MAT-244: Tutorial 3 : C.J. Adkins

Chapter 2 cont.

Theorem: If p & q are continuous on (a,b) & $t_0 \in (a,b)$, we have

$$\begin{cases} y' + py = q \\ y(t_0) = y_0 \end{cases}$$ has a unique solution on (a,b)

Theorem: If f & $\frac{\partial f}{\partial y}$ are continuous in $(\alpha,\beta)\times(\gamma,\delta)$ & $(t_0,y_0)\in(\alpha,\beta)\times(\gamma,\delta)$,

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$$ has a unique solution on (t_0-h, t_0+h), with some $h>0$

Remark: Note the second case allows for "local" solutions for non-linear 1st order O.D.E

Ex: (2.4-6) Where do solutions exist? $(\ln(t))y' + y = \cot(t)$, $y(2)=3$

Rewrite in S.F: $y' + \frac{1}{\ln(t)}y = \frac{\cot(t)}{\ln(t)}$, notice we need $\boxed{t>1}$ & avoid "blow ups" of $\ln(t)$

$\cot(t) = \frac{\cos(t)}{\sin(t)}$ needs to be finite, so $\sin(t)\neq 0$ & t needs to reach $t=2$. Also, $\ln(t)\neq 0$

Our Interval $=(1,\pi)$ since $2\in(1,\pi)$ & $\sin(t)\neq 0$ for $t\in(0,\pi)$

$\therefore$ Our solution exists on $(1,\pi)$ ▼

Ex: (2.4-#12) Where do solutions exist? $\frac{dy}{dt} = \overbrace{\frac{y\cos(t)}{1+y}}^{f(t,y)}$

Where is f cont? its easy to see with $(y,t)\in[\mathbb{R}\setminus\{-1\}\times\mathbb{R}]$ $(-\infty,-1)\cup(-1,\infty)$

Where is $\frac{\partial f}{\partial y}$ cont? $\frac{\partial f}{\partial y} = \frac{\cos(t)}{1+y} - \frac{y\cos(t)}{(1+y)^2} = \frac{\cos(t)}{(1+y)^2}$, continuous in same region!

$\therefore$ Either solutions live in $(-\infty,-1)\times\mathbb{R}$ or $(-1,\infty)\times\mathbb{R}$ (Remember, just a smaller region) ▼

Ex: (2.4-#33-Discontinuous Coefficients) Solve:

$$\begin{cases} y' + p(t)y = 0 \\ y(0) = 1 \end{cases} \text{ where } p(t) = \begin{cases} 2 & t\in[0,1] \\ 1 & t>1 \end{cases}$$

2 cases, ① if $t\in[0,1] \Rightarrow y' = -2y \Leftrightarrow y(t) = C\exp(-2t)$, $y(0)=1 \Rightarrow C=1$

② if $t>1 \Rightarrow y' = -y \Leftrightarrow y(t) = \tilde{C}\exp(-t)$, now we pick $\tilde{C}$ s.t y is cont on $\mathbb{R}_{\geq 0}$

We need to choose $\tilde{C} = 1/e$

$$\therefore\ y(t) = \begin{cases} \exp(-2t) & t\in[0,1] \\ \exp(-(t+1)) & t>0 \end{cases}$$ This solves the O.D.E

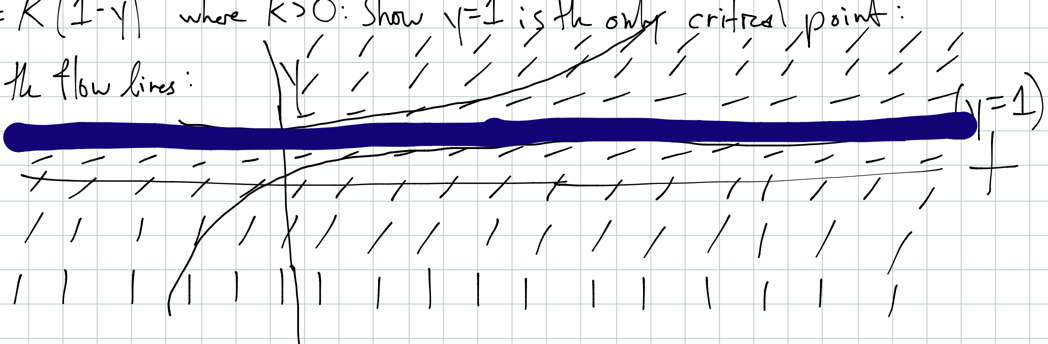
$\boxed{\text{Remark: It's impossible to make } y' \text{ cont}}$

Autonomous Equations:

Ex (2.5-#7 · Semistable Equilibrium Solutions)

a) $\frac{dy}{dt} = K(1-y)^2$ where $K>0$: Show $y=1$ is the only critical point:

Lets draw the flow lines:



Notice $y=1 \Rightarrow \frac{dy}{dt}=0$ (flat), if $y\neq 0$, $\frac{dy}{dt}>0$ (+ slope)

b) ▬ is the Equilibrium, notice that only solutions below the line will become stable.

c) lets solve the O.D.E:

$$\frac{dy}{dt} = k(1-y)^2 \iff \int \frac{dy}{(1-y)^2} = \int k\,dt \iff \frac{1}{1-y} = Kt + C$$

$$y(0)=y_0 \Rightarrow C = \frac{1}{1-y_0} \quad \therefore\ 1 + \frac{y_0-1}{(1-y_0)Kt+1} = y(t)$$

Ex (2.5-#17) Solve Grompertz Eq: $y' = ry\ln\left(\frac{K}{y}\right)$ s.t $y(0)=y_0$

The Eq is separable so... $\int \frac{dy}{y\ln(K/y)} = \int r\,dt \iff \int \frac{du}{u} = -rt + C \iff \ln\left(\ln\left(\frac{K}{y}\right)\right) = -rt + C$

let $u = \ln\left(\frac{K}{y}\right)$
$du = -\frac{dy}{y}$

$\iff \ln\left(\frac{K}{y}\right) = Ce^{-rt} \iff \frac{K}{\exp(Ce^{-rt})} = y(t)$, $y(0)=y_0 \Rightarrow C = \ln(K/y_0) \Rightarrow y(t) = K\exp\left(\ln\left(\frac{y_0}{K}\right)e^{-rt}\right)$

b) If $r = 0.71\,\frac{1}{\text{years}}$, $K = 80.5\times10^6$ kg, $y_0/K = 0.25$, what is $y(2)$? We plug in the numbers:

$y(2) = 80.5\times10^6\left(\exp\left(-2\ln(2)\,e^{-1.42}\right)\right) = \#$

c) Find τ s.t $y(\tau)=\frac{3}{4}K$ -

$$y(\tau)=\frac{3}{4}K=K\exp\left(\ln\left(\frac{y_0}{K}\right)e^{-r\tau}\right) \Leftrightarrow \ln\left(\frac{3}{4}\right)=\ln\left(\frac{y_0}{K}\right)e^{-r\tau} \Leftrightarrow \ln\left(\frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{y_0}{K}\right)}\right)\left(\frac{-1}{r}\right)=\tau$$

Ex: (2.8 #14) Consider $\phi_n(x)=2nxe^{-nx^2}$, $0\le x\le 1$

a) Show $\lim_{n\to\infty}\phi_n(x)=0\ \forall x\in[0,1]$: $\lim_{n\to\infty}\frac{2nx}{e^{nx^2}} \overset{\text{L'hopital}}{=} \lim_{n\to\infty}\frac{2x}{x^2e^{nx^2}}=0$ since $\lim_{n\to\infty} e^{n}\to\infty$

Therefore we have that $\int_0^1 \lim_{n\to\infty}\phi_n(x)\,dx=0$

b) Compute $\int_0^1 \phi_n(x)$: $\int_0^1 2nxe^{-nx^2}dx=\int_0^1 ne^{-nu}du=e^{-nu}\Big|_1^0=1-e^{-n}$

$u=x^2$, $du=2x\,dx$

Notice that $\lim_{n\to\infty}\int_0^1\phi_n(x)\,dx=\lim_{n\to\infty}(1-e^{-n})=\underline{1}\neq\int_0^1\lim_{n\to\infty}\phi_n(x)\,dx$

A Quick Review of the Picard-Lindelöf Theorem.

How to prove $y'=f(t,y)$, $y(t_0)=y_0$ has a solution? Integrate!

$\Rightarrow y(t)=y(t_0)+\int_{t_0}^{t}f(s,y(s))\,ds$ by FTC (this is a fixed point eq)

So we take an approximating sequence $\{\varphi_K\}$, where

$$\varphi_0(t)=y_0\ \&\ \varphi_{K+1}=y_0+\int_{t_0}^{t}f(s,\varphi_K(s))\,ds$$

By the Banach fixed point theorem we can show $\varphi_K\to y(t)$, as $K\to\infty$.

Note: We need f Lipschitz or f_y continuous.

Questions & Quiz

Note: take up Question #5 from Assignment 1