Tutorial 6 - MAT244 - C.J. Adkins

Repeated Roots, Non-homogeneous 2nd order

(where $ay''+by'+cy=0$)

Recall that we still haven't dealt with the case when $\Delta = b^2-4ac=0$ (discriminant). When this happens, we know that $\left(\frac{-b}{2a}=r\right)$

$$y(t)=e^{rt} \quad \text{solves} \quad ay''+by'+cy=0$$

What about the second solution? Lets try to find it by setting $y=v(t)e^{rt}$,

$\Rightarrow y' = v'e^{rt}+rve^{rt}$, $y''=v''e^{rt}+2rv'e^{rt}+r^2ve^{rt}$, plugging in gives

$$e^{rt}\left(av''+2arv'+ar^2v+bv'+brv+cv\right)=e^{rt}\Big(av''+v'\underbrace{(2ar+b)}_{r=\frac{-b}{2a}\Rightarrow 0}+v\underbrace{(ar^2+br+c)}_{0 \text{ by } r \text{ being root}}\Big)=0$$

$\Rightarrow v''=0$ (☺ easy to solve!)

Just integrate twice and we see $v(t)=At+B$

$$\Rightarrow y(t)=Ate^{rt}+Be^{rt} \quad \text{(these are the fundamental solutions!)}$$

Reduction of Order - If we know one solution to $y''+py'+qy=0$, call it y_1, then we can try $y_2=vy_1$.

$y_2'=v'y_1+vy_1'$, $y_2''=v''y_1+2v'y_1'+vy_1''$, if we plug this in we obtain:

$$y''+py'+qy=0 \Rightarrow v''y_1+v'(2y_1'+py_1)+v\underbrace{(y_1''+py_1'+qy_1)}_{0,\ \text{since } y_1 \text{ is a solution.}}=0$$

$$\Rightarrow v''=-v'\left(2\frac{y_1'}{y_1}+p\right)$$

$\therefore$ y_2 is a solution works if v is as above. This is a first order eq in disguise. Since

$$\frac{v''}{v'}=-\left(2\frac{y_1'}{y_1}+p\right) \overset{\text{integrate}}{\Rightarrow} \ln(v')=-\int\left(\frac{2y_1'}{y_1}+p\right)dt$$

$$\Rightarrow v'=\exp\left(-\int\left(\frac{2y_1'}{y_1}+p\right)dt\right)$$

$$\therefore v(t)=\int \exp\left(-\int\left(\frac{2y_1'}{y_1}+p\right)dt\right)dt=\int \frac{W}{y_1^2}dt$$

Ex (3.4 - #6) Solve: $y''-6y'+9y=0$

Notice $\Delta=0$ & $r=\frac{-b}{2a}=3$, by above we have $y(t)=Ae^{3t}+Bte^{3t}$ $\forall A,B\in\mathbb{R}$

Ex (3.4 - #14) Solve: $y''+4y'+4y=0$, $y(-1)=2$, $y(-1)=1$

Notice $\Delta=0$ & $r=\frac{-b}{2a}=-2$, $\therefore$ $y(t)=Ae^{-2t}+Bte^{-2t}$, Now we solve for A&B by the data.

$y(-1)=2=Ae^2-Be^2$, $y(1)=1=Ae^{-2}+Be^{-2}$ $\Rightarrow e^2=A+B$ & $2e^{-2}=A-B$ $\therefore A=\frac{e^2}{2}+e^{-2}$, $B=\frac{e^2}{2}-e^{-2}$

$$\therefore y(t) = \left(\frac{e^2}{2} + \frac{1}{e^2}\right)e^{-2t} + \left(\frac{e^2}{2} - \frac{1}{e^2}\right)te^{-2t}$$

Ex (3.4-#30) Reduce to a first order eq & find y_2

$$x^2y'' + xy' + \left(x^2 - \tfrac{1}{4}\right)y = 0 \quad, x>0 \quad, y_1 = \frac{\sin x}{\sqrt{x}}$$

Remember we try $y_2 = vy_1$, this gives

$$v' = \frac{W}{y_1^2}$$, The O.D.E in standard form is $y'' + \underbrace{\tfrac{1}{x}}_{p}y' + \left(1 - \frac{1}{4x^2}\right)y = 0$

$$v' = x\frac{\exp\left(-\int \overbrace{\tfrac{1}{x}}^{p} dx\right)}{\sin^2 x} = \frac{x}{\sin^2 x}\frac{1}{x} = \frac{1}{\sin^2 x} = \csc^2 x$$

Remark: Notice the power of Abel's theorem in action, we don't really have much work!

$$\therefore v' = \csc^2 x \Rightarrow v(x) = \cot(x) \Rightarrow y_2(x) = v(x)y_1(x) = \frac{\cos(x)}{\sin(x)}\frac{\sin(x)}{\sqrt{x}} = \boxed{\frac{\cos(x)}{\sqrt{x}}}$$

Note: It is also possible to find y_2 via Abel's Theorem:

$y'' + py' + qy = 0$ w/ y_1, we can use $W(y_1,y_2) = A\exp\left(-\int p\,dx\right) = y_1y_2' - y_2y_1'$

$$\Rightarrow \frac{W}{y_1^2} = \frac{y_2'}{y_1} - \frac{y_2y_1'}{y_1^2} = \left(\frac{y_2}{y_1}\right)' \Rightarrow y_1\int \frac{W}{y_1^2}dt = y_2$$

This is the same formula as before!

<u>Non-homogeneous Eq:</u> $y'' + py' + qy = g$

Remarks, Notice that y_1 & y_2 that solve $y'' + py' + qy = 0$ also work above!

$\therefore$ Our solution must take the form $y(t) = \underbrace{Ay_1 + By_2}_{\text{homogeneous part}} + \underbrace{\tilde{y}}_{\text{non-homogeneous}}$

To solve this we have, The Method of Undetermined Coefficients! (i.e Guess the Ans)
Only works if

$$g(t) = \begin{cases} P_n(t) = a_nt^n + \cdots + a_1t + a_0 \\ P_n(t)e^{\alpha t} \\ P_n(t)e^{\alpha t}\begin{cases}\sin(\beta t) \\ \cos(\beta t)\end{cases} \end{cases}$$

Ex (3.5-#10) Solve $y'' + y = 3\sin 2t + t\cos 2t$

First we solve the homogeneous part: $y'' + y = 0 \Rightarrow y_{hom}(t) = A\sin t + B\cos t$

Now for the non-homogeneous part: We guess $\tilde{y} = (b_1 t + b_0)\sin(2t) + (a_1 t + a_0)\cos(2t)$

why just first power of t? Because only one is on the R.H.S.

$\tilde{y}' = b_1 \sin(2t) + 2b_1 t\cos(2t) + 2b_0\cos(2t) + a_1\cos(2t) - 2a_1 t\sin(2t) - 2a_0\sin(2t)$

$\tilde{y}'' = 2b_1\cos(2t) + 2b_1\cos(2t) - 4b_1 t\sin(2t) - 4b_0\sin(2t) - 2a_1\sin(2t) - 2a_1\sin(2t) - 4a_1 t\cos(2t) - 4a_0\cos(2t)$

$= \cos(2t)(4b_1 - 4a_0) + \sin(2t)(-4a_1 - 4b_0) + t\cos(2t)(-4a_1) + t\sin(2t)(-4b_1)$

$\therefore \tilde{y}'' + \tilde{y} = \cos(2t)(4b_1 - 3a_0) + \sin(2t)(-4a_1 - 3b_0) + t\cos(2t)(-3a_1) + t\sin(2t)(-3b_1)$

Now we just match & solve! By R.H.S we see that:

$4b_1 - 3a_0 = 0 \Rightarrow \frac{4}{3}b_1 = a_0$, $-4a_1 - 3b_0 = 3$, $-3a_1 = 1 \Rightarrow a_1 = -\frac{1}{3}$, $-3b_1 = 0 \Rightarrow b_1 = 0$

$\Rightarrow a_0 = 0$ $\frac{4}{3} - 3b_0 = 3 \Rightarrow b_0 = -\frac{5}{9}$ If we put it all together we get

$\therefore y(t) = A\sin t + B\cos t - \frac{t}{3}\cos(2t) - \frac{5}{9}\sin(2t)$

Now for the <u>Variation of Parameters</u>

Find the homogeneous solutions to $y'' + py' + qy = g$ then suppose

$$y(t) = u(t)y_1(t) + v(t)y_2(t)$$

If we restrict that $u'y_1 + v'y_2 = 0$ ① we obtain the following after subbing in:

$$u'y_1' + v'y_2' = g \quad ②$$

①&② form 2 eq, 2 unknowns.

We can solve to find

$$u' = -\frac{y_2 g}{W(y_1, y_2)} \quad \& \quad v' = \frac{y_1 g}{W(y_1, y_2)} \Rightarrow \begin{aligned} u(t) &= -\int \frac{y_2 g}{W}dt + A \\ v(t) &= \int \frac{y_1 g}{W}dt + B \end{aligned}$$

Ex(3.6-#10) Solve: $y'' - 2y' + y = \frac{e^{t}}{1+t^2}$

Note: $W = e^{2t}$

Solve hom part: $y'' - 2y' + y = 0 \Rightarrow y_{hom} = Ae^t + Bte^t$

By Above formula for non-hom part we have $u(t) = -\int \frac{t}{1+t^2}dt = -\frac{1}{2}\ln(1+t^2) + A$

$$v(t) = \int \frac{dt}{1+t^2} = \text{Arctan}(t) + B$$

$\Rightarrow y(t) = Ae^t + Bte^t - \frac{e^t}{2}\ln(1+t^2) + te^t\,\text{Arctan}(t)$