Tutorial 8 - MAT244 - C. J. Adkins

Solving Hom linear systems w/ $a_{ij} \in \mathbb{C}$

i.e what are the solutions to $\dot{x} = Ax$? where $A \in M_{2\times2}(\mathbb{C})$

Well, if we find eigenvectors & eigenvalues of A, recall diagonalization methods

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = \begin{pmatrix} \vec{\lambda}_1 \cdots \vec{\lambda}_n \end{pmatrix} A \begin{pmatrix} \vec{\lambda}_1 \cdots \vec{\lambda}_n \end{pmatrix}^{-1} = \Lambda A \Lambda^{-1}$$

Thus if $\vec{x} = \Lambda^{-1} \vec{y}$

$$\dot{x} = Ax \iff \Lambda \dot{x} = \Lambda A x \iff \Lambda(\Lambda^{-1}\dot{y}) = \Lambda A \Lambda^{-1} y$$

$$\iff \dot{y} = Dy \quad ①$$

Expanding out ① gives $\begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix} = \begin{pmatrix} \lambda_1 y_1 \\ \vdots \\ \lambda_n y_n \end{pmatrix}$

Since these are all first order, we have call this $\vec{C}_i$

$$\vec{y}_i = \vec{B}_i \exp(\lambda_i t) \iff \vec{x}_i = \Lambda^{-1} \vec{B}_i \exp(\lambda_i t) = \vec{C}_i \exp(\lambda_i t)$$

To find what the $\vec{C}_i$'s should be, plug $\vec{x}_i$ back into the equation!

$$\dot{x} = Ax \Rightarrow \lambda_i \vec{C}_i \exp(\lambda_i t) = A\vec{C}_i \exp(\lambda_i t)$$

$$\therefore \lambda_i \vec{C}_i = A\vec{C}_i \quad \text{(Eigenvectors!!!)}$$

Thus the eigenvalues & vectors completely make up our solution.

<u>Ex</u> (7.5-29) Consider $ay'' + by' + cy = 0$ where $a, b, c \in \mathbb{R}$

We showed solution dependent on the roots of $ar^2 + br + c = 0$

a) Find an equivalent system: $(x_1 = y, x_2 = y')$

$$ay'' + by' + cy = 0 \iff y'' + \frac{b}{a}y' + \frac{c}{a}y = 0 \iff \dot{x} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix} \vec{x}$$

The eigenvalues of the matrix are

$$\det\begin{vmatrix} \lambda & -1 \\ \frac{c}{a} & \lambda+\frac{b}{a} \end{vmatrix} = \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} = P(\lambda)$$

Thus $P(\lambda)=0 \iff a\lambda^2+b\lambda+c=0$ (same equation from ch.3)

Ex (7.5-9) Solve

$$\dot{x} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} x$$

Find eigenvalue & vectors, $P(\lambda) = \det\begin{vmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1 = \lambda(\lambda-2)$

$\therefore P(\lambda)=0 \iff \lambda=0, \lambda=2$ Eigenvalues

Eigenvectors: $\lambda=0 \Rightarrow \vec{x} \in \ker\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \iff \vec{x} \in \operatorname{span}\begin{pmatrix} 1 \\ i \end{pmatrix}$, $\vec{\lambda}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$\lambda=2 \Rightarrow \vec{x} \in \ker\begin{pmatrix} -1 & i \\ -i & -1 \end{pmatrix} \iff \vec{x} \in \operatorname{span}\begin{pmatrix} 1 \\ -i \end{pmatrix}$, $\vec{\lambda}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

Thus, the solution to the system is:

$$\vec{x} = A\begin{pmatrix} 1 \\ i \end{pmatrix} + B\begin{pmatrix} 1 \\ -i \end{pmatrix}\exp(2t), \quad A,B \in \mathbb{C}$$

Note, that if we have complex eigenvalues (or even vectors) the same proceedure holds.

Terminology for systems behavior at $\vec{x}=0$, The 2 by 2 case.

1) $\lambda_1 = a$, $\lambda_2 = -a$, $a \in \mathbb{R} \Rightarrow \vec{x}=0$ is a saddle point
2) $\lambda_1 = a$, $\lambda_2 = b$, $\operatorname{sgn}(\lambda_1) = \operatorname{sgn}(\lambda_2) \Rightarrow \vec{x}=0$ is a node
3) $\lambda_1 = a+bi$, $\lambda_2 = c+di$, $a,b,c,d \in \mathbb{R}$, $a,b \neq 0 \Rightarrow \vec{x}=0$ is a spiral point.

Ex(7.6-13)

a) eigenvalues of $\dot{x} = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix} x \Rightarrow P(\lambda) = (\alpha-\lambda)^2 + 1 = \lambda^2 - 2\alpha\lambda + (\alpha^2+1)$

$P(\lambda) = 0 \Leftrightarrow \lambda = \alpha \pm \frac{\sqrt{4\alpha^2 - 4(\alpha^2+1)}}{2} = \alpha \pm i$

b) critical values of λ (when do changes happen)

well $\alpha = 0$ is a critical point since, $\alpha \neq 0 \Rightarrow$ spiral

Why? Since the solution is given by: first find eigenvectors!

$\lambda = \alpha + i \Rightarrow \vec{x} \in \operatorname{Ker}\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \Leftrightarrow x \in \operatorname{span}\begin{pmatrix} -i \\ 1 \end{pmatrix}$, thus $\lambda_{\alpha+i} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

$\lambda = \alpha - i \Rightarrow \vec{x} \in \operatorname{Ker}\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \Leftrightarrow \vec{x} \in \operatorname{span}\begin{pmatrix} i \\ 1 \end{pmatrix}$, thus $\lambda_{\alpha-i} = \begin{pmatrix} i \\ 1 \end{pmatrix}$

Aside

notice $\overline{\lambda_{\alpha+i}} = \lambda_{\alpha-i}$, this happens for complex roots since

if $A\vec{x} = \lambda\vec{x}$ & $\lambda_\pm = a \pm bi$, then $\overline{\lambda_+} = \lambda_-$ so $\overline{A\vec{x}} = \overline{\lambda\vec{x}} \Leftrightarrow \bar{A}\bar{\vec{x}} = \bar{\lambda}\bar{\vec{x}}$

thus: if $A = \bar{A} \Rightarrow \bar{\vec{x}}$ is an eigenvector for λ_- if $\vec{x}$ is an eigenvector for λ_+.

Back to solution

$\vec{x} = A\begin{pmatrix} i \\ 1 \end{pmatrix}\exp((\alpha-i)t) + B\begin{pmatrix} -i \\ 1 \end{pmatrix}\exp((\alpha+i)t)$, $A, B \in \mathbb{C}$

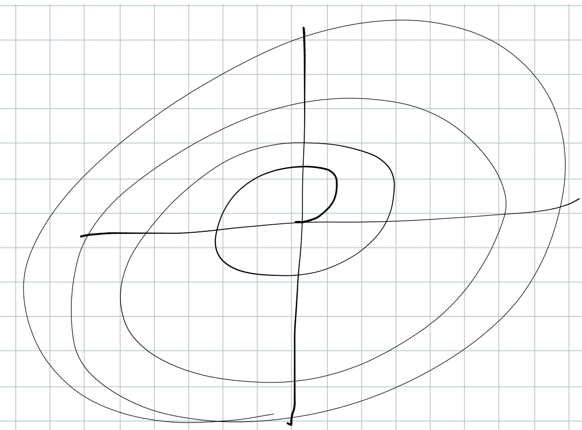
lets make this real valued! use $e^{i\theta} = \cos\theta + i\sin\theta$, thus

$\frac{\vec{x}}{e^{\alpha t}} = A\begin{pmatrix} i \\ 1 \end{pmatrix}(\cos t - i\sin t) + B\begin{pmatrix} -i \\ 1 \end{pmatrix}(\cos t + i\sin t)$

$= \tilde{A}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + \tilde{B}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$ where $\tilde{A} = A + B$, $\tilde{B} = i(A-B)$

$\therefore \vec{x} = e^{\alpha t}\left[\tilde{A}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + \tilde{B}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}\right]$

Obviously if $\alpha \neq 0$ we have growth or decay for the length (radius) while $\sin t$ & $\cos t$ rotate in circles.



spiral graph.

Note direction depends on $\tilde{A}, \tilde{B}$ $\tilde{A} \Rightarrow$ C.W, $\tilde{B} \Rightarrow$ C.C.W

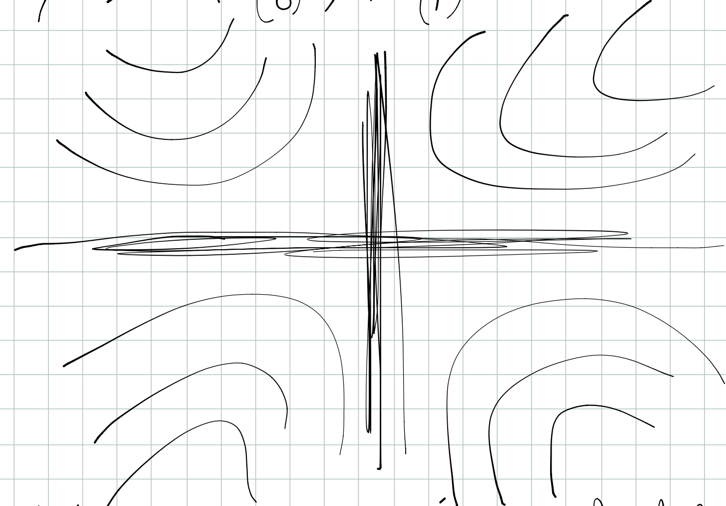
Now lets look at other solutions.

Saddle $\lambda_+ = -\lambda_- = \lambda \neq 0$ & $\lambda \in \mathbb{R}$, Solution look like

$$\vec{x} = A\vec{\lambda}_+ \exp(\lambda t) + B\vec{\lambda}_- \exp(-\lambda t)$$

For simplicity assume $\vec{\lambda}_+ = \begin{pmatrix}1\\0\end{pmatrix}$, $\vec{\lambda}_- = \begin{pmatrix}0\\1\end{pmatrix}$

$\Rightarrow$



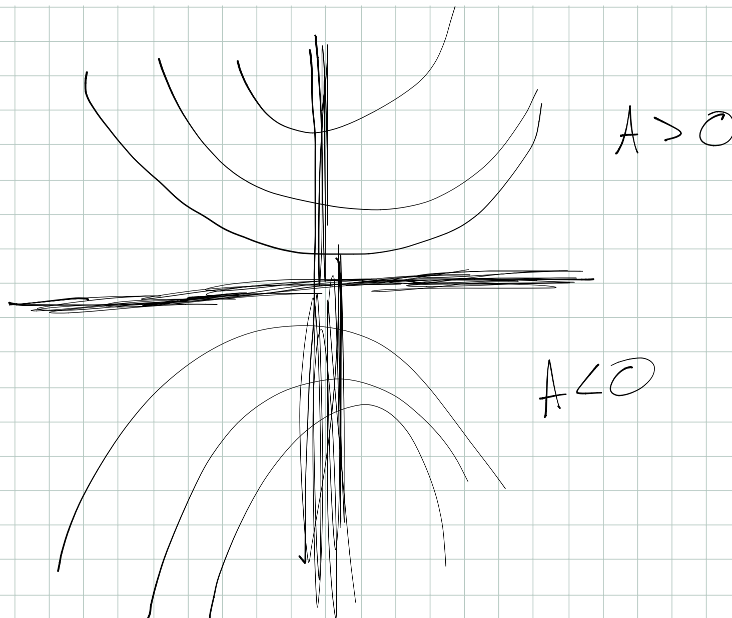
each quadrant corresponds to some pairing of $A, B > 0$, $A < 0, B > 0$, $A, B < 0$, $A > 0, B < 0$

which one is which in this case?

Node $\text{sgn}(\lambda_+) = \text{sgn}(\lambda_-)$ with $\lambda_+, \lambda_- \in \mathbb{R}$, Solution look like

$$\vec{x} \approx A\vec{\lambda}_\pm \exp(\lambda_\pm t)$$, since when $t \to 0$, one $\lambda_\pm$ dominates.

For simplicity assume $\lambda_+ = \lambda_- = \lambda$ & $\vec{\lambda}_+ = \begin{pmatrix}1\\0\end{pmatrix}$, $\vec{\lambda}_- = \begin{pmatrix}0\\1\end{pmatrix}$, thus



Quiz: Solve: $\dot{x} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} x$ in a real form.

Solution, well find eigenvalues.

$$P(\lambda) = \det(A - I\lambda) = \begin{vmatrix} 1-\lambda & 2 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 2 = \lambda^2 - 2\lambda + 3$$

$$\therefore P(\lambda) = 0 \iff \lambda_\pm = 1 \pm \frac{\sqrt{4 - 4\cdot 3}}{2} = 1 \pm \sqrt{2}i$$

eigenvectors?

$$\lambda = 1 + \sqrt{2}i \Rightarrow x \in \mathrm{Ker}\begin{pmatrix} -\sqrt{2}i & 2 \\ -1 & -\sqrt{2}i \end{pmatrix} \iff x \in \mathrm{span}\begin{pmatrix} -\sqrt{2}i \\ 1 \end{pmatrix} \Rightarrow \vec{x} = \begin{pmatrix} -\sqrt{2}i \\ 1 \end{pmatrix}$$

$$\lambda = 1 - \sqrt{2}i \Rightarrow x \in \mathrm{Ker}\begin{pmatrix} \sqrt{2}i & 2 \\ -1 & \sqrt{2}i \end{pmatrix} \iff x \in \mathrm{span}\begin{pmatrix} \sqrt{2}i \\ 1 \end{pmatrix} \Rightarrow \vec{x} = \begin{pmatrix} \sqrt{2}i \\ 1 \end{pmatrix}$$

Thus:

$$\frac{\vec{x}}{e^t} = A\begin{pmatrix} \sqrt{2}i \\ 1 \end{pmatrix} \exp(-\sqrt{2}it) + B\begin{pmatrix} -\sqrt{2}i \\ 1 \end{pmatrix} \exp(\sqrt{2}it)$$

$$= A\begin{pmatrix} \sqrt{2}i \\ 1 \end{pmatrix}\left[\cos\sqrt{2}t - i\sin\sqrt{2}t\right] + B\begin{pmatrix} -\sqrt{2}i \\ 1 \end{pmatrix}\left[\cos\sqrt{2}t + i\sin\sqrt{2}t\right]$$

$$= \tilde{A}\begin{pmatrix} \sqrt{2}\sin\sqrt{2}t \\ \cos\sqrt{2}t \end{pmatrix} + \tilde{B}\begin{pmatrix} \sqrt{2}\cos\sqrt{2}t \\ -\sin\sqrt{2}t \end{pmatrix} \quad \text{where } \tilde{A} = A + B, \ \tilde{B} = i(A - B)$$

$$\therefore \vec{x} = \tilde{A}e^t\begin{pmatrix} \sqrt{2}\sin\sqrt{2}t \\ \cos\sqrt{2}t \end{pmatrix} + \tilde{B}e^t\begin{pmatrix} \sqrt{2}\cos\sqrt{2}t \\ -\sin\sqrt{2}t \end{pmatrix} \quad \blacksquare$$