Tutorial 11 - MAT244 - C.J. Adkins

Series Solutions

We're back to O.D.E's in the form

$$y''+py'+qy=0 \quad \text{or} \quad P(x)y''+Q(x)y'+R(x)y=0$$

Terminology: If we have no blow ups, i.e $p(x)=\frac{Q(x)}{P(x)}$ & $q=\frac{R(x)}{P(x)}$ are continuous, then we say any point around x_0 is <u>ordinary</u>

If they are not continuous, i.e $P(x_0)=0$, we say it's a <u>singular point</u>

What are we going to try... series solutions about ordinary points:

$$y=\sum_{n=0}^{\infty} a_n(x-x_0)^n$$

Ex: (5.2-#3) Find the series that solves $y''-xy'-y=0$ about $x_0=1$

① Suppose $y(x)=\sum_{n=0}^{\infty} a_n(x-1)^n$

Aside: $\sum_{n=0}^{\infty} a_{n+1}(n+1)(x-1)^{n+1} = \sum_{n=1}^{\infty} a_n(n)(x-1)^n$

② Plug it into the equation:

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x-1)^n - x\sum_{n=0}^{\infty} a_{n+1}(n+1)(x-1)^n - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

Write x as $1+(x-1)$, thus

$$\sum_{n=1}^{\infty}\left[a_{n+2}(n+2)(n+1) - a_{n+1}(n+1) - a_n n - a_n\right](x-1)^n + 2a_2 - a_1 - a_0 = 0$$

③ Thus: $a_2=\frac{a_1+a_0}{2}$ & $a_{n+2}=\frac{a_{n+1}+a_n}{(n+2)}$ ⇐ This is called the <u>recurrence relation</u>

Now we need to find the pattern to rewrite "y"

④ i.e. $a_3=\frac{a_2+a_1}{3}=\frac{a_1+a_0}{6}+\frac{a_1}{3}=\frac{a_1}{2}+\frac{a_0}{6}$, $a_4=\frac{a_3+a_2}{4}=\frac{a_1}{8}+\frac{a_0}{24}+\frac{a_1}{8}+\frac{a_0}{8}=\frac{a_1}{4}+\frac{a_0}{6}$

$$\therefore y(x)=a_0+a_1(x-1)+\left(\frac{a_1}{2}+\frac{a_0}{2}\right)(x-1)^2+\left(\frac{a_1}{2}+\frac{a_0}{6}\right)(x-1)^3+\left(\frac{a_1}{4}+\frac{a_0}{6}\right)(x-1)^4+\cdots$$

⑤ $\Rightarrow y^{(1)}=1+\frac{1}{2}(x-1)^2+\frac{1}{6}(x-1)^3+\frac{1}{6}(x-1)^4+\cdots$ ← terms with a_0

$y^{(2)}=(x-1)+\frac{1}{2}(x-1)^2+\frac{1}{2}(x-1)^3+\frac{1}{4}(x-1)^4+\cdots$ ← terms with a_1

$\boxed{\text{Remark: } a_0, a_1 \in \mathbb{R} \text{ are the constants for the fundamental solutions}}$

No nice closed form :(

Ex(5.2-#10) Find the series solution for: $(4-x^2)y''+2y=0$, $x_0=0$

① Suppose $y(x)=\sum_{n=0}^{\infty} a_n x^n$

② Plug in: $\sum_{n=0}^{\infty}\left[4a_{n+2}(n+2)(n+1)x^n - a_{n+2}(n+2)(n+1)x^{n+2}\right] + \sum_{n=0}^{\infty} 2a_n x^n = 0$

③ $\Rightarrow 4a_{n+4}(n+4)(n+3) - a_{n+2}(n+2)(n+1) + 2a_{n+2} = 0$, $a_2 = -\frac{1}{4}a_0$, $a_3 = -\frac{1}{12}a_1$

$$\Rightarrow a_{n+2} = \frac{n(n-1) - 2}{4(n+2)(n+1)} a_n$$

④ Find the pattern (if any ☺)

$a_4 = 0$, $a_5 = \frac{a_3}{20} = -\frac{1}{20(12)}a_1$, $a_6 = 0$, $a_7 = \frac{3}{28}a_5 = -\frac{3}{28(20)(12)}a_1 \ldots$

$\Rightarrow y = a_0 + a_1 x - \frac{1}{4}a_0 x^2 - \frac{1}{12}a_1 x^3 + 0 - \frac{1}{20(12)}a_1 x^5 - \frac{3}{28(20)(12)}a_1 x^7 - \ldots$

⑤ $\therefore y_1 = 1 - \frac{1}{4}x^2$ ← a_0 terms

$y_2 = x - \frac{x^3}{12} - \frac{x^5}{20(12)} - \frac{3x^7}{28(20)(12)} - \ldots = x - \sum_{n=1}^{\infty}\frac{x^{2n+1}}{(2n+1)!!\,4^n}$ ← a_1 terms

← Just odd terms
exmp: $7!! = 7\cdot 5\cdot 3\cdot 1$

There is another method for this, namely Taylor Series.

Suppose $y'' + py' + qy = 0$ has $y = \sum_{n=0}^{\infty}\frac{y^{(n)}(x_0)}{n!}(x-x_0)^n$ ← call this a_n

Notice from the O.D.E, we have:

$$y''(x_0) = -p(x_0)y'(x_0) - q(x_0)y(x_0), \text{ i.e } a_2 = \frac{1}{2!}\left(-p(x_0)a_1 - q(x_0)a_0\right)$$

We can find the 3rd via differentiation, i.e

$$y''' = -py'' - (p'+q)y' - q'y \Rightarrow a_3 = \frac{1}{3!}\left(-2p(x_0)a_2 - (p'(x_0) + q(x_0))a_1 - q'(x_0)a_0\right)$$

.... What's the punch line...

__Theorem__: $y'' + py' + qy = 0$ has a solution $y = \sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 y_1(x) + a_1 y_2(x)$ if p, q are analytic at x_0 (i.e have derivatives around x_0). The radius of convergence is at least as large as p or q's.

Ex: (5.3-#2) Find the first 4 terms of the series solution of:

$$y'' + \sin x\, y' + \cos x\, y = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad x_0 = 0$$

Note, $y(0) = a_0 = 0$, $y'(0) = a_1 = 1$, from before

$a_2 = \frac{1}{2}\left(-p(0)a_1 - q(0)a_0\right) = 0$, $a_3 = \frac{1}{3!}\left(-2\cancel{p(0)a_2} - (p'(0) + q(0))a_1 - \cancel{q'(0)a_0}\right) = -\frac{1}{3}$

4th term, $y'''' = \left(-py''' - (2p'+q)y'' - (p''+2q')y' - q''y\right) \Rightarrow a_4 = \frac{1}{4!}\left(-0 - 0 - 0 - 0\right) = 0$

$\therefore\ y(x) = x - \frac{x^3}{3} + \cdots$ ∎

Ex(5.3-#8) Find the lower bound of the radius of convergence for $xy'' + y = 0,\ x_0 = 1$

$\Rightarrow y'' + \frac{1}{x} y = 0$, need to look at radius for $\frac{1}{x}$ at $x_0 = 1$

Well, $x = 0$ is singular, thus $R \geq |0-1| = 1$ ∎

Ex(5.3-#15) Let x & x^2 be solutions to $P(x)y'' + Q(x)y' + R(x)y = 0$. Can we say whether the point $x = 0$ is ordinary or singular. Prove the Answer.

Well, let $y = Ay_1 + By_2 = Ax + Bx^2 \Rightarrow y' = A + 2Bx$, $y'' = 2B$, thus

$2BP(x) + Q(x)(A + 2Bx) + R(x)(Ax + Bx^2) = 0$ for x about 0.

$\Rightarrow B(2P(x) + 2xQ(x) + x^2R(x)) = 0$

$A(Q(x) + R(x)x) = 0$

$\Rightarrow Q(x) = -R(x)x \Rightarrow 2P - 2x^2R + x^2R = 2P - x^2R = 0 \Leftrightarrow \frac{x^2R}{2} = P$

$\therefore$ The O.D.E takes the form:

$R(x)\left[\frac{x^2}{2}y'' - xy' + y\right] = 0 \Rightarrow y'' - \frac{2}{x}y' + \frac{2}{x^2} = 0 \Rightarrow 0$ is singular ∎