Tutorial 9 - MAT244 - C. J. Adkins

Fundamental & Defective Matrices

Call the solution to $\dot{x} = A(t)x$, $x^{(1)}, \dots, x^{(n)}$. Denote "the fundamental matrix" as

$$\Psi(t) = x^{(1)} \cdots x^{(n)} = \begin{pmatrix} x_1^{(1)} & \cdots & x_1^{(n)} \\ \vdots & \ddots & \vdots \\ x_n^{(1)} & \cdots & x_n^{(n)} \end{pmatrix}$$

Notice this means $x = c_1 x^{(1)} + \dots + c_n x^{(n)} = \Psi(t)\vec{c}$. For initial value problems, i.e. $x(t_0) = x_0$ we have

$$\vec{c} = \Psi^{-1}(t_0)x^0, \text{ thus } x = \Psi(t)\Psi^{-1}(t_0)x^0$$

Thus we have a 1st order matrix system $\Psi'(t) = A(t)\Psi(t)$.

We also have the "special" fundamental matrix denoted by: $\Phi(t) = \Psi(t)\Psi^{-1}(t_0)$

If we solve for $\Phi(t)$, we have $x = \Phi(t)x_0$.

Matrix Exponentials

$\dot{x} = Ax$ has "$x = \exp(At)$" as a solution, by "taylor expansion"

$$x = \exp(At) = I + \sum_{n=1}^{\infty} \frac{A^n t^n}{n!}$$

It follows that the above solves $\frac{d}{dt}x = Ax$ & $\exp(0) = I$, thus by uniqueness

$$\Phi = \exp(At) \quad \& \quad x = \exp(At)x^0$$

Ex, 7.7-#12 Solve: $\dot{x} = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} x$, $x(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ by using $\Phi(t)$

Well, find eigenvalues, $p(\lambda) = (\lambda+1)^2 + 4 = \lambda^2 + 2\lambda + 5 \Rightarrow \lambda_\pm = -1 \pm 2i$

eigenvectors can be found to be $\vec{\lambda}_+ = \begin{pmatrix} 2i \\ 1 \end{pmatrix}$, $\vec{\lambda}_- = \begin{pmatrix} -2i \\ 1 \end{pmatrix}$

$$\therefore \Psi(t) = \begin{pmatrix} 2ie^{(-1+2i)t} & -2ie^{(-1-2i)t} \\ e^{(-1+2i)t} & e^{(-1-2i)t} \end{pmatrix} \Rightarrow \Psi(0) = \begin{pmatrix} 2i & -2i \\ 1 & 1 \end{pmatrix} \Rightarrow \Psi^{-1}(0) = \frac{1}{4}\begin{pmatrix} -i & 2 \\ i & 2 \end{pmatrix}$$

$$\Rightarrow \Phi(t) = \Psi(t)\Psi^{-1}(0) = \frac{1}{4}\begin{pmatrix} 2\exp[(-1+2i)t] + 2\exp[(-1-2i)t] & i\exp[(-1+2i)t] - i\exp[(-1 \dots \\ i\exp[(-1-2i)t] - i\exp[(-1+2i)t] & 2\exp[(-1+2i)t] + 2\exp[(-1-2i)t] \end{pmatrix}$$

Thus: $\vec{x} = \Phi(t)\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ ∎

What about repeated roots? This is slightly deeper than what you may have seen, so we proceed with care.

We have $\dot{x} = Ax$ & we suppose $P_A(\lambda) = (\lambda - \lambda_k)^k \cdots$

λ_k is repeated k times!

If we can find k eigenvectors, then the diagonalization theorem holds & the fundamental solution follows.

If we can't find k eigenvectors, then the diagonalization theorem does not hold! But

Jordan Normal Form: $\begin{pmatrix} \lambda_1 & \delta_i & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \delta_i \\ 0 & & & \lambda_n \end{pmatrix}$ where $\delta = 1$ or 0

$\boxed{\text{Thm}}$ If alg multiplicity $\neq$ geo multiplicity for λ_k, then we can create eigenvectors to account for the difference. Namely if we only have n eigenvectors, we can create $k-n$ "generalized eigenvectors". Then for such an A that is "defective", we have

$$A = \Lambda J \Lambda^{-1}$$

where J is a Jordan block matrix.

I.e $J = \begin{pmatrix} D & 0 \\ 0 & J_{\lambda_k} \end{pmatrix}$ where $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$, $J_{\lambda_k} = \begin{pmatrix} \lambda_k & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_k \end{pmatrix}$

$\therefore$ for repeated roots, $\dot{x} = Ax \iff \dot{y} = Jy \quad (x = \Lambda^{-1} y)$

$$\Rightarrow \begin{pmatrix} y_1' \\ \vdots \\ y_n' \\ y_{1_k}' \\ \vdots \\ y_{k-1_k}' \\ y_{k_k}' \end{pmatrix} = \begin{pmatrix} \lambda_1 y_1 \\ \vdots \\ \lambda_n y_n \\ \lambda_k y_{1_k} + y_{2_k} \\ \vdots \\ \lambda_k y_{k-1_k} + y_k \\ \lambda_k y_{k_k} \end{pmatrix}$$

solving this system from the bottom up show us (using first order linear theory) that

$$\boxed{\vec{y}_{m_k} = \vec{C}_{m_k} t^m \exp(\lambda_k t)}$$

Now what about $\Lambda^{-1} \vec{C}_m$? We found they were eigenvalues before do we have something similar?

let $K=2$, then if λ is the eigenvalue, $\vec{\lambda}$ the eigenvector

$$\boxed{x^{(1)} = \vec{\lambda} \exp(\lambda t) \quad , \quad x^{(2)} = \vec{\lambda}\, t \exp(\lambda t) + \vec{\xi} \exp(\lambda t)}$$

where ξ is the generalized eigenvector, it satisfies $(A \cdot I\lambda)\xi = \vec{\lambda}$

Indeed $\dot{x} = Ax \Rightarrow \vec{\lambda} e^{\lambda t} + \cancel{\lambda \vec{\lambda} t e^{\lambda t}} + \lambda \vec{\xi} e^{\lambda t} = \cancel{A\vec{\lambda} t e^{\lambda t}} + A\xi e^{\lambda t}$

$$\Rightarrow \vec{\lambda} + \lambda \vec{\xi} = A\xi \Leftrightarrow \vec{\lambda} = (A - I\lambda)\xi$$

In general, we have, ξ_i where $(A - I\lambda)\xi_j = \vec{\lambda}_1 + \cdots + \vec{\lambda}_n + \sum_{i=0}^{j} \xi_i$
with this we can show

$$x^{(n-k)} = (\vec{\lambda}_1 + \cdots + \vec{\lambda}_n)\frac{t^k}{k}\exp(\lambda t), \ldots, x^{(n)} = \xi_k \exp(\lambda t)$$

(Ex. 7.8 - #7) Solve: $\dot{x} = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} x \quad , \quad x(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

① eigenvalues $P(\lambda) = \begin{vmatrix} 1-\lambda & -4 \\ 4 & -7-\lambda \end{vmatrix} = (\lambda - 1)(\lambda + 7) + 16 = \lambda^2 + 6\lambda + 9 = (\lambda + 3)^2$

$\therefore P(\lambda) = 0 \Leftrightarrow \lambda = -3$

② eigenvectors:

$$\operatorname{Ker}\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} = \operatorname{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \vec{\lambda} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

③ generalized eigenvectors

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\xi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \xi = \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}$$

④ Plug into formula

$$x(t) = A\begin{pmatrix} 1 \\ 1 \end{pmatrix}\exp(-3t) + Bt\begin{pmatrix} 1 \\ 1 \end{pmatrix}\exp(-3t) + B\begin{pmatrix} 1/4 \\ 0 \end{pmatrix} e^{-3t}$$

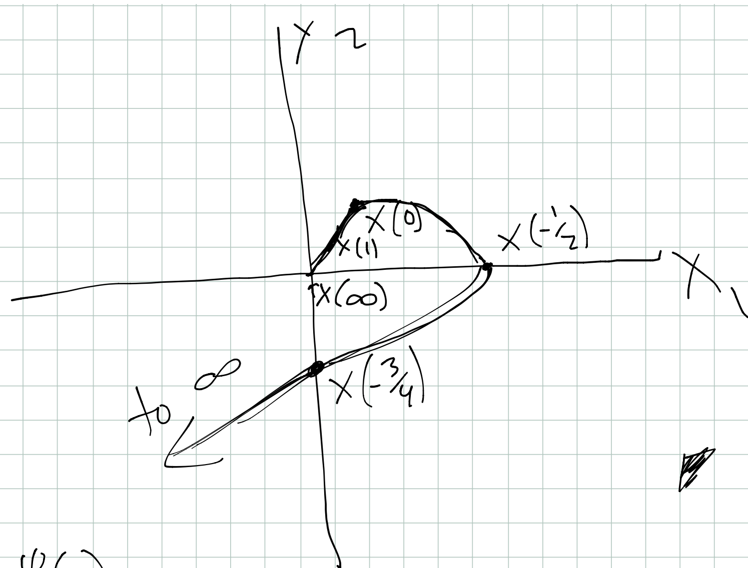
⑤ Initial Data

$$x(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} A + B/4 \\ A \end{pmatrix} \Rightarrow A = 2, B = 4$$

⑥ Done!

$$x(t) = \begin{pmatrix} 3 + 4t \\ 2 + 4t \end{pmatrix}\exp(-3t)$$

b) draw the trajectory of $x(t)$



Remark $\Lambda = \Psi(0)$

(Ex 7.8-#11) Solve:

$$\dot{x} = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{pmatrix} x \quad , \quad x(0) = \begin{pmatrix} -1 \\ 2 \\ -30 \end{pmatrix}$$

① eigenvalues are 1 & 2 (read off diagonal)

② eigenvectors are

$$\lambda = 1 \Rightarrow \text{Ker} \begin{pmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 3 & 6 & 1 \end{pmatrix} = \text{span} \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix} \quad , \quad \vec{x} = \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix}$$

$$\lambda = 2 \Rightarrow \text{Ker} \begin{pmatrix} -1 & 0 & 0 \\ -4 & -1 & 0 \\ 3 & 6 & 0 \end{pmatrix} = \text{span} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

③ generalized eigenvectors,

$$\begin{pmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 3 & 6 & 1 \end{pmatrix} \xi = \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix} \Rightarrow \xi = \begin{pmatrix} 1/4 \\ 7/8 \\ 0 \end{pmatrix}$$

④ plug into formula

$$x(t) = A \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix} t e^{t} + A \begin{pmatrix} 1/4 \\ 7/8 \\ 0 \end{pmatrix} e^{t} + B \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix} e^{t} + C \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t}$$

⑤ Initial data

$$x(0) = \begin{pmatrix} -1 \\ 2 \\ -30 \end{pmatrix} = \begin{pmatrix} A/4 \\ 7/8 A - B \\ 6B + C \end{pmatrix} \Rightarrow A = -4, \; B = -\tfrac{11}{2}, \; C = 3$$

⑥ Put it all together.

$$x(t) = \begin{pmatrix} -1 \\ 4t+2 \\ -24t-33 \end{pmatrix} \exp(t) + \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \exp(2t)$$

Quiz: If $\Psi(t) = \begin{pmatrix} 3e^{-t} & e^{t} \\ 2e^{-t} & e^{t} \end{pmatrix}$ for $\dot{x} = Ax$, find $\exp(At)$.

A/ Well, we know $\phi(t) = \Psi(t)\Psi^{-1}(0) = \exp(At)$ by uniqueness of the O.D.E solution. Thus:

$$\Psi(0) = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \Rightarrow \Psi^{-1}(0) = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

$$\therefore \exp(At) = \begin{pmatrix} 3e^{-t} & e^{t} \\ 2e^{-t} & e^{t} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 3e^{-t} - 2e^{t} & 3e^{t} - 3e^{-t} \\ 2e^{-t} - 2e^{t} & 3e^{t} - 2e^{-t} \end{pmatrix}$$