MAT237 - Tutorial 19 - 4 August 2015

1 Coverage

Divergence theorem and Stokes' theorem.

2 Problems

I suggest the following problems.

I think these will take up the whole tutorial, but if they don't you should feel free to augment them with another problem, likely just a computation.

The first problem here is not on the list, and is an example of applying Stokes' Theorem to a surface integral where you don't know it's a curl yet, which I like.

I don't expect anyone to easily get the answer to 3(c), but it's interesting to think about. I'll comment more below.

1. Let S be the part of the paraboloid defined by $z = x^2 + y^2$ satisfying $z \leq 4$, oriented so that normal vectors point upwards. Let $\mathbf{G}(x, y, z) = (-3xz^2, 0, z^3)$. Compute $\int\int_S \mathbf{G} \cdot \hat{\mathbf{n}}\, dA$.

2. (BL 13.4.8, with some extra stuff) Let $\mathbf{F} : \mathbb{R}^3 \setminus \{\mathbf{0}\} \to \mathbb{R}^3$ be the vector field

$$\mathbf{F}(x, y, z) = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}.$$

   (a) Show that $\nabla \cdot \mathbf{F} = 0$.

   (b) Let S_r be the sphere of radius $r > 0$ centred at the origin in $\mathbb{R}^3$. Compute $\int\int_{S_r} \mathbf{F} \cdot \hat{\mathbf{n}}\, dA$.

   (c) Compare your result with in (b) with your result in (a). Why does this not contradict the Divergence Theorem?

   (d) Show that $\mathbf{F}$ cannot be expressed as $\mathbf{F} = \nabla \times \mathbf{G}$ for any C^1 vector field $\mathbf{G} : \mathbb{R}^3 \setminus \{\mathbf{0}\} \to \mathbb{R}^3$.

3. (BL 13.5.5, with some extra stuff)

   (a) Let S_1 and S_2 be smooth surfaces in $\mathbb{R}^3$ whose boundaries, and the Stokes' orientations on them, coincide. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1 vector field. Show that

$$\int\int_{S_1} (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dA = \int\int_{S_2} (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dA.$$

   (b) Let S be an oriented smooth surface in $\mathbb{R}^3$ whose boundary is the unit circle in the xy-plane, and let $\mathbf{F}$ be a C^1 vector field on $\mathbb{R}^3$ whose curl has zero z-component. Show that

$$\int\int_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dA = 0.$$

(c) Is the result in part (a) still true if we replace $\nabla \times \mathbf{F}$ with an arbitrary C^1 vector field $\mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3$? What if we add some assumptions about the domain of the vector field?

3 Solutions and Comments

1. ***Solution***: We could do this directly, but that seems annoying. Instead we would like to use Stokes' theorem. To do this, we need to express $\mathbf{G}$ as the curl of a vector field. Given a vector field $\mathbf{F} = (F_1, F_2, F_3)$, we have that it's curl is:

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right).$$

For us, this means that we would like:

$$\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = -3xz^2, \quad \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} = 0, \quad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = z^3.$$

There are likely many ways to do this, but one particularly simple solution that jumps out at the reader is $\mathbf{F}(x, y, z) = (0, xz^3, 0)$. This is a C^1 vector field, and so we can apply Stokes' theorem:

$$\int\int_S \mathbf{G} \cdot \hat{\mathbf{n}}\, dA = \int\int_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dA = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{x}.$$

In our case, ∂S is the circle of radius 2 centred on the z-axis, living in the plane $z = 4$. We can parameterize this by $(2\cos(t), 2\sin(t), 4)$, and can therefore easily compute:

$$\begin{aligned} \oint_{\partial S} \mathbf{F} \cdot d\mathbf{x} &= \oint_{\partial S} xz^3\, dy \\ &= \int_0^{2\pi} (2\cos(t))(4)^3(2\cos(t)\, dt) \\ &= 256 \int_0^{2\pi} \cos^2(t)\, dt \\ &= 256\pi. \end{aligned}$$

2. ***Solution***: (a) A routine computation shows that

$$\frac{\partial F_1}{\partial x} = \frac{-2x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^{5/2}},$$

from which one can easily conclude from the symmetry of the function that $\nabla \cdot \mathbf{F}$ will equal 0.

(b) It is natural to work in spherical coordinates for this problem.

In this case we see that $\mathbf{F}$ always points in the radial direction, and so on S_r we have $\mathbf{F} \cdot \hat{\mathbf{n}} = \|F\| = \frac{1}{r^2}$. Therefore we can easily compute:

$$\int\int_{S_r} \mathbf{F} \cdot \hat{\mathbf{n}}\, dA = \int_0^{2\pi} \int_0^{\pi} \frac{1}{r^2} r^2 \sin(\phi)\, d\phi d\theta = \int_0^{2\pi} \int_0^{\pi} \sin(\phi)\, d\phi d\theta = 4\pi.$$

(c) This doesn't contradict the Divergence theorem because our vector field is not defined on the whole interior of S_r.

(d) If it could be expressed this way, we could apply Stokes' theorem to it, which would tell us:

$$\int\int_{S_r} F\cdot\hat{\mathbf{n}}\,dA = \int\int_{S_r}(\nabla\times\mathbf{G})\cdot\hat{\mathbf{n}}\,dA = \oint_{\partial S_r}\mathbf{G}\cdot d\mathbf{x}.$$

The last line integral above equals zero, since $\partial S_r = \emptyset$, contradicting our result in part (c).

3. ***Solution***: (a) This is immediate from Stokes' Theorem.

(b) This is a direct consequence of (a). Fix such an S and let D be the unit disk on the xy-plane in $\mathbb{R}^3$, given the Stokes' orientation such that the orientation on ∂D agrees with the orientation on ∂S. Then by (a) we have:

$$\int\int_S(\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}\,dA = 0 = \int\int_D(\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}\,dA = 0,$$

and the latter integral is zero, since the normal to D points directly up or down and by assumption $\nabla\times\mathbf{F}$ has zero z-component.

(c) The result in (a) is *not* true for an arbitrary vector field. Stoke's Theorem says that it's true for any vector field of the form $\mathbf{G} = \nabla\times\mathbf{F}$, so an equivalent question is: Can every C^1 vector field be expressed as the curl of another vector field?

The answer to this is no. It turns out that much more annoying version of the same argument in the proof of the Poincare lemma shows that a vector field defined on a star-shaped set can be expressed this way if and only if it is divergence-free. (We've already seen this condition fail on a set that isn't star-shaped in problem 2(d) above.) So, the result is true for an arbitrary divergence-free vector field on a star-shaped set.

We don't need to prove this though. We can however give a very simple counterexample to the result in (a) for a vector field with nonzero divergence. Let $\mathbf{G}(z,y,z) = (0,0,z)$, and S be the unit cube in $\mathbb{R}^3$. Write $S = S_1\cup S_2$, where S_1 is the unit square in the xy-plane (the bottom of S, and S_2 is the rest of S. Then $\partial S_1 = \partial S_2 =$ the unit square in the xy-plane, and we can easily calculate:

$$\int\int_{S_1}\mathbf{G}\cdot\hat{\mathbf{n}}\,dA = 0 \text{ and } \int\int_{S_2}\mathbf{G}\cdot\hat{\mathbf{n}}\,dA = 1.$$

Comments: Just regarding (c). I like this problem because it's the other "half" of the Poincare lemma in this context. They know the Poincare lemma in the form about showing curl-free vector fields can be expressed as gradients, but another consequence of the general lemma about differential forms (which they don't see, of course) is that divergence-free vector fields can be expressed as curls.

I'm sure that Tyler has and still plans to try to make the students understand that all of these integral theorems are "shadows" of a grand framework involving exterior derivatives and such, and this aspect of the Poincare lemma fits into that design nicely

A direct proof of this fact in this context can be done just like Tyler's proof of the version I mentioned, but it's much more annoying. The idea is that given a divergence-free vector field $\mathbf{F}$ on a star-shaped set S, you define a new vector field $\mathbf{G}$ by

$$(\mathbf{G}(\mathbf{x}))_i = \int_{L_x} (\mathbf{x} \times \mathbf{F}(\mathbf{x}))_i \, ds,$$

where L_x is the line joining the centre of the star-shaped set to $\mathbf{x}$. That's not so hard to write down, but verifying that the curl of this vector field equals $\mathbf{F}$ is a nightmare.