MAT237 - Tutorial 15 - 16 July 2015

1 Coverage

Iterated integrals.

2 Problems

I suggest the following problems. I won't have too much to comment about any of these, since they're all computational. For all of them, the only hard/interesting work comes in visualizing the regions over which we integrate.

1. (BL 12.3.10 (b)) Determine the integral of $f(x, y, z) = z$ over the region in $\mathbb{R}^3$ in the first octant bounded by the surfaces $x = 0$, $z = 0$, $y = 3x$, and $y^2 + z^2 = 9$.
2. (BL 12.3.11 (a)) Compute $\int\int_R \left[2 + x^2y^3 - y^2\sin(x)\right] dA$ where $R = \left\{ (x, y) \in \mathbb{R}^2 \,:\, |x| + |y| \leq 1 \right\}$.
3. (BL 12.3.11 (b)) Compute $\int\int_R \left[ax^2 + by^3 + \sqrt{a^2 - x^2}\right] dA$ where $R = \left\{ (x, y) \in \mathbb{R}^2 \,:\, |x| \leq a, |y| \leq b \right\}$.
4. (BL 12.3.12 (c)) Determine the integral of $f(x, y, z) = z$ over the region in the portion of the sphere $x^2 + y^2 + z^2 \leq 4$ in the first octant.

3 Solutions and Comments

1. ***Solution***: Since we're only in the first octant, we only need to worry about positive values of z, and so we can rearrange the equation of the cylinder into a more useful form without losing anything important: $z = \sqrt{9 - y^2}$. Then, the region over which we want to integrate can be described as:

$$S = \left\{ (x, y, z) \,:\, 0 \leq y \leq 3, 0 \leq x \leq \tfrac{y}{3}, 0 \leq z \leq \sqrt{9 - y^2} \right\}$$

or equivalently

$$S = \left\{ (x, y, z) \,:\, 0 \leq x \leq 1, 3x \leq y \leq 3, 0 \leq z \leq \sqrt{9 - y^2} \right\}$$

In the first case, we integrate to find:

$$\begin{aligned} \int_0^3 \int_0^{\frac{y}{3}} \int_0^{\sqrt{9-y^2}} z \, dz dx dy &= \frac{1}{2} \int_0^3 \int_0^{\frac{y}{3}} 9 - y^2 \, dx dy \\ &= \frac{1}{6} \int_0^3 9y - y^3 \, dy \\ &= \frac{1}{6} \left[\tfrac{9}{2}y^2 - \tfrac{1}{4}y^4\right]_0^3 \\ &= \frac{1}{6} \left(\frac{81}{2} - \frac{81}{4}\right) = \frac{27}{8} \end{aligned}$$

In the second case, we integrate to find:

$$\begin{aligned}
\int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z \, dz dy dx &= \frac{1}{2} \int_0^1 \int_{3x}^3 9 - y^2 \, dy dx \\
&= \frac{1}{2} \int_0^1 \left[9y - \tfrac{1}{3} y^3 \right]_{3x}^3 dx \\
&= \frac{1}{2} \int_0^1 27 - 9 - 27x + 9x^3 \, dx \\
&= \frac{1}{2} \left[18x - \tfrac{27}{2} x^2 + \tfrac{9}{4} x^4 \right]_0^1 \\
&= \frac{27}{8}
\end{aligned}$$

2. ***Solution***: The region $R = \left\{ (x,y) \in \mathbb{R}^2 \, : \, |x| + |y| \leq 1 \right\}$ looks like a diamond, bounded by the straight lines $y = x \pm 1$ and $y = -x \pm 1$. (Alternatively, it's the shape made by joining the four points $(\pm 1, 0)$ and $(0, \pm 1)$.)

   Breaking our integral up into three pieces, we see that we have to evaluate

   $$\int \int_R 2 \, dA + \int \int_R x^2 y^3 \, dA - \int \int_R y^2 \sin(x) \, dA$$

   For the first of these, the answer is simply twice the area of the region R. Since R is a rotated square with side length $\sqrt{2}$, this is easily seen to equal 4.

   For the second one, note that the function $x^2 y^3$ is odd in y. Since our region is symmetric about the x-axis, this integral will be zero.

   Similarly for the third one, note that the function $y^2 \sin(x)$ is odd in x. Since our region is symmetric about the y-axis, this integral will also be zero.

   So, the answer is 4.

3. ***Solution***: In this problem, the region $R = \left\{ (x,y) \in \mathbb{R}^2 \, : \, |x| \leq a, |y| \leq b \right\}$ is simply a rectangle, centred at the origin with side lengths $2a$ and $2b$. Like the last problem, there is some symmetry to exploit. Breaking our integral into pieces, we have:

   $$a \int \int_R x^2 \, dA + b \int \int_R y^3 \, dA + \int \int_R \sqrt{a^2 - x^2} \, dA.$$

   Immediately we notice that the second of these will be zero, since it's odd in y, and we will be integrating it from $y = -b$ to $y = b$ along a range of x.

   The remaining two are even in x, so we only need to compute the integrals on the part of

R to the right of the y-axis, for example, and double the result. We compute separately:

$$\begin{aligned}
a\int\int_R x^2\,dA &= a\int_{-b}^{b}\int_{-a}^{a} x^2\,dxdy \\
&= 2a\int_{-b}^{b}\int_{0}^{a} x^2\,dxdy \\
&= \frac{2a}{3}\int_{-b}^{b} a^3\,dy \\
&= \frac{4}{3}a^4 b
\end{aligned}$$

To solve the second integral we make the natural substitution $x = a\sin(t)$, and again exploit evenness to make our lives slightly easier.

$$\begin{aligned}
\int\int_R \sqrt{a^2-x^2}\,dA &= 2\int_0^a\int_{-b}^{b}\sqrt{a^2-x^2}\,dydx \\
&= 4b\int_0^a \sqrt{a^2-x^2}dx \\
&= 4b\int_0^{\arcsin(1)}\sqrt{a^2-a^2\sin^2(t)}\;(a\cos(t))\,dt \\
&= 4a^2b\int_0^{\frac{\pi}{2}}\cos^2(t)\,dt \\
&= 2a^2b\int_0^{\frac{\pi}{2}} 1+\cos(2t)\,dt \\
&= 2a^2b\left[t+\tfrac{1}{2}\sin(2t)\right]_0^{\frac{\pi}{2}} = \pi a^2 b
\end{aligned}$$

So putting it all together, our answer is $a^2b\left(\frac{4}{3}a^2+\pi\right)$.

4. ***Solution***: This will not be fun. It will serve as a reminder of how useful changes of coordinates are for later.

   The region in question is an 8th of a sphere of radius 2. It's obviously symmetric in its variables, so there's no preferred order of the variables when integrating. One way to express this region is as the set of points (x, y, z) such that:

$$\begin{aligned}
0 \le x \le 2 \\
0 \le y \le \sqrt{4-x^2} \\
0 \le z \le \sqrt{4-x^2-y^2}
\end{aligned}$$

   Accordingly, we must compute the following mess:

$$\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} z\,dzdydx$$

We get:

$$\begin{aligned}
\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} z \, dz dy dx &= \frac{1}{2} \int_0^2 \int_0^{\sqrt{4-x^2}} 4 - x^2 - y^2 \, dy dx \\
&= \frac{1}{2} \int_0^2 \left[(4-x^2)y - \frac{1}{3} y^3 \right]_0^{\sqrt{4-x^2}} dx \\
&= \frac{1}{3} \int_0^2 (4-x^2)^{3/2} \, dx \\
&= \frac{1}{3} \int_0^{\frac{\pi}{2}} (4 - 4\sin^2(t))^{3/2} \, (2\cos(t)) dt \\
&= \frac{16}{3} \int_0^{\frac{\pi}{2}} \cos^4(t) \, dt \\
&= \frac{8}{3} \int_0^{\frac{\pi}{2}} (1 + \cos(2t))^2 \, dt \\
&= \frac{8}{3} \int_0^{\frac{\pi}{2}} 1 + 2\cos(2t) + \frac{1}{2}(1 + \cos(4t)) \, dt \\
&= \frac{8}{3} \left[\frac{3}{2} + \sin(2t) + \frac{1}{8} \sin(4t) \right]_0^{\frac{\pi}{2}} \\
&= \frac{8}{3} \frac{3\pi}{4} = \frac{\pi}{2}
\end{aligned}$$