Tutorial 7 - MAT244 - C. J. Adkins

First Order Systems

What is a 1st order system? $\dot{x} = Ax$ / $x' = Ax$ / $\vec{x}' = A\vec{x}$ / ... ← different notations

where $A \in M_{n \times n}(\mathbb{R})$ & $\vec{x} : \mathbb{R}^n \to \mathbb{R}^n$

Since "A" is a matrix with constant coefficients, this corresponds to an nth order O.D.E

Quick Review of Linear Algebra:

let $A : M_{n\times n}(\mathbb{C}) \to M_{n\times n}(\mathbb{C})$ (n by n matrix whose elements are functions $\mathbb{C} \to \mathbb{C}$)
B : " $\to$ "

Multiplication: $AB = \begin{pmatrix} \text{———} \\ \\ \end{pmatrix} \begin{pmatrix} | & \\ & \end{pmatrix} = \begin{pmatrix} \cdot & \\ & \end{pmatrix}$

ith row · jth column = i,jth entry
(· ← "dot product" or inner product)

Dot Product: let $\vec{x} : \mathbb{C}^n \to \mathbb{C}^n$
$\vec{y}$: " $\to$ "

then $(\vec{x}, \vec{y}) = \vec{x} \cdot \vec{y} = \vec{x}^T \vec{y} = (x_1 \cdots x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \sum_{i=1}^{n} x_i y_i$

Addition $A + B = \begin{pmatrix} a_{ij} \end{pmatrix} + \begin{pmatrix} b_{ij} \end{pmatrix} = \begin{pmatrix} a_{ij} + b_{ij} \end{pmatrix}$

Inverse $\iff$ det $\neq 0$

① By means of co-factor matrix, $C_{ij} = (-1)^{i+j} M_{ij}$ ← Minor (det of remaining matrix missing ith row & jth col)

$\Rightarrow A^{-1} = \frac{1}{\det A} C^T$

② Row Operations
- Switch rows
- Multiply by nonzero scaler
- Add row together

Then, $A | I = I | A^{-1}$ by row Operations.

Note, this looks like

$$A|I = \left(\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix} \middle| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right) = \left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \middle| \; A^{-1} \right)$$

"A"

Note, integration & differentiation work term by term:

Ex (7.2 - #21) If:

$$A = \begin{pmatrix} e^{t} & 2e^{-t} & e^{2t} \\ 2e^{t} & e^{-t} & -e^{2t} \\ -e^{t} & 3e^{-t} & 2e^{2t} \end{pmatrix}, \quad B = \begin{pmatrix} 2e^{t} & e^{-t} & 3e^{2t} \\ -e^{t} & 2e^{-t} & e^{2t} \\ 3e^{t} & -e^{-t} & -e^{2t} \end{pmatrix}$$

a) $$A + 3B = \begin{pmatrix} 8e^{t} & 5e^{-t} & 10e^{2t} \\ -e^{t} & 7e^{-t} & 2e^{2t} \\ 8e^{t} & 0 & e^{2t} \end{pmatrix}$$

b) $$AB = \begin{pmatrix} 2e^{2t} - 2 + 3e^{3t}, & 1 + 4e^{-2t} - e^{t}, & 3e^{3t} + 2e^{t} - e^{4t} \\ 4e^{2t} - 1 - 3e^{3t}, & , & \\ , & , & \end{pmatrix}$$ ← fill in the rest :)

c) $$\frac{dA}{dt} = A' = \dot{A} = \begin{pmatrix} e^{t} & -2e^{-t} & 2e^{2t} \\ 2e^{t} & -e^{-t} & -2e^{2t} \\ -e^{t} & -3e^{-t} & 4e^{2t} \end{pmatrix}$$

d) $$\int_0^1 A\,dt = \begin{pmatrix} \int_0^1 e^{t}dt & \int_0^1 2e^{-t}dt & \int_0^1 e^{2t}dt \\ 2\int_0^1 e^{t}dt & \int_0^1 e^{-t}dt & -\int_0^1 e^{2t}dt \\ -\int_0^1 e^{t}dt & 3\int_0^1 e^{-t}dt & 2\int_0^1 e^{2t}dt \end{pmatrix}$$ ← so messy

$$= \begin{pmatrix} e-1, & 2(1-e), & \frac{1}{2}(e^2-1) \\ 2(e-1), & (1-e), & \frac{1}{2}(1-e^2) \\ (1-e), & 3(1-e), & (e^2-1) \end{pmatrix}$$

Eigenvalues & eigenvectors!

eigenvector ≈ "fixed point", eigenvalue ≈ scalar of how "fixed point" changes

i.e. $A\vec{x} = \lambda\vec{x}$ ← fixed point (on both sides!)

↑ eigenvalue

lets solve this.

$$Ax = \lambda x \iff Ax - \lambda x = 0 \iff (A - I\lambda)\vec{x} = 0$$

$\vec{x} = 0$ isn't the solution $\iff \det(A - I\lambda) = 0$ (why? want a singular matrix)

$P(\lambda) = \det(A - I\lambda)$ is called the characteristic equation. Its roots are the eigen values.

Vectors of the kernel of $(A - I\lambda)\vec{x}$ with $\lambda =$ eigenvalue are called eigenvectors. Thus $A\vec{x} = \lambda\vec{x}$

Ex (7.3-#20). Find eigen (values & vectors) for $\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$

By above we look at $\begin{vmatrix} 1-\lambda & \sqrt{3} \\ \sqrt{3} & -1-\lambda \end{vmatrix} = 0$ (note $|A| = \det A$)

$P(\lambda) = -(1-\lambda)(1+\lambda) - 3 = \lambda^2 - 4$, $\therefore P(\lambda) = 0 \iff \lambda = \pm 2$ ← eigenvalues

Eigenvectors?

$\lambda = 2 \Rightarrow \vec{x} \in \operatorname{Ker}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{pmatrix} \iff \vec{x} \in \operatorname{span}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$, $\therefore \vec{x} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$

$\lambda = -2 \Rightarrow \vec{x} \in \operatorname{Ker}\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \iff \vec{x} \in \operatorname{span}\begin{pmatrix} -\frac{1}{\sqrt{3}} \\ 1 \end{pmatrix}$, $\therefore \vec{x} = \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ 1 \end{pmatrix}$

the eigenvectors!

Ex (7.3-#3) Suppose $A = A^* = \bar{A}^T$ (Hermitian), let λ be an eigenvalue for $\vec{x}$.

a) Show $(A\vec{x}, \vec{x}) = (\vec{x}, A\vec{x})$

By Def: $(A\vec{x}, \vec{x}) = \overline{(A\vec{x})}^T \vec{x} = \bar{\vec{x}}^T \bar{A}^T \vec{x} = \bar{x}^T A \vec{x} = (\vec{x}, A\vec{x})$ (Hermitian)

b) Show $\lambda(x,x) = \bar{\lambda}(x,x)$, use fact $A\vec{x} = \lambda\vec{x}$ thus:

$\lambda(\vec{x}, \vec{x}) = (\vec{x}, \lambda\vec{x}) = (\vec{x}, A\vec{x}) = (A\vec{x}, \vec{x}) = \bar{\lambda}(\vec{x}, \vec{x})$

c) Show $\lambda = \bar{\lambda}$,

we know $\vec{x} \neq 0 \Rightarrow (x,x) = \|x\|^2 \neq 0$ so by b) $\lambda\|x\|^2 = \bar{\lambda}\|x\|^2 \iff \lambda = \bar{\lambda}$ ∎

Back to O.D.E's!

Consider the Homogeneous System: $\dot{x} = Ax$

let $\vec{x} = c_1 x^{(1)} + \cdots + c_n x^{(n)}$, $x^{(k)}$ are our fundamental solutions

To the system:

$$\begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

We can define the Wronskian as $W[x^{(1)} \dots x^{(n)}] = \det(x^{(1)} \dots x^{(n)})$

Abels theorem takes the form $W[x^{(1)} \dots x^{(n)}] = C\exp(\int \text{Trace}(A))$

$\text{Trace}(A) = p_{11} + p_{22} + \dots + p_{nn}$

Ex (7.4-#6) Consider: $x^{(1)} = \begin{pmatrix} t \\ 1 \end{pmatrix}$ & $x^{(2)} = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$

a) Compute $W[x^{(1)} x^{(2)}]$

By def $W = \det \begin{pmatrix} t & t^2 \\ 1 & 2t \end{pmatrix} = 2t(t) - (1)t^2 = t^2$

can check from the Wronskian too!

b) Where are $x^{(1)}$ & $x^{(2)}$ linearly independent?

Well, lin indep $\iff a x^{(1)} + b x^{(2)} = 0 \iff a = b = 0$

Thus we see $a x^{(1)} + b x^{(2)} = \begin{pmatrix} t(a+bt) \\ a + 2bt \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $t = 0$ & $a = 0 \Rightarrow \vec{0}$ thats it :)

$\therefore$ linear independent on $(-\infty, 0)$, $(0, \infty)$

c) What can we say about the coeffcients in the system of homogeneous D.E satisfied by $x^{(1)}$ & $x^{(2)}$?

Well... There must be a singularity at $t = 0$ (i.e not continuous around $t=0$) ✓

d) Find the system of equations for $x^{(1)}$ & $x^{(2)}$

Well... let $\vec{x} = x^{(1)} + x^{(2)}$, then $\dot{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2t \\ 2 \end{pmatrix} = \begin{pmatrix} 1+2t \\ 2 \end{pmatrix}$

We want $\dot{x} = Ax = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \vec{x}$

$$\therefore \begin{pmatrix} 1+2t \\ 2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} t+t^2 \\ 1+2t \end{pmatrix} = \begin{pmatrix} a_{11}(t+t^2) + a_{12}(1+2t) \\ a_{21}(t+t^2) + a_{22}(1+2t) \end{pmatrix}$$

$$\Rightarrow a_{11} = 0, \ a_{12} = 1, \ \& \ \boxed{\begin{matrix} 0 = a_{21}t + a_{22} \\ 2 = a_{21}t^2 + a_{22}2t \end{matrix}} \Rightarrow a_{21} = \frac{\tilde{a}_{21}}{t^2}, \ a_{22} = \frac{\tilde{a}_{22}}{t}$$

$$\Rightarrow \begin{matrix} \tilde{a}_{21} = -\tilde{a}_{22} \\ 2 = \tilde{a}_{21} + 2\tilde{a}_{22} \end{matrix} \Rightarrow \begin{matrix} \tilde{a}_{21} = -2 \\ \tilde{a}_{22} = 2 \end{matrix} \Rightarrow \dot{x} = \begin{pmatrix} 0 & 1 \\ -\frac{2}{t^2} & \frac{2}{t} \end{pmatrix} x \text{ is the system} \blacksquare$$

Ex (7.4 - #4) If $x_1 = y$, $x_2 = y'$ then

ⓐ $y'' + py' + qy = 0 \iff \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ ⓑ

Show if $x^{(1)}$ & $x^{(2)}$ are fundamental solutions of ⓑ & y_1 & y_2 are fundamental solutions of ⓐ, then

$$W[y_1, y_2] = c\, W[x^{(1)}, x^{(2)}] \quad (c \text{ constant})$$

Homework Question :) , Proceed by definition.

$$W[x^{(1)}, x^{(2)}] = \det\begin{pmatrix} x^{(1)} & x^{(2)} \end{pmatrix} = \det\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = x_{11}x_{22} - x_{12}x_{21}$$

Since the above systems are equivalent

$$\Rightarrow \quad y_1 = A x_{11} + B x_{12}$$
$$y_2 = C x_{11} + D x_{12}$$

$$W[y_1, y_2] = y_1 y_2' - y_2 y_1' = (Ax_{11} + Bx_{12})(Cx_{21} + Dx_{22}) - (Cx_{11} + Dx_{12})(Ax_{21} + Bx_{22})$$

$$= ACx_{11}x_{21} + ADx_{11}x_{22} + BCx_{12}x_{21} + BDx_{12}x_{22}$$
$$- CAx_{11}x_{21} - ADx_{12}x_{21} - BCx_{11}x_{22} - DBx_{12}x_{22}$$

$$= AD\, W[x^{(1)}, x^{(2)}] - BC\, W[x^{(1)}, x^{(2)}]$$

$$= (AD - BC)\, W[x^{(1)}, x^{(2)}] \quad ■$$

$(AD - BC)$ ← determinate of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix}$