MAT237 - Tutorial 15 - 21 July 2015

1 Coverage

Iterated integrals.

2 Problems

I suggest the following problems. Like last time, I won't have too much to comment about any of these, since they're all computational. Not all of these are from the big list either, since I found that section somewhat lacking in different sorts of examples.

1. Let R be the region to the left of the y-axis bounded between the curves $x = 1 - y^2$ and $x = 8(1 - y^2)$. Compute $\int\int_R \frac{y^2}{x}\, dA$.

2. (BL 12.4.6) Let $a, b, c > 0$, and let R be the region bounded inside the ellipse $a^2x^2 + b^2y^2 = c^2$. Compute $\int\int_R x^2\, dA$.

3. (BL 12.4.8 (b)) (I think this problem is more interesting without part (a) giving it away.) Compute $\int_{\mathbb{R}} e^{-x^2}\, dx$.

   (Hint: Consider the function $e^{-x^2-y^2}$ on $\mathbb{R}^2$.)

4. Let R be the region enclosed by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 2 + y$. Evaluate $\int\int\int_R 4y\, dV$.

3 Solutions and Comments

1. ***Solution***: The region in question is the space between two parabolas, which intersect at the points $(0, \pm 1)$. The most natural choice of coordinate transformation to me seems to be the one given by $(x, y) = (v(1 - u^2), u)$, since for a given fixed v you get exactly all the intermediary parabolas. u is bounded between -1 and 1, and v between 1 and 8.

   The Jacobian of this transformation is:

   $$\left|\det \begin{pmatrix} -2uv & 1-u^2 \\ 1 & 0 \end{pmatrix}\right| = |1 - u^2| = 1 - u^2$$

where the last equality is because $-1 \leq u \leq 1$ in R. We can now compute:

$$\begin{aligned}
\int\int_R \frac{y^2}{x}\, dA &= \int_1^8 \int_{-1}^1 \frac{u^2}{v(1-u^2)}(1-u^2)\, dudv \\
&= \int_1^8 \int_{-1}^1 \frac{u^2}{v}\, dudv \\
&= \frac{2}{3}\int_1^8 \frac{1}{v}\, dv \\
&= \frac{2}{3}\left[\log|v|\right]_1^8 = 2\log(2).
\end{aligned}$$

2. ***Solution***: First, let's put the equation of our ellipse into standard form:

$$\frac{a^2}{c^2}x^2 + \frac{b^2}{c^2}y^2 = 1.$$

This now looks like the equation of a circle of radius 1 in coordinates $(u, v) = (\frac{c}{a}x, \frac{c}{b}y)$, which we can then model with polar coordinates as usual. Rather than do two coordinate transformations, we combine them into one:

$$(x, y) = \left(\frac{c}{a}r\cos(\theta), \frac{c}{b}r\sin(\theta)\right),$$

where as we would expect $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 1$. The Jacobian of this transformation is:

$$\left|\det\begin{pmatrix} \frac{c}{a}\cos(\theta) & -\frac{c}{a}r\sin(\theta) \\ \frac{c}{b}r\sin(\theta) & \frac{c}{b}r\cos(\theta)\end{pmatrix}\right| = \left|\frac{c^2}{ab}r^2\cos^2(\theta) + \frac{c^2}{ab}r^2\sin^2(\theta)\right| = \frac{c^2}{ab}r^2.$$

We are now ready to compute:

$$\begin{aligned}
\int\int_R x^2\, dA &= \int_0^{2\pi}\int_0^1 \left(\frac{c}{a}r\cos(\theta)\right)^2\left(\frac{c^2}{ab}r^2\right)\, drd\theta \\
&= \frac{c^4}{a^2b}\int_0^{2\pi}\int_0^1 r^3\cos^2(\theta)\, drd\theta \\
&= \frac{c^4}{4a^2b}\int_0^{2\pi}\cos^2(\theta)\, d\theta \\
&= \frac{\pi c^4}{4a^3b}.
\end{aligned}$$

3. ***Solution***: As we know, the function e^{-x^2} has no elementary antiderivative, so there's no way to compute this integral directly in $\mathbb{R}$, so it's not the first year calculus question it appears to be... Instead, we consider the hint and think about integrating $e^{-x^2-y^2}$ in $\mathbb{R}^2$. First, why is this helpful? Well:

$$\int\int_{\mathbb{R}^2} e^{-x^2-y^2}\, dA = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-x^2}e^{-y^2}\, dydx = \left(\int_{\mathbb{R}} e^{-x^2}\, dx\right)\left(\int_{\mathbb{R}} e^{-y^2}\, dy\right) = \left(\int_{\mathbb{R}} e^{-x^2}\, dx\right)^2.$$

So we see that the number we're looking for is the square root of $\int\int_{\mathbb{R}^2} e^{-x^2-y^2}\, dA$. So let's compute that!

The most natural choice of coordinates here are polar coordinates, since in these coordinates the exponent of our integrand is very simple. So using the usual $(x, y) = (r\cos(\theta), r\sin(\theta))$, we get:

$$\int\int_{\mathbb{R}^2} e^{-x^2-y^2}\, dA = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\, dr d\theta$$

which is now very much tractable. Indeed:

$$\begin{aligned}\int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\, dr d\theta &= \frac{1}{2}\int_0^{2\pi}\left[-e^{-r^2}\right]_0^{\infty} d\theta \\ &= \frac{1}{2}\int_0^{2\pi} d\theta = \pi.\end{aligned}$$

From this we conclude the wonderful formula:

$$\int_{\mathbb{R}} e^{-x^2}\, dx = \sqrt{\pi},$$

4. ***Solution***: The region R can be described as:

$$R = \left\{ (x, y, z) \,:\, 0 \leq x^2 + y^2 \leq 1,\, 0 \leq z \leq 2 + y \right\}.$$

Cylindrical coordinates are the most natural ones for the job: $(x, y, z) = (r\cos(\theta), r\sin(\theta), z)$. The bounds on x and y in the description of R above are very natural to express in these coordinates, so it remains only to express the bounds on z. This is simple enough:

$$0 \leq z \leq 2 + y \Rightarrow 0 \leq z \leq 2 + r\sin(\theta).$$

Now we can simply compute:

$$\begin{aligned}\int\int\int_R 4y\, dV &= 4\int_0^{2\pi}\int_0^1\int_0^{2+r\sin(\theta)} (r\sin(\theta))\, r\, dz dr d\theta \\ &= 4\int_0^{2\pi}\int_0^1 r^2\sin(\theta)(2 + r\sin(\theta))\, dr d\theta \\ &= 8\int_0^{2\pi}\int_0^1 r^2\sin(\theta)\, dr d\theta + 4\int_0^{2\pi}\int_0^1 r^3\sin^2(\theta)\, dr d\theta \\ &= \frac{8}{3}\int_0^{2\pi}\sin(\theta)\, d\theta + \int_0^{2\pi}\sin^2(\theta)\, d\theta \\ &= 0 + \pi = \pi.\end{aligned}$$