\mu'(t)}{\mu'(t)}y(t) = g(t)$$

So this $\mu(t)$, will help us "factor" the ODE if

$$\mu'(t) = p(t)\mu(t)$$

For this reason, we'll call $\mu(t)$ and integrating factor. We may solve for an explicit formula using the separable method. Let's do this

$$\mu' = p\mu \iff \int \frac{d\mu}{\mu} = \int p(t)dt \implies \ln|\mu| = \int p(t)dt \implies \boxed{\mu(t) = \exp\left[\int p(t)dt\right]}$$

$\mu(t)$ will always allow us to factor the ODE. More explicitly we have a formula for $y(t)$ since

$$\frac{d}{dt}(\mu(t)y(t)) = \mu(t)g(t) \implies \boxed{y(t) = \frac{1}{\mu(t)}\int \mu(t)g(t)dt}$$

This formula completely solves this case. As an aside, notice as long as we can make sense of the formula we have uniqueness and existence for a solution y.

(c) **Exact Equations** Suppose that M, N, M_y, N_t are continuous in some open box, then

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} \implies M(t,y) + N(t,y)y' = 0 \quad \text{has a solution}$$

Why does this happen? You can think of this as black magic for the moment, or if you have some interest in mathematics you can look up something called the exterior differential d, and it satisfies

$$F(x,y) = Const \iff d(F(x,y) = d(Const) \iff \underbrace{\frac{\partial F}{\partial x}}_{M} dx + \underbrace{\frac{\partial F}{\partial y}}_{N} dy = 0$$

We call "functions" of this form exact. Notice that the derivative condition is simply the condition that the partials match, i.e.

$$M_y = N_x \iff \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$$

Thus the solution to the ODE should be the sum of the linearly independent parts of the integral of each derivative, we denote this special sum as $\oplus$ as you would with vectors as you've seen in linear algebra.

$$F(x,y) = \int M dx \oplus \int N dy \implies \boxed{\int M dx \oplus \int N dy = C \quad \text{where} \quad C \in \mathbb{R}}$$

where the boxed formula is a implicit solution to the ODE.

(d) **Homogeneous Equations** In this context, we actually mean that

$$M(t,y)dx + N(t,y)dy = 0$$

with M and N homogeneous. A function is said to be homogeneous of degree n if we have that

$$f(\lambda t, \lambda y) = \lambda^n f(t,y) \quad \text{where} \quad \lambda \in \mathbb{R}$$

Notice in this case that the ratio of scaled M and N is fixed. i.e.

$$\frac{M(t,y)}{N(t,y)} = \frac{M(\lambda t, \lambda y)}{N(\lambda t, \lambda y)}$$

Now if we fix $\lambda = 1/t$, we see that

$$\frac{M(t,y)}{N(t,y)} = \frac{M(1,y/t)}{N(1,y/t)} = f(y/t)$$

This leads us to consider the change of variables $y/t = v$. We see

$$M(t,y)dx + N(t,y)dy = 0 \iff \frac{M(t,y)}{N(t,y)} + y' = 0 \iff f(v) + xv' + v = 0$$

We can solve the ODE in v since it is separable, we see the solution takes the form

$$\int \frac{dv}{f(v)+v} = \int \frac{dt}{t} \implies \ln|t| = \int \frac{dv}{f(v)+v}$$

or, in our original variables we see

$$\boxed{\ln|t| = \int \frac{tN(t,y)}{tM(t,y)+yN(t,y)} d\left(\frac{y}{t}\right)}$$

Which isn't very helpful, but it shows that we can always find a solution if the integral makes sense.

Example(Separable) - Find the general solution to

$$y^2\sqrt{1-x^2}dy = \arcsin(x)dx$$

Notice this is in differential form, it is equivalent to

$$y^2\sqrt{1-x^2}y' = \arcsin(x)$$

To solve it, notice the ODE is separable. Thus

$$y^2\sqrt{1-x^2}y' = \arcsin(x) \iff \int y^2 dy = \int \frac{\arcsin(x)}{\sqrt{1-x^2}}dx \implies \frac{y^3}{3} = \frac{\arcsin^2(x)}{2} + C \quad \text{where} \quad C \in \mathbb{R}$$

The C is just the integration constant. It means that we have a 1-parameter family of solution to the ODE. (Note: The integral is solved using u-sub with $u = \arcsin(x)$.) Thus our solution to the ODE is

$$y(x) = \sqrt[3]{\frac{3\arcsin^2 x}{2} + C}$$

Example(1st Order Linear) - Solve the IVP

$$ty' + (t+1)y = t \quad y(\ln(2)) = 1$$

To use our formula for an integrating factor, we need to rewrite the ODE in standard form. Namely

$$y' + \frac{t+1}{t}y = 1$$

Thus our integrating factor is

$$\mu(t) = \exp\left[\int \frac{t+1}{t}dt\right] = te^t$$

Using our formula, we see the general solution is

$$y(t) = \frac{1}{te^t}\int te^t dt = \frac{1}{te^t}(e^t(t-1)+C) = 1 - \frac{1}{t} + \frac{C}{te^t} \quad \text{where} \quad C \in \mathbb{R}$$

where we used the integration by parts formula to evaluate the integral. The initial condition (data) states we want the solution such that $y(\ln(2)) = 1$. This forces our choice of C, namely

$$y(\ln(2)) = 1 - \frac{1}{\ln(2)} + \frac{C}{2\ln(2)} = 1 \iff C = 2$$

Thus the solution to the initial value problem(IVP) is

$$y(t) = 1 - \frac{1}{t} + \frac{2}{te^t}$$

Example(1st Order Linear) - Find the general solution to

$$y' + \frac{1}{t}y = 3\cos(2t)$$

We use our integrating factor method, i.e.

$$\mu(t) = \exp\left[\int p(t)dt\right] = t$$

Thus via the formula for $y(x)$ we have

$$y(x) = \frac{1}{t}\int 3t\cos(2t)dt = \frac{3}{4t}\left(2t\sin(2t) + \cos(2t) + C\right) \quad \text{where} \quad C \in \mathbb{R}$$

using integration by parts.

Example(Exact Equation) - Find the general solution to

$$\underbrace{\frac{x}{(x^2+y^2)^{3/2}}}_{=M} + \underbrace{\frac{y}{(x^2+y^2)^{3/2}}}_{=N} y' = 0$$

It's easy to verify that

$$M_y = \frac{-3xy}{(x^2+y^2)^{5/2}} = N_x \implies \text{Exact!}$$

Using the formula we derived, we see

$$\begin{aligned} F(x,y) = \int M dx \oplus \int N dy &= \int \frac{x}{(x^2+y^2)^{3/2}}dx \int \frac{y}{(x^2+y^2)^{3/2}}dy \\ &= -\frac{1}{2}\left(\frac{1}{\sqrt{x^2+y^2}} \oplus \frac{1}{\sqrt{x^2+y^2}}\right) \\ &= -\frac{1}{2\sqrt{x^2+y^2}} \end{aligned}$$

Thus the implicit solution to the ODE is (note that we hid the factor in front in the constant)

$$\frac{1}{\sqrt{x^2+y^2}} = C \quad \text{where} \quad C \in \mathbb{R}$$

Exact Integrating Factors Suppose that we have

$$M(t,y)dt + N(t,y)dy = 0$$

and the equation if not exact, i.e. $M_y \neq N_t$. The goal here is find out what we may multiply the equation by to try and make it exact. There isn't exactly (haha) a systematic approach to always finding an integrating factor ($\mu(x, y)$) to make the equation exact, but we have some special cases. Suppose that μ is a function of y alone. This means that we want the following ODE to be exact

$$\underbrace{\mu(y)M}_{\tilde{M}}\,dt + \underbrace{\mu(y)N}_{\tilde{N}}\,dy = 0$$

For this to be exact, we need ($\tilde{M}_y = \tilde{N}_x$)

$$\frac{\partial}{\partial y}(\mu(y)M) = \mu' M + \mu M_y = \mu(y)N_t$$

$$\implies \frac{\mu'(y)}{\mu(y)} = \frac{N_t - M_y}{M}$$

This equation tells us that this is a integrating factor if $(N_t - M_y)/M$ is a function only of y. If we assume this, we see our integrating factor takes the form

$$\boxed{\mu(y) = \exp\left[\int \frac{N_t - M_y}{M}\,dy\right]}$$

Similarly, we may repeat the same argument to show $\mu(t)$ is an integrating factor if $(M_y - N_t)/N$ is a function only of t. In this case we'd see

$$\boxed{\mu(x) = \exp\left[\int \frac{M_y - N_t}{N}\,dt\right]}$$

As an exercise, find the condition for $\mu(xy)$ to be an integrating factor. **Remark! There isn't always just one integrating factor, consider** $2\sin(y) + x\cos(y)y' = -1$**, you can check that** $\mu(x) = x$ **or** $\mu(y) = 1/\sqrt{2\sin(y)+1}$ **both work**.

Example(Exact Integrating Factor) - Find the general solution to

$$\underbrace{y}_{M} + \underbrace{(2x - ye^y)}_{N}\,y' = 0$$

We see the equation is not exact, since

$$M_y = 1 \quad \& \quad N_x = 2$$

But notice that the derivatives are similar (i.e. constants), so maybe we could find an integrating factor. From the derivative of an exact integrating factor, we see

$$\frac{N_x - M_y}{M} = \frac{1}{y}$$

which is just a function of y. Thus

$$\mu(y) = \exp\left[\int \frac{N_x - M_y}{M}\,dy\right] = y$$

Let's check that y makes the new ODE exact, we have

$$\underbrace{y^2}_{=\tilde{M}} + \underbrace{(2xy - y^2e^y)}_{=\tilde{N}}\,y' = 0$$

Clearly, we have

$$\tilde{M}_y = 2y = \tilde{N}_x$$

which means the ODE is exact. Therefore, our solution is a level set of

$$\begin{aligned} F(x,y) = \int \tilde{M}dx \oplus \int \tilde{N}dy = \int y^2 dx \oplus \int (2xy - y^2e^y)dy \\ = y^2x \oplus xy^2 - (y^2 - 2y + 2)e^y \\ = xy^2 - (y^2 - 2y + 2)e^y \end{aligned}$$

i.e. the (implicit) general solution is

$$xy^2 - (y^2 - 2y + 2)e^y = C \quad \text{where} \quad C \in \mathbb{R}$$

Example(Discontinuous Coefficients) - Solve the IVP

$$y' + p(t)y = 0, \quad y(0) = 1 \quad \text{where} \quad p(t) = \begin{cases} 2 & t \in [0,1] \\ 1 & t > 1 \end{cases}$$

By definition of a solution, we want it to be continuous, i.e. we'll have to glue pieces together. We have two cases, $t \in [0,1]$ and $t > 1$. For $t \in [0,1]$ we're solving

$$y' + 2y = 0 \implies y_1(t) = C_1 e^{-2t} \quad \text{where} \quad C_1 \in \mathbb{R}$$

The initial data implies that $C_1 = 1$. For $t > 1$, we have

$$y' + y = 0 \implies y_2(t) = C_2 e^{-t}$$

The constant here is determine by making the function continuous. We want

$$\lim_{t \to 1^+} C_2 e^{-t} = y_1(1) = e^{-2} \implies C_2 = e^{-1}$$

Thus the continuous solution to the IVP is

$$y(t) = \begin{cases} e^{-2t} & t \in [0,1] \\ e^{-(t+1)} & t > 1 \end{cases}$$

Notice

Autonomous Equations Basically Autonomous ODE is one that doesn't depend on t,x, etc. i.e. the independent variable. Characteristics of systems of this type will be critical points(solutions), or equilibrium. We call a critical point, a value y_0 such that the autonomous system satisfies

$$y' = f(y_0) = 0$$

In situations like this it makes sense to talk about stability of a critical point. **Stable critical points** are points such that a perturbation like

$$y(t_0) = y_0 \pm \epsilon$$

will eventually fall back into the y_0 state for $\epsilon > 0$ (not too big so it passes through another critical point). More specifically, this means that

$$\boxed{f(y_0 + \epsilon) < 0 \quad \& \quad f(y_0 - \epsilon) > 0 \implies \text{Stable Critical Point}}$$

Unstable critical points are exactly the opposite, i.e.

$$\boxed{f(y_0+\epsilon)>0 \quad \& \quad f(y_0-\epsilon)<0 \implies \text{Unstable Critical Point}}$$

Then we have the mixed case, stable on one side, unstable on the other. These are called **Semi-stable critical points** and have the property that

$$\boxed{\text{sgn}(f(y_0+\epsilon)=\text{sgn}(f(y_0-\epsilon)) \implies \text{Semi-Stable Critical Point}}$$

Variation of Parameters Suppose we consider the homogeneous 1st Order linear ODE, namely

$$y'(t)+p(t)y(t)=0$$

We may find the solution to this via separation of variables, it is

$$y(t)=C\exp\left[-\int p(t)dt\right] \quad \text{where} = CI(t) \quad C\in\mathbb{R}$$

with $I(t)=\exp\left[-\int p(t)dt\right]$. Now suppose we consider the non-homogeneous case, with $g(t)$ not identically zero.

$$y'(t)+p(t)y(t)=g(t)$$

We'll find the solution by varying the constant on the homogeneous solution, i.e. making C into a function $C(t)$. Plugging this into the ODE gives us (note that $I'(t)=-pI(t)$)

$$y'+py=C'I+CI'+pCI=C'I\underbrace{-pCI+pCI}_{=0}=g \implies C'(t)=\frac{g(t)}{I(t)}$$

Which means we may find $C(t)$ which solves the equation if we integrate. We see

$$C(t)=\int\frac{g(t)}{I(t)}dt$$

Notice that $I(t)=1/\mu(t)$, the inverse of our integrating factor. Which results in the same formula as we deduced earlier.

$$y(t)=C(t)I(t)=\frac{1}{\mu(t)}\int\mu(t)g(t)dt$$

Euler's Method(1st Order Taylor Approximation) Recall the definition of slope, i.e. rise over run, and call $y(x_n)=y_n$. We have

$$\frac{\Delta y}{\Delta x}=\frac{y_{n+1}-y_n}{x_{n+1}-x_n}\approx y_n' \quad \text{if } x_{n+1}-x_n \text{ is small}$$

Define ϵ as our step size between points, i.e. $x_{n+1}-x_n=\epsilon$ for all n. Now consider the 1st order ODE

$$y'=f(x,y)$$

If our ϵ is small enough, it's not a bad approximation to assume that

$$y_n'=\frac{y_{n+1}-y_n}{\epsilon}=f(x_n,y_n) \implies y_{n+1}=y_n+\epsilon f(x_n,y_n)$$

which is exactly a taylor expansion up to first order. This gives me a computational method to solve an ODE in an iterative method (of course assuming you give me $y(x_0)=y_0$, i.e. the initial data). This works since we're essentially flowing along the direction field created from the ODE for ϵ time. Gluing together all this "slope" lines approximates the solution (as long as f and $\partial_y f$ are continuous [or Lipschitz]).

Example(Convergence of Euler's Method) Consider the IVP

$$y' = 1 - t + y, \quad y(t_0) = y_0$$

Clearly we may solve this using the methods talked about earlier to obtain

$$y(t) = (y_0 - t_0)e^{t-t_0} + t$$

But let's see how we'd implement Euler's Method here. Using the idea above, define a step size ϵ starting from the initial point t_0, i.e. $t_n = t_0 + n\epsilon$. We have that

$$y_{n+1} = y_n + \epsilon f(t_n, y_n) = y_n + \epsilon(1 - t_n + y_n) = (1+\epsilon)y_n + \epsilon(1 - t_n)$$

By using the above formula recursively, we can get everything in terms of the initial data. i.e. prove by induction that

$$y_n = (1+\epsilon)^n(y_0 - t_0) + t_n$$

The base case is immediate, so assume true for n and prove that $n+1$ holds. We leave this as an exercise. Now fix $t > t_0$ and take

$$\epsilon = \frac{(t - t_0)}{n} \implies t_n = t$$

Consider the limit of the solution now as our step size ϵ approaches 0, i.e. $n \to \infty$. We have

$$\lim_{n\to\infty} y(t_n) = \lim_{n\to\infty}\left(1 + \frac{(t-t_0)}{n}\right)^n (y_0 - t_0 + t_n = (y_0 - t_0)e^{t-t_0} + t$$

using the fact that

$$\lim_{n\to\infty}\left(1 + \frac{a}{n}\right)^n = e^a$$

this shows that as our steps get finer, we'll recover the actual solution.

The Picard-Lindelöf Theorem(Existence and Uniqueness) This theorem will give us existence and uniqueness to the specific 1st order ODE. Suppose that $f(t, y(t))$ and $\partial y f(t, y(t))$ are continuous in a box B centred around (t_0, y_0). Then the IVP

$$y' = f(t, y), \quad y(t_0) = y_0$$

has a unique solution locally around (t_0, y_0).

This basically states that we can always find a solution if $f(t, y)$ is nice enough. The idea of the proof is as follows. Turn the differential equation into an integral equation

$$\int_{t_0}^{t} y'(s)ds = \int_{t_0}^{t} f(s, y(s))ds \implies y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds$$

We want to show that the integral equation has a fixed point y, since if it does, that y is a solution to the ODE. To do this we can define an approximating sequence (these are usual called **Picard Iterations**)

$$\phi_0 = y_0 \quad \& \quad \phi_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, \phi_n(s))ds$$

We want to show that this sequence convergences to something(i.e. the solution), so we first show the distance is shrinking between iteration of this sequence for small enough t. Which is basically done with the following bound

$$\begin{aligned}\max_{t\in B}|\phi_{i+1}(t)-\phi_{j+1}(t)| = \max_{t\in B}\left|\int_{t_0}^{t}[f(s,\phi_i(s))-f(s,\phi_j(s))]ds\right| &\leqslant \max_{t\in B}\int_{t_0}^{t}|f(s,\phi_i(s))-f(s,\phi_j(s))|ds \\ K=\max_{t\in B}\frac{\partial f}{\partial y} \implies &\leqslant K\max_{t\in B}\int_{t_0}^{t}|\phi_i(s)-\phi_j(s)|ds \\ &\leqslant K\max_{t\in B}(t-t_0)|\phi_i(t)-\phi_j(t)|\end{aligned}$$

Then taking B such that $K(t-t_0)=\tilde{K}<1$ for all $t\in B$. This shows that this a contraction map, which is a fancy way of saying distances are shrinking, i.e.

$$\max_{t\in B}|\phi_{n+1}(t)-\phi_n(t)| \leqslant \tilde{K}^n \max_{t\in B}|\phi_1(t)-\phi_0(t)|$$

One may check that $|\phi_1(t)-\phi_0(t)|<\infty$, which means

$$\lim_{n\to\infty}\max_{t\in B}|\phi_{n+1}(t)-\phi_n(t)|=0$$

Which shows there is a limit point. You can think of the ϕ_n like an finite taylor expansion approximating the solution that converges to the actual solution in the limit. Uniqueness of this solution we leave as an exercise.

Exercise(Existence)-Where do solutions exist?

$$\frac{dy}{dt}=\frac{y\cos(t)}{1+y}$$

From the statement of the PIcard-Lindelöf Theorem, we need to check where f and $\partial_y f$ are continuous. Since

$$f(t,y(t))=\frac{y\cos(t)}{1+y}$$

we see that f is continuous everywhere but the "blow up" at $y=-1$, thus f continuous on $(t,y)\in\mathbb{R}\times\mathbb{R}\setminus\{-1\}$. We have that

$$\frac{\partial f}{\partial y}=\frac{\cos(t)}{(1+y)^2}$$

which is continuous everywhere but $y=-1$ again, thus $\partial_y f$ is continuous on $(t,y)\in\mathbb{R}\times\mathbb{R}\setminus\{-1\}$. Thus we have that solutions may live in either

$$\mathbb{R}\times(-\infty,-1) \quad \text{or} \quad \mathbb{R}\times(-1,\infty)$$

Remember that it may just be on a subset of this for given (t_0,y_0).

Exercise(Existence)-Where do solutions exist?

$$y'\ln(t)+y=\cot(t), \quad y(2)=3$$

Since this is 1st order linear, let's rewrite this in the standard form

$$y'+\frac{1}{\ln(t)}y=\frac{\cot(t)}{\ln(t)}$$

From our previous algebra, we know

$$y(t) = \frac{1}{\mu(t)} \int \mu(t) g(t)$$

Solves the ODE. As long as the integrals make sense, we'll have a solution. Check $\mu(t)$ first,

$$\mu(t) = \exp\left[\int \frac{dt}{\ln(t)}\right]$$

We have a singularity around $t = 1$, since $\ln(1) = 0$. This means t must be larger than 1 (since our initial data starts at $t_0 = 2$.). The other integral we have to worry about is

$$\int \cot(t)\mu(t)dt$$

cotangent has singularities at $t = n\pi$ with $n \in \mathbb{Z}$. Since $2 < \pi$, We see that our solution will be defined on $t \in (1, \pi)$.

Tutorial #5 & #6

MAT 244 – ODE – Summer 2013

2nd Order Equations

2nd Order Linear O.D.E Second order linear ordinary differential equations take the form

$$y''(t) + p(t)y'(t) + q(t)y(t) = \begin{cases} 0 & \Longleftarrow \text{ Homogeneous} \\ g(t) & \Longleftarrow \text{ Non-Homogeneous} \end{cases}$$

How do we solves equations of this form? It depends on quite a few things, but let's stress the importance of the word linear. Suppose that y_1 and y_2 solve the homogeneous problem, then we have that their linear combination is also a solution. Let's check this by plugging $y = Ay_1 + By_2$, $A, B \in \mathbb{R}$, into the ODE. We have

$$\begin{aligned} y'' + py' + qy =& Ay_1'' + By_2'' + p(Ay_1' + By_2') + q(Ay_1 + By_2) \\ =& A\underbrace{(y_1'' + py_1' + qy_1)}_{=0} + B\underbrace{(y_2'' + py_2' + qy_2)}_{=0} \\ =& 0 \end{aligned}$$

Thus any linear combination of solutions is a solution. We'll only consider the simpler cases in this course, starting with Homogeneous with constant coefficients. It's convenient to write this case as

$$ay'' + by' + cy = 0 \quad \text{where} \quad a, b, c \in \mathbb{R}$$

instead of solving this directly, let's try to guess the solution here(later we'll have a more constructive approach with 1st order systems which will show that nth order ODE's have n linearly independent solutions). Guess

$$y(t) = e^{\lambda t}$$

as a solution with $\lambda \in \mathbb{C}$. If we plug this into the ODE we see $y(x)$ is a solution if

$$e^{\lambda t}(a\lambda^2 + b\lambda + c) = 0$$

Since the exponential is never zero, our only hope is that λ is a root of the quadratic polynomial. This polynomial has a special name, it is called the characteristic equation

$$P(\lambda) = a\lambda^2 + b\lambda + c$$

Clearly λ is a root if

$$\lambda_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which means that $e^{\lambda_\pm x}$ is a solution. We write $\lambda_\pm$ cause there are two possibilities of the choice of sign taken for a solution. It turns out that for 2nd Order ODE, we'll have two "fundamental solutions", and these two roots, λ_+ and λ_- correspond to the general form of our solutions. Though this of course depends on the quantity inside the square root, call it the discriminate (Δ), i.e.

$$\Delta = b^2 - 4ac$$

We see three cases:

(a) $\Delta > 0$**, Distinct Roots** In this case we have that both roots λ_+ and λ_- are real, so we have a general solution of

$$y(t) = Ae^{\lambda_+ t} + Be^{\lambda_- t} \quad \text{where} \quad A, B \in \mathbb{R}$$

(b) $\Delta < 0$**, No Real Roots** In this case we have that both are roots are complex number since we're taking the square root of a negative number. For now let's just call $\sqrt{-1} = i$ and without going into too much detail, we state Euler's formula (some consider this one of the best formula's in mathematics since it relates $1, 0, \pi, i$ and e with $\theta \equiv \pi$)

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Note: One can visualize this as the (x, y) position of a circle of radius one given in terms of the angle θ. Back to the matter of our imaginary roots, we'll call them

$$\lambda_\pm = \underbrace{\frac{-b}{2a}}_{=\eta} \pm i \underbrace{\frac{\sqrt{4ac - b^2}}{2a}}_{=\xi} = \eta \pm i\xi$$

Using additivity of the exponential, $e^{a+b} = e^a e^b$, we have that our complex valued solution takes the form:

$$y(x) = Ae^{(\mu + i\xi)t} + Be^{(\mu - i\xi)t} = e^{\mu t}\left(Ae^{i\xi t} + Be^{-i\xi t}\right) \quad \text{where} \quad A, B \in \mathbb{C} \cong \mathbb{R} + i\mathbb{R}$$

This solves the ODE algebraically, but it'd be nice if we have a real valued solution. It turns out that for this to happen, we need $y(x) = \overline{y(x)}$, where the complex conjugate is defined as

$$z = \mu + i\xi \quad \& \quad \overline{z} = \mu - i\xi$$

This condition implies that

$$y(x) = e^{\mu t}\left(Ae^{i\xi t} + \overline{A}e^{-i\xi t}\right) \quad A \in \mathbb{C}$$

is real valued, but the form still looks complex valued. To make things look real, we invoke Euler's Formula and notice:

$$\begin{aligned} y(x) =& e^{\mu t}\left(Ae^{i\xi t} + \overline{A}e^{-i\xi t}\right) \\ =& e^{\mu t}\left(A(\cos(\xi t) + i\sin(\xi t)) + \overline{A}(\cos(\xi t) - i\sin(\xi t))\right) \\ =& e^{\mu t}\left(\underbrace{(A + \overline{A})}_{=\tilde{A}}\cos(\xi t) + \underbrace{i(A - \overline{A})}_{=\tilde{B}}\sin(\xi t)\right) \\ =& e^{\mu t}\left(\tilde{A}\cos(\xi t) + \tilde{B}\sin(\xi t)\right) \end{aligned}$$

which is a real valued solution if with $\tilde{A}, \tilde{B} \in \mathbb{R}$. As a final remark, notice that

$$\tilde{A} = 2\Re(A) \quad \& \quad \tilde{B} = -2\Im(A)$$

i.e. $\tilde{A}$ is twice the real part of A and $\tilde{B}$ is twice the negative imaginary part of A. This wraps up the complex roots case.

(c) $\Delta = 0$**, Repeated Roots** In this case, it seems as if we only have one solution to the ODE, namely

$$\lambda = -\frac{b}{2a} \quad \& \quad y(t) = Ae^{\lambda t} \quad \text{where} \quad A \in \mathbb{R}$$

There are a few ways to derive the second solution, but we'll go over a case known as **Reduction of Order** . Consider the ODE,

$$y'' + py' + qy = 0$$

and suppose that y_1 solves the ODE. Try

$$y(t) = u(t)y_1(t)$$

as a solution. We'll find that there is actually a very nice expression for $u(t)$. We have that

$$y' = u'y_1 + uy_1' \quad \& \quad y'' = u''y_1 + 2u'y_1' + uy_1''$$

via product rule. Substituting this into the ODE, we obtain (with a bit of rearranging)

$$u''y_1 + u'(2y_1' + py_1) + u\underbrace{(y_1'' + py_1' + qy_1)}_{=0} = 0$$

we see the u term drop out since y_1 is a solution by assumption. We're effectively left with a 1st order ODE

$$u'' + u'\left(2\frac{y_1'}{y_1} + p\right) = 0$$

this is easily solvable using separability. Noting that $\frac{d}{dt}\ln(f(t)) = f'/f$ via chain rule. We obtain

$$\begin{aligned} u'' + u'\left(2\frac{y_1'}{y_1} + p\right) = 0 \iff \int \frac{du'}{u'} &= \int \left(-2\frac{y_1'}{y_1} - p\right) dt \\ \implies \ln|u'| &= \ln\left(\frac{1}{y_1^2}\right) - \int p(t)dt \\ \implies u(t) &= \int \frac{A \exp\left(-\int p(t)dt\right)}{y_1^2} dt \quad \text{where} \quad A \in \mathbb{R} \end{aligned}$$

Thus we see that the second solution to the ODE may be found via

$$y(t) = y_1(t) \int \frac{A \exp\left(-\int pdt\right)}{y_1^2} dt$$

In this case of the repeated root, we have $p = b/a$ and $y_1 = \exp(-\frac{b}{2a}t)$. Plugging this into the formula gives

$$y(t) = e^{-\frac{b}{2a}t} \int A\, dt = A\underbrace{te^{-\frac{b}{2a}t}}_{=y_2} + B\underbrace{e^{-\frac{b}{2a}t}}_{=y_1} \quad \text{where} \quad A, B \in \mathbb{R}$$

That finishes up the third case.

Now that we've seen each case derived, we should have a feel for what we're looking for in terms of solutions for 2nd order homogeneous constant coefficients. Let's look at some examples to see how we may apply what we've done.

Example($\Delta > 0$) - Solve the IVP

$$y'' + 8y' - 9y = 0, \quad y(1) = 1, \quad y'(1) = 0$$

Via the above which we just talked about, we see that the characteristic equation for this ODE is

$$P(\lambda) = \lambda^2 + 8\lambda - 9 = (\lambda + 9)(\lambda - 1)$$

From our factorization it is clear that are two roots are $\lambda_1 = -9$ and $\lambda_2 = 1$. Thus the general solution to this ODE is

$$y(t) = Ae^{-9t} + Be^t, \quad A, B \in \mathbb{R}$$

Given the initial data, this gives us 2 equations, 2 unknowns. Namely

$$Ae^{-9} + Be = 1 \quad \& \quad -9Ae^{-9} + Be = 0$$

Solving this for A and B will give you that the solution to the IVP is

$$y(t) = \frac{1}{10}\left(9e^{t-1} + e^{9(1-t)}\right)$$

As you can see, our general formulas are doing a bulk of the work!

Example($\Delta < 0$) - Solve the IVP

$$y'' + 2y' + 2y = 0, \quad y\left(\frac{\pi}{4}\right) = 2, \quad y'\left(\frac{\pi}{4}\right) = -2$$

We find the characteristic equation, in this case we have

$$P(\lambda) = \lambda^2 + 2\lambda + 2$$

It's easy to check that $\Delta < 0$ and find that

$$\lambda_{\pm} = -1 \pm i$$

We know the general solution takes the form

$$y(t) = e^{-t}\left(A\cos(t) + B\sin(t)\right) \quad \text{where} \quad A, B \in \mathbb{R}$$

Now to solve for the coefficients, we plug in the initial data. We again have 2 equations, 2 unknowns. Namely

$$e^{-\frac{\pi}{4}}\left(\frac{A+B}{\sqrt{2}}\right) = 2 \quad \& \quad -2e^{-\frac{\pi}{4}}\left(\frac{B}{\sqrt{2}}\right) = -2$$

It's easy to solve and find the solution to the IVP is

$$y(t) = \sqrt{2}e^{\frac{\pi}{4}-t}\left(\cos(t) + \sin(t)\right)$$

Example($\Delta = 0$) - Solve the IVP

$$y'' + 4y' + 4y = 0, \quad y(-1) = 2, \quad y(1) = 1$$

As we've been seeing, we first find the characteristic equation. It clearly is

$$P(\lambda) = \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2$$

Their is only one root to the above, it is

$$\lambda = -2$$

Thus, using are previous construction we know the general solution is

$$y(t) = Ate^{-2t} + Be^{-2t} \quad \text{where} \quad A, B \in \mathbb{R}$$

Now we find A, B using the initial data. Plug and chug, 2 equations, 2 unknowns

$$-Ae^2 + Be^2 = 2 \quad \& \quad Ae^{-2} + Be^{-2} = 1$$

We find the solution to the IVP is

$$y(t) = \left(\frac{e^4 - 2}{2e^2}\right) te^{-2t} + \left(\frac{e^4 + 2}{2e^2}\right) e^{-2t}$$

Euler Equations As a special case of what we're talking about, and an example of how we may change variables in ODE, consider

$$t^2 y''(t) + \alpha t y'(t) + \beta y(t) = 0, \quad \text{where} \quad \alpha, \beta \in \mathbb{R}, \quad t > 0$$

Take the change of variables $t \to \ln(t)$, i.e. $x = \ln(t)$. Using chain rule, we may calculate $y'(x)$ and $y''(x)$ in terms of $y'(t)$ and $y''(t)$. We have

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = y'(x) \cdot \frac{1}{t} \quad \& \quad \frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{y'(x(t))}{t}\right) = y''(x) \cdot \frac{1}{t^2} - y'(x) \cdot \frac{1}{t^2}$$

Substituting this into the above ODE gives a constant coefficient one!

$$y''(x) + (\alpha - 1)y'(x) + \beta y(x) = 0$$

This means all our previous work applies to this case as well. As an exercise for yourself, convert the general 3 cases over to the Euler Equations. You'll see that it corresponds roughly to trying t^λ as a solution instead of $e^{\lambda t}$. This is because

$$e^{\lambda x} = e^{\lambda \ln t} = e^{\ln t^\lambda} = t^\lambda$$

Example(Euler Equations) - Find the general solution to

$$t^2 y'' + 7ty' + 10y = 0, \quad t > 0$$

From the remark above, we substitute t^λ into the equation to find our characteristic equation. We see the equation becomes

$$t^\lambda(\lambda(\lambda - 1) + 7\lambda + 10) = 0$$

Since $t > 0$, the only way for this to be a solution is that λ is a root of the characteristic equation:

$$P(\lambda) = \lambda(\lambda - 1) + 7\lambda + 10 = \lambda^2 + 6\lambda + 10$$

We see the roots of the quadratic are

$$\lambda_\pm = -3 \pm i$$

From the exercise of converting over the formulas in the Euler case, we see the general solution is

$$y(t) = \frac{1}{t^3}\Big(A\sin(\ln(t)) + B\cos(\ln(t))\Big) \quad \text{where} \quad A, B \in \mathbb{R}, \quad t > 0$$

Wronskian - A tool for Linear Independence The Wronskian (W) is defined as

$$W[y_1, y_2] = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1 y_2 - y_1' y_2$$

where y_1, y_2 solve $y'' + py' + qy = 0$. Notice that the Wronskian is a function of t if y_1, y_2 are functions of t. It has the following very remarkable property

$$\boxed{y_1, y_2 \text{ are linearly independent} \iff W[y_1, y_2](t) \neq 0 \quad \forall t \quad \text{(where defined)}}$$

This means that if y_1 and y_2 solve a 2nd order linear equation with $W[y_1, y_2] \neq 0$ then $y(t) = Ay_1 + By_2$ is the general solution.

Abel's Formula - Formula for the Wronskian If y_1 and y_2 solve $y'' + py' + qy$ with p and q continuous on an interval I. Then we have

$$W[y_1, y_2](t) = A \exp\left[-\int p(t)dt \right]$$

This is easily derived by showing that $W' + pW = 0$, then solving this ODE. We leave it as an exercise.

Reduction of Order - Through the Wronskian We derived a formula for the reduction of order method. We'll now see this is contained within the Wronskian itself! Let's set ourselves in the same situation, we know y_1 solves $y'' + py' + qy = 0$. The Wronskian via Abel's Formula gives us a way to compute W without y_2, i.e. we may solve y_2 through the definition of the Wronskian.

$$\begin{aligned} y_1 y_2' - y_1' y_2 = W[y_1, y_2] &\implies \frac{d}{dt}\left(\frac{y_2}{y_1} \right) = \frac{y_2'}{y_1} - \frac{y_1' y_2'}{y_1^2} = \frac{W[y_1, y_2]}{y_1^2} \\ &\implies \frac{y_2}{y_1} = \int \frac{W[y_1, y_2]}{y_1^2} dt \\ &\implies y_2(t) = y_1(t) \int \frac{W[y_1, y_2](t)}{y_1^2} dt \end{aligned}$$

which is exactly the formula we derived earlier.

Example(Reduction of Order) Given y_1, find y_2 if

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{4} \right) y = 0, \quad x > 0, \quad y_1(x) = \frac{\sin x}{\sqrt{x}}$$

Recall the formula we derived,

$$y(x) = y_1 \int \frac{W}{y_1^2} dx$$

We may compute W by reducing the ODE to the standard form of

$$y'' + \underbrace{\frac{1}{x}}_{=p} y' + \left(1 - \frac{1}{4x^2} \right) y = 0$$

Thus, via Abel's formula we have

$$W[y_1, y_2](x) = A \exp\left(-\int \frac{dx}{x} \right) = A \exp\left(\ln \frac{1}{x} \right) = \frac{A}{x} \quad \text{where} \quad A \in \mathbb{R}$$

Now we evaluate the integral in the formula we derived

$$y(x) = y_1 \int \frac{W}{y_1^2} dx = Ay_1 \int \frac{dx}{\sin^2 x} = y_1 \int \csc^2 x \, dx = Ay_1 \cot x + By_1 = \frac{A\cos x + B\sin x}{\sqrt{x}}$$

with $A, B \in \mathbb{R}$ and B is our integration constant. We see that if you take $B = 0$ we'll get the "second fundamental part" of the solution. Thus our second fundamental solution is

$$y_2(x) = \frac{\cos x}{\sqrt{x}}$$

Example(Wronskian and Fundamental Solutions) Consider $y'' - y' - 2y = 0$, one can check that $y_1 = e^{-t}$ and $y_2 = e^{2t}$. These two solutions form a fundamental set for the ODE since

$$W[y_1, y_2] = y_1 y_2' - y_1' y_2 = 3e^t \neq 0 \quad \forall t \in \mathbb{R} \implies [y_1, y_2] \text{ are fundamental}$$

Since the ODE is linear, we know that $y_3 = -2y_2, y_4 = y_1 + 2y_2$, and $y_5 = 2y_1 - 2y_3$ are also solutions. Let's check some pairs to see if they are fundamental.

1. Is $[y_1, y_3]$ fundamental? We see it is since $W[y_1, y_3] = -2W[y_1, y_2] \neq 0$.
2. Is $[y_2, y_3]$ fundamental? We see it isn't since $W[y_2, y_3] = -2W[y_2, y_2] = 0$
3. Is $[y_1, y_4]$ fundamental? We see it is since $W[y_1, y_4] = \underbrace{W[y_1, y_1]}_{=0} + 2W[y_1, y_2] \neq 0$
4. Is $[y_4, y_5]$ fundamental? We see it isn't since

$$y_5 = 2y_1 + 4y_2 = 2y_4 \implies W[y_4, y_5] = 2W[y_4, y_4] = 0$$

Exact 2nd Order Equations We say

$$P(x)y'' + Q(x)y' + R(x)y = 0 \quad \text{is exact} \iff [P(x)y']' + [f(x)y]' = 0$$

Let's find the conditions on P, Q, R for exactness. Notice that

$$[P(x)y']' + [f(x)y]' = P'y' + Py'' + f'y + fy' = P(x)y'' + (P' + f)y' + f'y = 0$$

This implies that $f' = R$, i.e.

$$f = \int R(x)dx$$

We see that $Q = P' + \int R(x)dx$. This implies we need

$$P'' - Q' + R = 0 \text{ for exactness}$$

Non-Homogeneous Equations We'll introduce some basic methods to solve some non-homogeneous 2nd order linear equations here. Namely

$$y'' + py' + qy = g$$

Notice that if y_1 and y_2 that solve the homogeneous equation will not change a particular solution to the above ODE. Thus the general solution to a non-homogeneous 2nd order linear ODE must take the form

$$y(t) = \underbrace{Ay_1(t) + By_2(t)}_{fundamental} + \underbrace{y_p(t)}_{particular} \quad \text{where} \quad A, B \in \mathbb{R}$$

The Method of Undetermined Coefficients, i.e. Guess the Answer This is a very special way of solving simple expressions for $g(t)$. Namely if $g(t)$ takes the form:

$$g(t) = \begin{cases} P_n(t) = a_n t^n + \ldots + a_1 t + a_0 & \text{Polynomials} \\ P_n(t)e^{\alpha t} & \text{Exponentials /Polynomials} \\ P_n(t)e^{\alpha t}\begin{cases} \sin(\beta t) \\ \cos(\beta t) \end{cases} & \text{Trig/Exp/Poly} \end{cases}$$

Then we may just guess that form as the answer for the particular solution, and reduce the problem to a system of equations and solve for the constant. **Careful!!! if the fundamental solution already contains one of the terms you're guessing, you must add a t to make it a repeated root for this method to work!**

Example(Undetermined Coefficients) - Find the general solution to

$$y'' + y = 3\sin 2t + t\cos 2t$$

First we solve the homogeneous part. Clearly, $y'' + y = 0$ gives

$$y(t) = A\sin t + B\cos t + y_p(t)$$

We see that since $g(t)$ takes one of the nice forms mentioned above, it is possible to guess the answer. We guess

$$y_p(t) = c_1\sin(2t) + (c_2 t + c_3)\cos(2t) \quad \text{where} \quad c_1, c_2, c_3 \in \mathbb{R}$$

since we see $t\cos(2t)$ we guess up to a first order polynomial. Substitute this into the equation and plug and chug to deduce :

$$LHS = y_p'' + y_p = (-3c_1 - 4c_2)\sin(2t) + (-3tc_2 + c_3)\cos(2t)$$

$$RHS = 3\sin 2t + t\cos 2t$$

By comparing coefficients, we see we have

$$-3c_1 - 4c_2 = 3 \quad \& \quad -3c_2 = 1 \quad \& \quad c_3 = 0$$

thus we have

$$c_2 = -\frac{1}{3} \quad \& \quad c_1 = -\frac{5}{9} \quad \& \quad c_3 = 0$$

This means the general solution to the ODE is

$$y(t) = A\sin t + B\cos t - \frac{5}{9}\sin(2t) - \frac{t}{3}\cos(2t)$$

Variation of Parameters Suppose that y_1 and y_2 solve $y'' + py' + qy = 0$, then suppose that $y = A(t)y_1 + B(t)y_2$ (i.e. make the constants functions, vary the parameters) solves

$$y'' + py' + qy = g$$

It turns out we have flexibility in the constraints of the resulting system, so suppose in addition that $A'y_1 + B'y_2 = 0$ it is possible to obtain the following after some algebra

$$A' = -\frac{y_2 g}{W[y_1, y_2]} \quad \& \quad B' = \frac{y_1 g}{W[y_1, y_2]}$$

i.e.

$$y(t) = -y_1(t)\int \frac{y_2 g}{W[y_1, y_2]}dt + y_2(t)\int \frac{y_1 g}{W[y_1, y_2]}dt$$

solves the non-homogeneous ODE. Where $W[y_1, y_2]$ is exact since we're given y_1, y_2.

Example(Variation of Parameters) - Find the general solution to

$$y'' - 2y + y = \frac{e^t}{1+t^2}$$

We first solve the homogeneous component to obtain that

$$y(t) = A\underbrace{e^t}_{=y_1} + B\underbrace{te^t}_{=y_2} + y_p(t)$$

By definition of the Wronskian, we have that

$$W[y_1, y_2](t) = y_1 y_2' - y_1' y_2 = e^{2t}$$

Now we may compute the integrals in the formula given above. We have

$$A(t) = -\int \frac{y_2 g}{W[y_1, y_2]} dt = -\int \frac{t}{1+t^2} dt = -\frac{1}{2}\ln(1+t^2) + A$$

$$B(t) = \int \frac{y_1 g}{W[y_1, y_2]} dt = \int \frac{dt}{1+t^2} = \arctan(t) + B$$

with $A, B \in \mathbb{R}$ as the integration constants. Thus we see the general solution to the ODE is

$$y(t) = Ae^t + Bte^t - \frac{e^t}{2}\ln(1+t^2) + te^t \arctan(t)$$

Tutorial 7,8 & 9

MAT 244 – ODE – Summer 2013

First Order Systems

First Order Linear Systems As we've seen with 1st and 2nd order ODE we have two cases of homogeneous and non-homogeneous. We'll now consider matrix differential systems and show these contain nth order ODE. We define a first order linear system as

$$\dot{x}(t) = A(t)x(t)$$

where $x(t) : \mathbb{R} \to \mathbb{R}^n$, $A(t) : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ and $\dot{x} \equiv \frac{d}{dt}x(t)$. Since the system is linear, it turns out the linear combination fundamental solutions form the general solution. Thus we're looking for

$$x(t) = c_1 x^{(1)} + \ldots + c_n x^{(n)} \quad \text{where} \quad c_1, \ldots, c_n \in \mathbb{R}$$

Notice that we may write a homogeneous nth order equation as an **equivalent system** (matrix form) by

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \ldots + p_0(t)y = 0 \iff \dot{x} = \begin{pmatrix} 0 & 1 & 0 & \ldots & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & \ldots & 1 & 0 \\ -p_0 & \ldots & \ldots & \ldots & \ldots & -p_{n-1} \end{pmatrix} x \quad \text{where} \quad x(t) = \begin{pmatrix} y(t) \\ y'(t) \\ \vdots \\ \vdots \\ y^{(n-1)}(t) \end{pmatrix}$$

We'll see like the characteristic equation and exponential solutions appear naturally here.

The Wronskian Just as before, we'd like a notation of linear independence of our solutions to see how complete they are. In this case, we'll group together are fundamental solutions into a matrix, and call it a fundamental matrix $X(t)$ (word play in action right there). I.e. If x is the general solution to $\dot{x} = Ax$, then

$$x(t) = c_1 x^{(1)} + \ldots + c_n x^{(n)} = \underbrace{\begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ x^{(1)} & \ldots & \ldots & x^{(n)} \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}}_{X(t)} \underbrace{\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}}_{=c} = X(t)c$$

In this form we've removed all the constants into a constant vector c. Recall from linear algebra that all columns of a matrix are linearly independent if and only if it's determinant is non-zero. Thus, we see the ideal definition of the Wronskian is the following

$$W[x^{(1)}, \ldots, x^{(n)}](t) = \det X(t)$$

Let's check how this agrees with our definition of Wronskian from earlier. Notice that this is much more natural in the differential system setting.

Example(Wronskian) Suppose that $y''+py'+qy=0$ and $\dot{x}=Ax$ are equivalent systems(have a look above for a reminder). We'll show that

$$W[y_1,y_2](t) = Const * W[x^{(1)},x^{(2)}](t)$$

One may check that equivalent systems have the property that

$$y_1 = Ax_{11}+Bx_{12} \quad \& \quad y_2 = Cx_{11}+Dx_{12} \quad \text{where} \quad A,B,C,D\in\mathbb{R}$$

where our fundamental matrix is defined as

$$X(t) = \begin{pmatrix} x^{(1)} & x^{(2)} \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

Then by definition of our old Wronskian, we have

$$\begin{aligned} W[y_1,y_2](t) =& y_1y_2' - y_1'y_2 \\ =& (Ax_{11}+Bx_{12})(Cx_{21}+Dx_{22}) - (Ax_{21}+Bx_{22})(Cx_{11}+Dx_{12}) \\ =& (AD-BC)(x_{11}x_{22}-x_{12}x_{21}) \\ =& \begin{vmatrix} A & B \\ C & D \end{vmatrix}\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \\ =& \begin{vmatrix} A & B \\ C & D \end{vmatrix} W[x^{(1)},x^{(2)}](t) \end{aligned}$$

Example(Construction of a System from Solutions) Consider the vector functions

$$x^{(1)} = \begin{pmatrix} t \\ 1 \end{pmatrix} \quad \& \quad x^{(2)} = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$$

Are the functions linearly independent? We check the Wronskian

$$W[x^{(1)},x^{(2)}](t) = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = t^2$$

If $t>0$ our vectors are linearly independent (or $t<0$). If these were solutions to some first order system, we'd expect there to be some singularity at $t=0$ (due to the degeneracy). Let's create the system to check this intuition. We want $X(t)$ to solve $\dot{X}=A(t)X$, since $W[x^{(1)},x^{(2)}](t)\neq 0$ for $t>0$(The inverse X^{-1} makes sense), we have that

$$\dot{X}X^{-1} = A$$

We easily compute that

$$\dot{X}X^{-1} = \frac{1}{t^2}\begin{pmatrix} 1 & 2t \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2t & -t^2 \\ 1 & t \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -2/t^2 & 2/t^2 \end{pmatrix} = A$$

Thus $X(t)$ is the fundamental solution to

$$\dot{x} = \begin{pmatrix} 0 & -1 \\ -2/t^2 & 2/t^2 \end{pmatrix} x \quad \text{where} \quad t>0$$

A Formula for the Determinant Suppose $X(t)$ solves $\dot{X} = AX$ with initial data $X(t_0) = X_0$. Then

$$W[x^{(1)}, \ldots, x^{(n)}](t) = \det X(t) = \det X_0 \ \exp\left[\int_{t_0}^{t} \mathrm{tr} A(x) dx\right]$$

Solving Homogeneous Linear systems with Constant Coefficients We want to construct the solutions to $\dot{x} = Ax$ with $A \in M_{n\times n}(\mathbb{C})$. To do this, recall the following from linear algebra fact: Suppose that A is not defective(this means that A has enough eigenvectors to span a full basis), then there exists a matrix D consisting of the eigenvalues of A on the diagonal and a matrix Λ consisting of the eigenvectors of A as the columns. Together, they satisfy

$$D = \begin{pmatrix} \lambda_1 & 0 & \ldots & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & \ldots & \ldots & \ldots & \lambda_n \end{pmatrix} = \begin{pmatrix} \vec{\lambda}_1 & \ldots & \vec{\lambda}_n \end{pmatrix}^{-1} A \begin{pmatrix} \vec{\lambda}_1 & \ldots & \vec{\lambda}_n \end{pmatrix} = \Lambda A \Lambda^{-1}$$

Thus, if we consider the change of variables $x = \Lambda y$, we obtain the following

$$\dot{x} = Ax \iff \Lambda\dot{y} = A\Lambda y \iff \dot{y} = \Lambda^{-1}A\Lambda y \iff \dot{y} = Dy$$

The system in y is trivial to solve since it decouples into first order ODE. Namely we have

$$\dot{y} = Dy \implies \frac{dy_i}{dt} = \lambda_i y \quad i \in \{1, \ldots, n\}$$

These equations are all separable, this means that we have

$$y_i(t) = c_i e^{\lambda_i t} \quad i \in \{1, \ldots, n\} \quad c_i \in \mathbb{C}$$

This is why we've been guessing exponential solutions as previously. This doesn't quite solve our original problem though, we still need to send it back to x via $x = \Lambda y$. Expanding this out shows

$$x(t) = c_i\vec{\lambda}_1 e^{\lambda_1 t} + \ldots + c_n\vec{\lambda}_1 e^{\lambda_1 t} = \sum_{i=1}^{n} c_i\vec{\lambda}_i e^{\lambda_i t}$$

This effectively reduces the problem of solving a homogeneous linear system with constant coefficients to finding the eigenvalues and eigenvectors of A.

Example(2nd order Homogeneous with Constant Coefficients as a System) From our previous discussion, its obvious that (given $a \neq 0$)

$$ay'' + by' + cy = 0 \iff y'' + \frac{b}{a}y' + \frac{c}{a}y = 0 \iff \dot{x} = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix} x \quad \text{where} \quad x = \begin{pmatrix} y \\ y' \end{pmatrix}$$

This will explain why we called the 2nd order linear characteristic equation, well... a characteristic equation. The eigenvalues of the above matrix are given by the roots of

$$P(\lambda) = \begin{vmatrix} \lambda & -1 \\ c/a & \lambda + b/a \end{vmatrix} = \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a}$$

Thus we have

$$P(\lambda) = 0 \iff a\lambda^2 + b\lambda + c = 0$$

which is the equation we used previously. Now we see where the name comes from.

Example(2×2 Matrix system with Constant Coefficients) Solve

$$\dot{x} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} x$$

As we've seen from the above, the general solution reduces to finding the eigenvalues and eigenvalues of A. Let's compute the characteristic equation.

$$P(\lambda) = \det(A - I\lambda) = \begin{vmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{vmatrix} = \lambda(\lambda - 2)$$

Thus the eigenvalues are $\lambda = 0$ and $\lambda = 2$. Let's find the eigenvectors by checking the kernels now. For $\lambda = 0$ we have

$$\ker \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = \text{span}\begin{pmatrix} 1 \\ i \end{pmatrix} \implies \vec{\lambda_0} = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

For $\lambda = 2$ we have

$$\ker \begin{pmatrix} -1 & i \\ -i & -1 \end{pmatrix} = \text{span}\begin{pmatrix} 1 \\ -i \end{pmatrix} \implies \vec{\lambda_2} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Thus the solution to the system is

$$x(t) = c_1 \begin{pmatrix} 1 \\ i \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{2t} \quad \text{where} \quad c_1, c_2 \in \mathbb{C}$$

2×2 **System Types and Terminology** Just as with the 2nd order linear case with constant coefficients, there are 3 types of solutions for 2×2 matrices with constant coefficients(with one extra degenerate case). Following the same argument as before, it all comes down to the characteristic equation. Consider

$$\dot{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} x \quad \text{where} \quad a, b, c, d \in \mathbb{R}$$

Then we have the characteristic equation as

$$P(\lambda) = \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = \lambda^2 - \underbrace{(a+d)}_{\text{trace}A}\lambda + \underbrace{ad - bc}_{\det A}$$

We see the roots of this equation are

$$\lambda_{\pm} = \frac{a+d}{2} \pm \frac{\sqrt{(a+d)^2 - 4(ad - bc)}}{2} = \frac{a+d}{2} \pm \frac{\sqrt{(a-d)^2 - 4bc}}{2}$$

Again we have three possibilities for the eigenvalues, but lets see what exactly they entail.

(a) **Saddle** Let's start with the case with the trace not equal to zero, $p = a + d \neq 0$, and the discriminant $\Delta = (a - d)^2 - 4bc > 0$, is positive, and $|\text{tr}(A)| < \Delta$. From the above calculations, this implies that the eigenvalues have different signs. I.e.

$$\text{sgn}(\lambda_+) \neq \text{sgn}(\lambda_-)$$

Then the solution to the system takes the form

$$x(t) = A\vec{\lambda_+}e^{\lambda_+ t} + B\vec{\lambda_-}e^{\lambda_- t} \quad \text{where} \quad A, B \in \mathbb{R}$$

Assume that $\lambda_- < 0$ for the sake of convenience here. We see the following asymptotics (behaviour at infinity).

$$x(t) \propto \vec{\lambda_+} \quad \text{when} \quad t >> 0 \quad \& \quad x(t) \propto \vec{\lambda_-} \quad \text{when} \quad t << 0$$

since $\exp(x) \to 0$ as $x \to -\infty$. Notice that we only care about the direction of the flow (i.e. the vector), the constant doesn't play a major role which is why we leave it as simply proportional to ($\propto$).

(b) **Proper Node** In this case we have that $p \neq 0$, $\Delta > 0$ and $|\mathrm{tr}(A)| > \Delta$ i.e.

$$\mathrm{sgn}(\lambda_+) = \mathrm{sgn}(\lambda_-)$$

but notice that $\lambda_+ \neq \lambda_-$. In this case the solution to the system takes the form

$$x(t) = A\vec{\lambda_+}e^{\lambda_+ t} + B\vec{\lambda_-}e^{\lambda_- t} \quad \text{where} \quad A, B \in \mathbb{R}$$

as before, but the asymptotics are different here. Assume that $\lambda_+ > 0$ and that $\lambda_+ > \lambda_-$. Then we see that

$$x(t) \propto \vec{\lambda_+} \quad \text{when} \quad t >> 0 \quad \& \quad \lim_{t \to -\infty} x(t) = 0$$

(c) **Spiral** In this case we have that $p \neq 0$ and $\Delta < 0$. i.e.

$$\lambda_\pm = p \pm i\sqrt{|\Delta|}$$

we'll have a solution that takes the form

$$x(t) = e^{pt}\left[A\vec{\lambda_+}e^{i\sqrt{|\Delta|}t} + \overline{A}\vec{\lambda_-}e^{-i\sqrt{|\Delta|}t}\right] \quad \text{where} \quad A \in \mathbb{C}$$

noting that $\overline{\vec{\lambda_+}} = \vec{\lambda_-}$, we may use Euler's identity to find a closed form of the above. Assume

$$\lambda_+ = \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{pmatrix} \quad \text{where} \quad x_i, y_i \in \mathbb{R}$$

then if we substitute Euler's identity in, we obtain

$$x(t) = e^{pt}\left[A\begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{pmatrix}\left(\cos(\sqrt{|\Delta|}t) + i\sin(\sqrt{|\Delta|}t)\right) + \overline{A}\begin{pmatrix} x_1 - iy_1 \\ x_2 - iy_2 \end{pmatrix}\left(\cos(\sqrt{|\Delta|}t) - i\sin(\sqrt{|\Delta|}t)\right)\right]$$

Gathering like terms reveals

$$x(t) = e^{pt}\left[(A + \overline{A})\begin{pmatrix} x_1\cos(\sqrt{|\Delta|}t) - y_1\sin(\sqrt{|\Delta|}t) \\ x_2\cos(\sqrt{|\Delta|}t) - y_2\sin(\sqrt{|\Delta|}t) \end{pmatrix} + i(A - \overline{A})\begin{pmatrix} x_1\sin(\sqrt{|\Delta|}t) + y_1\cos(\sqrt{|\Delta|}t) \\ x_2\sin(\sqrt{|\Delta|}t) + y_2\cos(\sqrt{|\Delta|}t) \end{pmatrix}\right]$$

We know that $A + \overline{A} = \tilde{A} \in \mathbb{R}$ and $i(A - \overline{A} = \tilde{B} \in \mathbb{R}$, thus we usually write the real form of the solution as

$$x(t) = e^{pt}\left[\tilde{A}\begin{pmatrix} x_1\cos(\sqrt{|\Delta|}t) - y_1\sin(\sqrt{|\Delta|}t) \\ x_2\cos(\sqrt{|\Delta|}t) - y_2\sin(\sqrt{|\Delta|}t) \end{pmatrix} + \tilde{B}\begin{pmatrix} x_1\sin(\sqrt{|\Delta|}t) + y_1\cos(\sqrt{|\Delta|}t) \\ x_2\sin(\sqrt{|\Delta|}t) + y_2\cos(\sqrt{|\Delta|}t) \end{pmatrix}\right]$$

notice solutions of this type rotate around on ellipses with some given expansion(or dilation) corresponding to the $\exp(pt)$ term in the front. It's easy to see if $p = 0$, the solution will always rotate around a fixed orbit.

(d) **Improper Node** In this case we'll have $\Delta = 0$, which one can show means the matrix is defective and the previous diagonalization argument falls apart. We'll cover this in detail later on.

Example(Spiral System) Solve the following system in real form

$$\dot{x} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} x$$

To begin, find the eigenvalues of the characteristic equation.

$$P(\lambda) = \begin{vmatrix} 1-\lambda & 2 \\ -1 & 1 \end{vmatrix} = (1-\lambda)^2 + 2 = \lambda^2 - 2\lambda + 3$$

Thus, we see through the quadratic equation that

$$P(\lambda_\pm) = 0 \iff \lambda_\pm = 1 \pm \sqrt{2}i$$

The eigenvectors corresponding to the system are found by checking the kernel. We see, for λ_+,

$$\ker(A - I\lambda_+) = \ker \begin{pmatrix} -\sqrt{2}i & 2 \\ -1 & -\sqrt{2}i \end{pmatrix} = \text{span} \begin{pmatrix} -\sqrt{2}i \\ 1 \end{pmatrix} \implies \vec{\lambda_+} = \begin{pmatrix} -\sqrt{2}i \\ 1 \end{pmatrix}$$

Notice if the eigenvalues are complex, and the matrix is real. i.e. we have that $\overline{\lambda_+} = \lambda_-$ and $\overline{A} = A$, this means

$$Ax = \lambda_+ x \implies \overline{A\vec{x}} = \overline{\lambda_+ \vec{x}} \iff A\overline{\vec{x}} = \lambda_- \overline{\vec{x}}$$

This means that the eigenvector corresponding to λ_- is just the complex conjugate of the eigenvector corresponding to λ_+, i.e.

$$\vec{\lambda_-} = \overline{\vec{\lambda_+}} = \begin{pmatrix} \sqrt{2}i \\ 1 \end{pmatrix}$$

Thus

$$x(t) = e^t \left[A \begin{pmatrix} \sqrt{2}i \\ 1 \end{pmatrix} e^{-\sqrt{2}it} + \overline{A} \begin{pmatrix} -\sqrt{2}i \\ 1 \end{pmatrix} e^{\sqrt{2}it} \right] \quad \text{where} \quad A \in \mathbb{C}$$

is the real valued solution. Calling upon Euler's Identity we may deduce that

$$x(t) = e^t \left[\tilde{A} \begin{pmatrix} \sqrt{2}\sin\sqrt{2}t \\ \cos\sqrt{2}t \end{pmatrix} + \tilde{B} \begin{pmatrix} \sqrt{2}\cos\sqrt{2}t \\ -\sin\sqrt{2}t \end{pmatrix} \right] \quad \text{where} \quad \tilde{A}, \tilde{B} \in \mathbb{R}$$

With the usual $\tilde{A} = A + \overline{A}$ and $\tilde{B} = i(A - \overline{A})$.

Example(Which way does a spiral system spin?) One could always solve the system and just check the answer, but let's show we can deduce this from the matrix itself (once we know it's a spiral type of course). From our previous analysis, we basically just have to compute x_1, y_1, x_2, y_2 corresponding to the page before. We compute the eigenvector to find this explicitly. As a exercise, you may find that

$$\vec{\lambda}_+ = \left(-\frac{-a + d + \sqrt{(a-d)^2 + 4bc}}{2c}, 1 \right) \quad \& \quad \vec{\lambda}_- = \left(-\frac{-a + d - \sqrt{(a-d)^2 + 4bc}}{2c}, 1 \right)$$

effectively finding x_1, y_1, x_2, y_2. Since the system is a spiral, we must have $bc < 0$ (or else the square root isn't imaginary). Thus we have two cases, consider the case when $b > 0$ and $c < 0$. We then have that $y_1 > 0$ from the above. What does this mean in terms of our previous formula (i.e. do we spiral clockwise (cw) or counter clockwise (ccw)). You should see that this means we're cw. The other case is when $b < 0$ and $c > 0$, this case is the opposite direction, i.e. when we're ccw. The punch line of this is that you just have to check the sign of the b: $b > 0$ means cw and $b < 0$ means ccw.

Matrix Exponentials Consider $\dot{x}(t) = Ax(t)$ when A is not a function of t and define the formal series expansion of the exponential as

$$x(t) = \exp(At) = I + \sum_{n=1}^{\infty} \frac{A^n t^n}{n!}$$

Notice that this function solves the system since

$$\frac{dx}{dt} = \frac{d}{dt}\exp(At) = \sum_{n=1}^{\infty} \frac{A^n t^{n-1}}{(n-1)!} = A\sum_{n=1}^{\infty} \frac{A^{n-1}t^{n-1}}{(n-1)!} = A\sum_{n=0}^{\infty}\frac{A^n t^n}{n!} = Ax$$

We also see that $\exp(0) = I$, thus, since $x(t) = X(t)\vec{c}$ previously, by uniqueness of solution to the IVP we immediately have

$$\exp(At) = X(t)X^{-1}(0)$$

some call the special fundamental solution at t_0 the fundamental solution s.t. $\Phi(t_0) = I$, thus by linearity we have the following identities

$$\exp(A(t-t_0)) = \Phi_{t_0}(t) = X(t)X^{-1}(t_0)$$

Example(Matrix Exp) Solve the IVP in the special fundamental solution form.

$$\dot{x} = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

One may find the characteristic equation to find the eigenvalues,

$$P(\lambda) = \lambda^2 + 2\lambda + 5 \implies \lambda_\pm = -1 \pm 2i$$

The eigenvectors are easily found to be

$$\vec{\lambda_+} = \begin{pmatrix} 2i \\ 1 \end{pmatrix} \quad \& \quad \vec{\lambda_-} = \begin{pmatrix} -2i \\ 1 \end{pmatrix}$$

Thus, one may deduce that a fundamental solution to the system is (via our previous computations)

$$X(t) = e^t \begin{pmatrix} -2\sin 2t & 2\cos 2t \\ -\cos 2t & \sin 2t \end{pmatrix}$$

Thus, the exponential solution or special fundamental matrix is

$$\Phi_0(t) = X(t)X^{-1}(0)$$

We have

$$X^{-1}(t) = -\frac{e^{-t}}{2}\begin{pmatrix} \sin 2t & 2\cos 2t \\ -\cos 2t & -2\sin 2t \end{pmatrix} \implies X^{-1}(0) = \frac{1}{2}\begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$$

Therefore

$$\Phi_0(t) = \begin{pmatrix} \cos 2t & 4\sin 2t \\ \frac{1}{2}\sin 2t & \cos 2t \end{pmatrix}$$

Notice we may check our computation with $\Phi(0) = I$. Thus the solution to the IVP is

$$x(t) = \Phi_0(t)\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Variation of Parameters If we have non-homogeneous system, such as

$$\dot{x}(t) = A(t)x(t) + g(t)$$

There is a formula just as in the 2nd order case. It's most easily written interns of the fundamental solution to the homogeneous system. Let $X(t)$ be the fundamental solution of $\dot{X} = A(t)X$, then we have that

$$x(t) = X\vec{c} + X\int X^{-1}(t)g(t)dt$$

solves the non-homogeneous system. This is easily verified since

$$\dot{x}(t) = \dot{X}\vec{c} + \dot{X}\int X^{-1}(t)g(t)dt + X(X^{-1}g(t)) = A\left(X\vec{c} + X\int X^{-1}(t)g(t)dt\right) + g(t) = Ax(t) + g(t)$$

Furthermore, if $A(t)$ is constant valued, we have that

$$x(t) = \Phi_{t_0}(t)x(t_0) + \int_{t_0}^{t} \Phi_{t_0}(t-s)g(s)ds$$

Prove this as an exercise.

Jordan Normal Form and Defective Matrices Recall that a defective matrix is one in which the algebraic multiplicity does not match the geometric. i.e. the Eigenbasis is incomplete. More specially, this will only happen in cases of repeated roots (i.e. algebraic side of things), i.e.

$$P(\lambda) = (\lambda - \lambda_m)^k \dots$$

Here the eigenvalue λ_m is repeated k times and thus has an algebraic multiplicity of k. If we can only find $r < k$ eigenvectors, we need a way to generate something to complete the basis in a nice way. It turns out we may do this, and it is called Jordan Normal Form. The theorem states we may create $k - r$ generalized eigenvectors to account for the deficiency, such that for some Λ consisting of the eigenvectors and generalized eigenvectors we have

$$J = \begin{pmatrix} \lambda_1 & \delta_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \delta_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \delta_{n-1} \\ 0 & \dots & \dots & \dots & \lambda_n \end{pmatrix} = \Lambda A \Lambda^{-1}$$

where $\delta_i = 0$ or 1 depending on if λ_i is accounted in the Eigenbasis. Furthermore, we may section these into blocks of all the same eigenvalue. For all blocks that have no deficiency, they are just diagonal. Otherwise, we'll have 1's where the generalized eigenvectors (ordered). We may create these generalized eigenvectors ($\lambda_m^{g_i}$) by finding a solution to

$$(A - I\lambda_m)\lambda_m^g = \vec{\lambda_m} \quad \& \quad (A - I\lambda_m)\lambda_m^{g_{i+1}} = \lambda_m^{g_i}$$

One we have this, it's convenient to decompose J into it's Jordan Blocks (sub-matrices in which the diagonal elements are all the same in each block). In general, we'll have diagonal matrices D_{λ_m} and upper diagonal matrices (with 1's just above the diagonal) J_{λ_m}, together we can say

$$J = \begin{pmatrix} D_{\lambda_1}/J_{\lambda_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & D_{\lambda_k}/J_{\lambda_k} \end{pmatrix}$$

where / means or. We solved the case with D_{λ_m} already, the new guy is J_{λ_m}. So suppose

$$J_{\lambda_k} = \begin{pmatrix} \lambda_k & 1 & 0 & \dots & 0 \\ 0 & \lambda_k & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 1 \\ 0 & \dots & \dots & \dots & \lambda_k \end{pmatrix}$$

Using the change of variables $y = \Lambda x$, we have that $\dot{x} = Ax$ becomes $\dot{y} = J_{\lambda_k} y$. Thus the system solutions can be constructed from the top down by considering $y = (y_1, \dots, y_n)$ and take all but the first zero, then all but the 1 and 2 zero, all the way to n. i.e.

$$y^{(1)} = \begin{pmatrix} y_1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \implies \dot{y}_1 = \lambda_k y_1 \implies y_1 = e^{\lambda_k t}$$

$$y^{(2)} = \begin{pmatrix} y_2 \\ y_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \implies \begin{cases} \dot{y}_2 = \lambda_k y_2 + y_1 \\ \dot{y}_1 = \lambda_k y_1 \end{cases} \implies \begin{cases} y_2 = te^{\lambda_k t} \\ y_1 = e^{\lambda_k t} \end{cases}$$

you can check that we have the following by induction

$$y^{(n)} = \begin{pmatrix} y_n \\ y_{n-1} \\ \vdots \\ \vdots \\ y_1 \end{pmatrix} \implies \begin{cases} y_n' = \lambda_k y_n + y_{n-1} \\ \vdots \\ y_2' = \lambda_k y_2 + y_1 \\ y_1' = \lambda_k y_1 \end{cases} \implies \boxed{y_n = \frac{t^{n-1}}{(n-1)!} e^{\lambda_k t}}$$

Noting that we call $y(t) = \sum_{i=1}^{n} c_i y^{(i)}$, with $c_i \in \mathbb{R}$, we conclude that

$$y(t) = \begin{pmatrix} \sum_{i=1}^{n} c_i y_i \\ \sum_{i=1}^{n-1} c_i y_i \\ \vdots \\ c_2 y_2 + c_1 y_1 \\ c_1 y_1 \end{pmatrix}$$

Thus the solution to the system is just $x(t) = \Lambda y(t)$. Let's look at the 2 by 2 case to briefly see an explicit formula.

2×2 **Repeated Roots Formula** Suppose that A is defective and $\dot{x} = Ax$. From our above remarks, we saw that the solution to a defective system is given by

$$x(t) = \Lambda y(t) = \begin{pmatrix} \vec{\lambda_1} & \vec{\lambda_1^g} \end{pmatrix} \begin{pmatrix} c_1 te^{\lambda_1 t} + c_2 e^{\lambda_1 t} \\ c_1 e^{\lambda_1 t} \end{pmatrix} = \boxed{c_2 \vec{\lambda_1} e^{\lambda_1 t} + c_1 e^{\lambda_1 t} (\vec{\lambda_1} t + \vec{\lambda_1^g})}$$

Example(Repeated Roots) Solve

$$\dot{x} = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} x \quad \text{where} \quad x(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

As always, we first find the eigenvalues from the characteristic equation.

$$P(\lambda) = \begin{vmatrix} 1-\lambda & -4 \\ 4 & -7-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 9 = (\lambda+3)^2$$

Thus $\lambda = -3$ is a repeated root. Next up we have the eigenvectors:

$$\ker \begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} = \text{span} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \implies \vec{\lambda} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

notice we're missing an eigenvector for the basis, so we generate a generalized eigenvector from

$$(A - I\lambda)\vec{\lambda^g} = \vec{\lambda} \implies \begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \vec{\lambda}^g = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \implies \vec{\lambda}^g = \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}$$

Thus, via our previous formula, we see the solution to the system is

$$x(t) = e^{-3t} \left[A \begin{pmatrix} 1 \\ 1 \end{pmatrix} + B \begin{pmatrix} t + 1/4 \\ t \end{pmatrix} \right] \quad \text{where} \quad A, B \in \mathbb{R}$$

Plugging in the initial data implies that

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} A + B/4 \\ A \end{pmatrix} \implies A = 2, B = 4$$

Thus the solution to the IVP is

$$x(t) = \begin{pmatrix} 3 + 4t \\ 2 + 4t \end{pmatrix} e^{-3t}$$

Example(3×3 System) Find the general solution to

$$\dot{x} = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{pmatrix} x$$

As always, start off by finding the characteristic equation and the eigenvalues. In this case, since the matrix is diagonal, we have that

$$P(\lambda) = (\lambda - 1)^2(\lambda - 2)$$

Thus the eigenvalues are 1 and 2 with algebraic multiplicity of 2 and 1 respectively. The eigenvector for $\lambda = 2$ is found by looking at the kernel

$$\ker(A - 2I) = \ker \begin{pmatrix} -1 & 0 & 0 \\ -4 & -1 & 0 \\ 3 & 6 & 0 \end{pmatrix} = \text{span} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \implies \vec{\lambda}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda = 1$ we have

$$\ker(A - I) = \ker \begin{pmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 3 & 6 & 1 \end{pmatrix} = \text{span} \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix} \implies \vec{\lambda}_1 = \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix}$$

We're missing an eigenvector for $\lambda = 1$, so we have to generate a generalized one. We see

$$(A - I)\vec{\lambda}_1^g = \vec{\lambda}_1 \implies \begin{pmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 3 & 6 & 1 \end{pmatrix} \vec{\lambda}_1^g = \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix} \implies \vec{\lambda}_1^g = \begin{pmatrix} 1/4 \\ 7/8 \\ 0 \end{pmatrix}$$

Thus, using our previous formulas for diagonal and Jordan blocks we obtain

$$x(t) = e^t \left[A \begin{pmatrix} 1/4 \\ -t + 7/8 \\ 6t \end{pmatrix} + B \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix} \right] + C \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t} \quad \text{where} \quad A, B, C \in \mathbb{R}$$

Example(Fundamental Matrices and Matrix Exponentials) Find $\exp(At)$ if for $\dot{x} = Ax$ we have

$$X(t) = \begin{pmatrix} 3e^{-t} & e^t \\ 2e^{-t} & e^t \end{pmatrix}$$

Recalling that we had uniqueness of solution, we know that

$$\Phi(t) = X(t)X^{-1}(0) = \exp(At)$$

We compute the inverse at 0 using the inversion formula to obtain

$$X(0) = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \implies X^{-1}(0) = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

Thus the exponential is computed to be

$$\exp(At) = \begin{pmatrix} 3e^{-t} & e^t \\ 2e^{-t} & e^t \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 3e^{-t} - 2e^t & 3e^t - 3e^{-t} \\ 2e^{-t} - 2e^t & 3e^t - 2e^{-t} \end{pmatrix}$$

Autonomous Systems Just as previously, we call autonomous systems things that take the form

$$\dot{x}(t) = F(x)$$

i.e. no t dependence on the vector valued function F. We call a critical point of the system, a point x_0 s.t.

$$\boxed{F(x_0) = 0 \iff x_0 \text{ is a critical point}}$$

Example(Shifted Linear System)

Locally Linear System For autonomous systems, we may centre them around their critical points via a change of basis (i.e. change of variables) to obtain

$$\dot{x} = F(x) = Ax + \mathcal{O}(x^2)$$

Thus for x around 0, we have that

$$\dot{x} \approx Ax$$

which is a first order approximation of the system around the critical point and talk about some of the local properties without issue. Suppose you have $\dot{x} = F(x)$ and x_0 is a critical point. We have to methods of linearizing the system. The first one is to perform the change of variables and only take the first order terms. i.e. let $y = x - x_0$, so we have

$$\dot{x} = F(x) \iff \dot{y} = F(y + x_0) = Ay + \mathcal{O}(y^2)$$

The other is to recall Taylor's formula, and see

$$F(x) = F(x_0) + \text{Jac}_F(x_0)(x - x_0) + \mathcal{O}(x^2)$$

where $\text{Jac}(x_0)$ is the Jacobian of F at x_0. Notice if x_0 is a critical point, the first term $F(x_0)$ drops out. For the sake of having this written somewhere, we have (if $F(x) = (f(x,y), g(x,y))$)

$$\text{Jac}_F(x,y) = \begin{pmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{pmatrix}$$

Example(Linearizing with change of variables) Find the critical points of the system and linearize around one of them. What type of system does it look like locally?

$$x' = x + y^2, \quad y' = x + y \quad \text{or} \quad \dot{x} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} x + \begin{pmatrix} y^2 \\ 0 \end{pmatrix}$$

Clearly the point $(x,y) = (0,0)$ is a critical point, and the linearized system around it is given by the first bit of the above. i.e.

$$\dot{x} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} x$$

is the linearized system around $(0,0)$. We see it is a improper node since the eigenvalues are both 1 and the matrix is lower diagonal. We also have that

$$y' = 0 \implies x = -y, \quad 0 = x + y^2 \implies 0 = x + x^2 = x(1+x) \implies x = -1$$

i.e. $(-1,1)$ is also a critical point. The linearized system can be found via changing variables, $(x,y) \to (x+1, y-1)$. We see that this implies

$$x' \to (x-1) + (y-1)^2 = x + -2y + y^2, \quad y' \to (x-1) + (y+1) = x + y \implies \dot{x} = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} x + \begin{pmatrix} y^2 \\ 0 \end{pmatrix}$$

Thus the linearized system around $(-1,1)$ is

$$\dot{x} = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} x$$

Example(Linearizing with Jacobian) Find the critical points and Linearize

$$x' = (1+x)\sin y, \quad y' = 1 - x - \cos y$$

in this case we see the Jacobian method is more useful. First lets find the critical points, we have that

$$x' = 0 \iff (1+x)\sin y = 0 \iff x = -1 \quad \text{or} \quad y = n\pi \quad n \in \mathbb{Z}$$

For y', we see if $x = -1$

$$2 - \cos y = 0$$

which is impossible. If $y = n\pi$, we have that

$$\begin{cases} -x & n \text{ even} \\ 2-x & n \text{ odd} \end{cases}$$

Thus, our critical points are $(0, 2n\pi)$ for the even guys, and $(2, (2n+1)\pi)$ for the odd guys. Our Jacobian in this case is

$$\text{Jac}(x,y) = \begin{pmatrix} \sin y & (1+x)\cos y \\ -1 & \sin y \end{pmatrix}$$

Thus the linearized system around $(0, 2n\pi)$ is

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x$$

and around $(2, (2n+1)\pi)$ we have

$$\dot{x} = \begin{pmatrix} 0 & -3 \\ -1 & 0 \end{pmatrix} x$$

Trajectories of Autonomous Systems For 2d autonomous systems it is possible to find a equation for the trajectories by solving the corresponding first order ODE. If $\dot{x} = F(x) = (f(x,y), g(x,y))$, then

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{g}{f}$$

is an ODE for the trajectories.

Example(Trajectories) Find the trajectory for

$$x' = ay, \quad y' = -bx, \quad a, b, > 0, \quad x(0) = \sqrt{a}, \quad y(0) = 0$$

Well, from the above we have

$$\frac{dy}{dx} = \frac{-bx}{ay} \iff \int y dy = -\int \frac{b}{a} x dx \iff y^2 = const - \frac{b}{a} x^2$$

The initial condition implies that

$$\frac{x^2}{a} + \frac{y^2}{b} = 1$$

i.e. the trajectories are ellipses in this case. We could also have verified this by find eigenvalues and eigenvectors with the previous method to see the system is of type centre.

$$\dot{x} = \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix} x \implies P(\lambda) = \lambda^2 + ab \implies \lambda_{\pm} = \pm i\sqrt{ab}$$

Tutorial 11

MAT 244 – ODE – Summer 2013

Series Methods

Taylor Series meets ODE Recall from Calculus the following representation of continuous functions. If $f \in C^{n+1}(\mathbb{R})$, then we have the series expansion centered at x_0,

$$f(x) = \sum_{m=1}^{n} \frac{f^{(m)}(x_0)}{m!}(x - x_0)^m + \underbrace{\mathcal{O}((x - x_0)^{m+1})}_{\text{error term}}$$

This representation allows us to utilize an ODE in the following way, given an n-th order equation, we may always rewrite the DE as

$$y^{(n)}(x) = F(x, y, \ldots, y^{(n-1)})$$

i.e. we may compute $y^{(n)}(x_0)$ given everything in F. If we differentiate the the above in x, we obtain

$$y^{(n+1)}(x) = \frac{dF}{dx}(x, y, \ldots, y^{(n)})$$

Repeating this idea, we see that we may recursively find the k-th derivative by differentiating the ODE. Where we centre the expansion (i.e. x_0) is important! We'll start with the idea of **ordinary points**, these are points s.t. $F(x, \cdot)$ has no "blow ups" around x_0, i.e. it is continuous around x_0. In such case we say x_0 is ordinary. If F isn't continuous about x_0, we say that x_0 is a **singular point**.

Recurrence of series coefficients This will be our other main tool to compute series solutions. It relies on the fact that x^n and x^m are linearly independent for $n \neq m \in \mathbb{N}$. This method relies on the assumption that the solution takes the form of a series, i.e.

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

solves $y^{(n)}(x) = F(x, y, \ldots, y^{(n-1)})$. Since we've assumed it a solution, it should solve the ODE. With a bit of algebra, in the cases we'll work with, we'll be able to rewrite the resulting form as

$$\sum_{n=0}^{\infty} R_n (x - x_0)^n = 0$$

Where R_n is some constant that comes from the simplification of $F(x, y, \ldots, y^{(n-1)})$. By linear independence, we see that $y(x)$ will be a solution if

$$R_n = 0 \quad \forall n \in \mathbb{N}$$

This is called the **recurrence formula** (or equation) for the system. It's called this since it gives a method to compute the unknown coefficients a_n recursively. Since this case is a little more situational, lets have a look at an example.

Example(Recurrence Formula) Find the series that solves

$$y'' - xy' - y = 0 \quad \text{about} \quad x_0 = 1$$

Just as we said above, we assume that $y(x) = \sum a_n (x-1)^n$ is a solution and plug this into the ODE. We obtain

$$y''-xy'-y = 0 \implies \left(\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x-1)^n\right) - x\left(\sum_{n=0}^{\infty} a_{n+1}(n+1)(x-1)^n\right) - \left(\sum_{n=0}^{\infty} a_n(x-1)^n\right) = 0$$

Note: A common trick that comes up with this method is to write $x = (x-1)+1$. Using this trick, we now see that we may group the like terms as

$$\sum_{n=0}^{\infty} \underbrace{[a_{n+2}(n+2)(n+1) - a_{n+1}(n+1) - (n+1)a_n]}_{R_n}(x-1)^n = 0$$

Thus we see the recurrence formula, this gives us a method to compute the

$$R_n = a_{n+2}(n+2)(n+1) - a_{n+1}(n+1) - (n+1)a_n = 0 \implies \boxed{a_{n+2} = \frac{a_{n+1}+a_n}{n+2}}$$

Now we have a recurrence relation behind all the coefficients in the series solution. Since we weren't given a_0 or a_1 (i.e. the initial data), we'll leave the general solution in terms of this constants. This allows us to distinguish between the fundamental solutions of the system. More explicitly, we see

$$y(x) = a_0 \underbrace{\left(1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 + \ldots\right)}_{y_1} + a_1 \underbrace{\left((x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \ldots\right)}_{y_2}$$

What written above is the solution up to 4th order, i.e. terms of $(x-1)^4$. We haven't explicitly written terms with higher order, so to avoid the dots and introduce some nice notation we'll use $\mathcal{O}((x-1)^5)$ to denote terms of order 5 and higher. **There isn't always a nice closed form solution**, so we may only as for terms up to some order.

Example(Recurrence Formula) Find the series solution for

$$(4-x^2)y'' + 2y = 0 \quad \text{about} \quad x_0 = 0$$

Just as we've done previously, assume that $y(x) = \sum a_n x^n$ is a solution and plug it into the ODE. We obtain

$$\left(\sum_{n=0}^{\infty} 4a_{n+2}(n+2)(n+1)x^n - a_{n+2}(n+2)(n+1)x^{n+2}\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0$$

We may rewrite the above as (showing that we have flexibility in reenumeration)

$$\sum_{n=-2}^{\infty} \left[4a_{n+4}(n+4)(n+3) - a_{n+2}(n+2)(n+1) + 2a_{n+2}\right] x^{n+2} = 0$$

Thus we see the recurrence formula clearly in this form, and we deduce that (noting that we may do this by linear independence once again)

$$a_{n+2} = \frac{n(n-1)-2}{4(n+2)(n+1)}a_n \quad \forall n \in \mathbb{N}$$

Here we notice there is a pattern to the numbers. We find that

$$y(x) = a_0\left(1 - \frac{x^2}{4}\right) + a_1\left(x - \sum_{n=1}^{\infty} \frac{x^{2n+1}}{(2n+1)!!4^n}\right)$$

where the double factorial is just the product of odd terms (i.e. $(2n+1)!! = (2n+1)(2n-1)\dots 5*3*1)$. Which is the general solution to the ODE and the two fundamental solutions are visible with their general constants a_0 and a_1.

Convergence of the Series Solution We won't concern ourselves with this technically since you've covered this in a previous calculus course. There is one result we'd like to state via the Taylor Series method. We see that $y^{(n)}(x) = F(x, y, \dots, y^{(n-1)})$ has a series solution around x_0 provide that the coefficients of F are analytic around x_0 (i.e. the derivatives of F make sense around x_0). The radius of convergence is at least as large as the minimum of the coefficients.

Example(Taylor Series) Find the first 4 terms of the series solution to

$$y'' + \sin(x)y' + \cos(x)y = 0 \quad y(0) = 0, y'(0) = 1 \quad \text{about} \quad x_0 = 0$$

In this example we have that initial data, i.e. an IVP, so we know that $a_0 = 0$ and $a_1 = 1$. Keeping up with the notation above, we have

$$y'' = -\sin(x)y' - \cos(x)y = F(x, y, y')$$

Via Taylor's Theorem, we know that if $y(x) = \sum a_n x^n$, we have

$$a_n = \frac{f^{(n)}(0)}{n!}$$

So via the ODE, we have that

$$a_2 = \frac{y''(0)}{2} = \frac{-\sin(0)y'(0) - \cos(0)y(0)}{2} = 0$$

To find a_3, we just have to take a derivative, we see

$$y''' = \frac{dF}{dx} = \sin(x)y - 2\cos(x)y' - \sin(x)y''$$

Thus we see that

$$a_3 = \frac{y'''(0)}{6} = -\frac{1}{3}$$

To find a_4, we take another derivative. We see that

$$y^{(4)} = \cos(x)y + 3\sin(x)y' - 3\cos(x)y'' - \sin(x)y'''$$

Obviously we have that $a_4 = 0$ since everything drops, this means the solution up to fourth order is just

$$y(x) = x - \frac{x^3}{3} + \mathcal{O}(x^5)$$

Also, since sine and cosine are analytic everywhere, we have that the radius of convergence of the above solution is ∞, i.e. all $x \in \mathbb{R}$ make sense.

Example(Radius of convergence) Find a lower bound for the raids of convergence for

$$xy'' + y \quad \text{about} \quad x_0 = 1$$

The above form isn't standard, we rewrite to find that

$$y'' = -\frac{1}{x}y$$

i.e. there is a singularity at $x_s = 0$. This means that at worst, the convergence of the solution will be $R \geqslant |x_s - x_0| = 1$

Example Let x and x^2 be solutions to $P(x)y'' + Q(x)y' + R(x)y = 0$. Can we say whether the point $x = 0$ is ordinary or singular. Prove the answer.
Well lets check the form of the ODE. Let $y = Ay_1 + By_2 = Ax + Bx^2$, this of course means that

$$y' = A + 2Bx \quad \& \quad y'' = 2B$$

Plugging this into the system implies we have

$$2BP(x) + Q(x)(A + 2Bx) + R(x)(Ax + Bx^2) = 0 \implies \left\{ \begin{array}{c} A[Q(x) + R(x)x] = 0 \\ B[2P(x) + 2xQ(x) + x^2R(x)] = 0 \end{array} \right.$$

Lets reduce the above equations. We see that

$$Q(x) = -R(x)x$$

So let's plug this into the second equation.

$$2P(x) - 2x^2R(x) + x^2R(x) = 2P(x) - x^2R(x) = 0 \implies \frac{x^2R(x)}{2} = P(x)$$

This means the ODE must take the form

$$x^2R(x)\left[y'' - \frac{2}{x}y' + \frac{2}{x^2}y\right] = 0$$

In this form we clearly see that $x = 0$ is singular.

Miscellaneous Techniques

MAT 244 – ODE – Summer 2013

Misc - Techniques

Calculus Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are C^1 (continuous and differentiable) at $x \in \mathbb{R}$. Then we have

The Limit We say $\lim_{x\to c} f(x) = L$ if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all x around c such that $0 < |x - c| < \delta$, we have that $|f(x) - L| < \epsilon$.

Derivative The derivative of a function at x is defined as

$$f'(x) \equiv \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \equiv \lim_{y\to x} \frac{f(y) - f(x)}{y - x}$$

Product Rule The derivative of the product fg is given by

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

Chain Rule The derivative of their composition is given by

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

Integration

Integration by Substitution This exploits the chain rule, we have

$$\int g'(x)f(g(x))dx = \int f(u)du$$

Integration by Parts This exploits the product rule, we have

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$$

Linear Algebra Let $A, B : M_{n\times n}(\mathbb{C}) \to M_{n\times n}(\mathbb{C})$ (a n by n matrix whose elements are functions $M_{i,j} : \mathbb{C} \to \mathbb{C}$.)

Inner Product of Vectors Let $\vec{x} : \mathbb{C}^n \to \mathbb{C}^n$ and $\vec{y} : \mathbb{C}^n \to \mathbb{C}^n$ (we may drop the vector hat, and write x alone when the context is clear). The inner product of x and y is (in various notations)

$$\underbrace{(x,y)}_{\text{lazyman}} = \underbrace{\langle x,y\rangle}_{\text{inner}} = \underbrace{x\cdot y}_{\text{dot}} = \underbrace{\overline{x}^T y}_{\text{matrix}} \equiv \sum_{i=1}^{n} \overline{x_i} y_i$$

Multiplication of Matrices We may multiply matrices using the following rule:

$$AB = C \quad \text{where} \quad C_{i,k} = \sum_{j=1}^{n} A_{i,j}B_{j,k}$$

Notice that this rule can be viewed as taking the inner product(dot product) of ith row of A with the kth row B(in the real sense!). This number makes up the ith jth entry in C. Naturally, the only way this definition will make sense is if the inner product of these "vectors" does. i.e If A is a $n \times m$ matrix, and B is a $m \times p$ matrix, $AB = C$ makes sense and is a $n \times p$ matrix.

Addition of Matrices We may add matrices using the following rule:

$$A + B = C \quad \text{where} \quad C_{i,j} = A_{i,j} + B_{i,j}$$

i.e, it only makes sense to add matrices of the same dimensions, and then it is done element wise.

Determinant of Matrices There is a special quantity that appears while working with square matrices (i.e. $n \times n$ matrices), and it is called the determinate. The 2×2 case will be the most common we'll deal with, we define the determinate of A ($\det A$) as

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The general definition is a little cumbersome, and not needed for our purposes. We'll accept the Leibniz formula as a definition. The determinate of a $n \times n$ matrix A as

$$\det(A) = \sum_{\sigma\in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)}$$

where S_n is the symmetric group on n letters and σ is a chosen permutation of the letters. The sign of σ, $\text{sgn}(\sigma)$ is either $+1$ or -1 if the permutation is even or odd respectively(number of swaps from original word). One may see this as a recursive formula of the 2×2 case, by summing over smaller matrix determinants where a row and column is removed with $(-1)^{i+j}$ sign and iterating until the 2×2 case.

Example(Determinate of a 3×3 matrix) Using the Leibniz formula, we have that

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \sum_{\sigma\in S_3} \text{sgn}(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)}$$

There are 6 ways to rearrange 123, $S_3 = \{123, 231, 312, 132, 213, 321\}$, where the first 3 have even signs (two swaps or no swap) and the last 3 have odd signs (one swap). Thus we have

$$\det(A) = \underbrace{A_{1,1}A_{2,2}A_{3,3}}_{123} + \underbrace{A_{1,2}A_{2,3}A_{3,1}}_{231} + \underbrace{A_{1,3}A_{2,1}A_{3,2}}_{312} - \underbrace{A_{1,1}A_{2,3}A_{3,2}}_{132} - \underbrace{A_{1,2}A_{2,1}A_{3,3}}_{213} - \underbrace{A_{1,3}A_{2,2}A_{3,1}}_{321}$$

Inverse of Matrices It is not always possible to find an inverse to a matrix, but it is always possible if it's determinate is none zero. i.e.

$$\boxed{\det A \neq 0 \iff A^{-1} \quad \text{exists}}$$

There is a nice formula for the 2×2 case, and it is

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

More generally, we have to define a co-factor matrix, by $C_{i,j} = (-1)^{i+j} M_{i,j}$ where $M_{i,j}$ is called the minor of $A_{i,j}$, which is the determinant of the resulting matrix with the ith row and jth column removed. Then

$$A^{-1} = \frac{1}{\det A} C^T$$

Integration and Differentiation of Matrices This is identical to calculus in one variable, except we do everything term by term. i.e.

$$\frac{d}{dt} A(t) = \frac{dA_{i,j}}{dt} \quad \& \quad \int A dt = \int A_{i,j} dt$$

Example(Add,Mult,Diff,Int) Suppose we have two matrix function $A, B : M_{3\times3}(\mathbb{R}) \to M_{3\times3}(\mathbb{R})$:

$$A(t) = \begin{pmatrix} e^t & 2e^{-t} & e^{2t} \\ 2e^t & e^{-t} & -e^{2t} \\ -e^t & 3e^{-t} & 2e^{2t} \end{pmatrix} \quad \& \quad B(t) = \begin{pmatrix} 2e^t & e^{-t} & 3e^{2t} \\ -e^t & 2e^{-t} & e^{2t} \\ 3e^t & -e^{-t} & -e^{2t} \end{pmatrix}$$

(Addition) we have that $A + 3B$ is

$$A + 3B = \begin{pmatrix} e^t & 2e^{-t} & e^{2t} \\ 2e^t & e^{-t} & -e^{2t} \\ -e^t & 3e^{-t} & 2e^{2t} \end{pmatrix} + \underbrace{\begin{pmatrix} 6e^t & 3e^{-t} & 9e^{2t} \\ -3e^t & 6e^{-t} & 3e^{2t} \\ 9e^t & -3e^{-t} & -3e^{2t} \end{pmatrix}}_{3B} = \begin{pmatrix} 8e^t & 5e^{-t} & 10e^{2t} \\ -e^t & 7e^{-t} & 2e^{2t} \\ 8e^t & 0 & e^{2t} \end{pmatrix}$$

(Multiplication) we have that AB is (as an exercise fill in the questions marks)

$$AB = \begin{pmatrix} e^t & 2e^{-t} & e^{2t} \\ 2e^t & e^{-t} & -e^{2t} \\ -e^t & 3e^{-t} & 2e^{2t} \end{pmatrix} \begin{pmatrix} 2e^t & e^{-t} & 3e^{2t} \\ -e^t & 2e^{-t} & e^{2t} \\ 3e^t & -e^{-t} & -e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} - 2 + 4e^{3t} & 1 + 4e^{-2t} - e^t & 3e^{3t} + 2e^t - e^{4t} \\ 4e^{2t} - 1 - 3e^{3t} & ? & ? \\ ? & ? & ? \end{pmatrix}$$

(Differentiation) we have that A' is

$$\frac{d}{dt} A = A' = \dot{A} = \begin{pmatrix} e^t & -2e^{-t} & 2e^{2t} \\ 2e^t & -e^{-t} & -2e^{2t} \\ -e^t & -3e^{-t} & 4e^{2t} \end{pmatrix}$$

(Integration) we have that $\int A dt$ is

$$\int_0^1 A(t) dt = \begin{pmatrix} \int_0^1 e^t dt & \int_0^1 2e^{-t} dt & \int_0^1 e^{2t} dt \\ \int_0^1 2e^t dt & \int_0^1 e^{-t} dt & \int_0^1 -e^{2t} dt \\ \int_0^1 -e^t dt & \int_0^1 3e^{-t} dt & \int_0^1 2e^{2t} dt \end{pmatrix} = \begin{pmatrix} e - 1 & 2(1 - e) & \frac{1}{2}(e^2 - 1) \\ 2(e - 1) & 1 - e & \frac{1}{2}(1 - e^2) \\ 1 - e & 3(1 - e) & e^2 - 1 \end{pmatrix}$$

Eigenvalues and Eigenvectors We may think of these matrices as a linear operator(map) that takes in a vector and spits out a vector. When this map sends something to the same dimensional space (i.e. a matrix which is $n \times n$), it makes sense to take about "fixed points" through the map (vectors that don't change the way they're pointing, but maybe the length changed). In the context we mean

$$A\vec{x} = \lambda\vec{x}$$

i.e. we hit $x \in \mathbb{C}^n$ with the map $A : \mathbb{C}^n \to \mathbb{C}^n$, and got x back up to some scalar $\lambda \in \mathbb{C}$. Such x are called **eigenvectors** and the λ that accomplishes this is called an **eigenvalue**. It turns out these fixed points are an ideal tool to analysis how our linear operator A works. This will make our life easier in solving linear systems of differential equations, so let's find out how to find such objects. Well

$$Ax = \lambda x \iff Ax - \lambda x = 0 \iff (A - I\lambda)x = 0$$

If the map $(A - I\lambda)$ is invertible, then we obviously have

$$x = (A - I\lambda)^{-1}0 = 0$$

which is a trivial fixed point. So the obvious restriction to impose is that $(A - I\lambda)$ is not invertible. From our discussion about determinate and inverses earlier, we know this is equivalent to

$$P(\lambda) = \det(A - I\lambda) = 0$$

We'll call $P(\lambda)$ the **characteristic equation** of A. The roots of this equation correspond to the eigenvalues of A. Since the matrix is non-invertible with such a λ, it turns out that the kernel (things that get mapped to zero) of $(A - I\lambda)$ is non-trivial. The basis of this kernel will correspond to our eigenvectors. i.e.

$$x \in \ker(A - I\lambda)$$

Example(Eigenvalues and Eigenvectors) Find the eigenvalues and eigenvectors for

$$A = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

From the above, we look at the characteristic equation which is given by

$$P(\lambda) = \det(A - I\lambda) = \begin{vmatrix} 1-\lambda & \sqrt{3} \\ \sqrt{3} & -1-\lambda \end{vmatrix} = \lambda^2 - 4 = (\lambda - 2)(\lambda + 2)$$

Thus, clearly the eigenvalues (i.e. the roots) are $\lambda_\pm = \pm 2$. To find the eigenvectors, we look at the kernel of the map $(A - I\lambda)$, i.e. the things that get sent to zero. For $\lambda = 2$, we have

$$\ker(A - 2I) = \ker \begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{pmatrix} = \text{span}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \implies x_2 = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$$

For $\lambda = -2$, we have

$$\ker(A + 2I) = \ker \begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} = \text{span}\begin{pmatrix} -\frac{1}{\sqrt{3}} \\ 1 \end{pmatrix} \implies x_{-2} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$$

Notice that any vector in the span will work as our eigenvector since the only difference is a constant which we may pull out.

Example(Hermitian Operators) Suppose that A is hermitian, this means that $A = A^* \equiv \overline{A}^T$. Then let $\lambda \in \mathbb{C}$ be an eigenvalue for an eigenvector x. Notice that

$$\langle Ax, x\rangle = \langle x, Ax\rangle$$

This is true since the definition of the inner product implies

$$\langle Ax, x\rangle = \overline{(Ax)}^T x = \underbrace{\overline{x}^T \overline{A}^T x = \overline{x}^T Ax}_{A \text{ is hermitian}} = \langle x, Ax\rangle$$

Furthermore, the eigenvalue λ is necessarily real ($\lambda \in \mathbb{R}$) since

$$\lambda(x, x) = (x, \lambda x) = (x, Ax) = (Ax, x) = (\lambda x, x) = \overline{\lambda}(x, x)$$

To finish, note that x is an eigenvector which means $x \neq 0$ i.e the length $(x, x) = ||x||^2 > 0$. By the above equality we then have $\lambda = \overline{\lambda}$, which is equivalent to saying $\lambda \in \mathbb{R}$.