MAT237 - Tutorial 3 - 28 May 2015

1 Coverage

Continuity: They now know everything they need from this section of the notes and the Big List, including about uniform continuity.
Compactness: They've covered this whole section in the notes.

2 Problems

I suggest the following problems. Some are from the Big List, some not. I'll give you a quick idea of the solution if it seems necessary, and discuss what's worth stressing or notable connections to other things.

1. (BL 4.5, plus the counterexample.) Let $S \subseteq \mathbb{R}^n$, let $f : S \to \mathbb{R}^m$ be a uniformly continuous function, and let $\{x_k\} \subseteq S$ be a Cauchy sequence. Show that $\{f(x_k)\}$ is a Cauchy sequence. Give an example to show that this need not be true if f is just continuous.

2. (BL 4.6, corrected to be possible, plus the counterexample.) Let $S \subseteq \mathbb{R}^n$ and let $f : S \to \mathbb{R}$ be a uniformly continuous function. Show there exists a unique continuous function $\bar{f} : \bar{S} \to \mathbb{R}$ extending f. Give an example to show that this need not be possible if f is not uniformly continuous. Even continuous and bounded is not enough.

3. A subset $S \subseteq \mathbb{R}^n$ is called *discrete* if for every $x \in S$, there is an $\epsilon > 0$ such that $B(\epsilon, x)$ contains no other points from S. Show that a discrete set S is compact if and only if it's finite.

3 Solutions and Comments

1. ***Solution***: Fix $\epsilon > 0$. We want an N such that for all $m, n > N$, $\|f(x_n) - f(x_m)\| < \epsilon$. We know that f is uniformly continuous, so there is a $\delta > 0$ such that if $\|x - y\| < \delta$, then $\|f(x) - f(y)\| < \epsilon$. Since $\{x_k\}$ is a Cauchy sequence, we can find N be such that for all $m, n > N$, $\|x_n - x_m\| < \delta$. Then by the choice of δ, this means $\|f(x_n) - f(x_m)\| < \epsilon$, as required.

   This need not be true if f is continuous but not uniformly continuous. For example, let $S = (0, \infty) \subseteq \mathbb{R}$, and $f : S \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$. Then $\{\frac{1}{k}\}$ is a Cauchy sequence, but $\{f(x_k)\} = \{k\}$ is not.

   Comments: I suspect that anyone with a good intuitive idea of what Cauchy sequences and uniformly continuous functions are should get what's going on here, but that won't be many of the students. The core of this is just that uniformly continuous functions send points that are close together to points that are close together, and that what the second "close together" means doesn't depend on the points. Otherwise this proof is just bookkeeping and moving ϵs and δs around between different definitions. This is the sort of proof where I would try to stress the fact that you don't have to have any ideas to do it. You can just write down the definitions and see what to do. It's entirely manipulation of definitions.

The counterexamples with non-uniformly continuous functions really get at the core of what it means to be continuous but not uniformly continuous, which in turn get at the core of why uniform continuity is worth talking about. I think people are skeptical about a version of continuity that super well behaved functions like x^2 or e^x don't satisfy, and an example like this is what will convince people it's worth thinking about.

2. ***Solution***: It remains to assign values to the points in $\bar{S} \setminus S$, so fix an x in there (if there are no such x's, there's nothing to do). We know from our study of sequences that there is a sequence $\{x_k\} \subseteq S$ such that $x_k \to x$. Then $\{f(x_k)\}$ is a Cauchy sequence in $\mathbb{R}$ by the previous exercise, and so it converges to some L. Then we define $f(x) = L$. (An $\epsilon/3$ argument shows this is well-defined.)

   First, $\bar{f} : \bar{S} \to \mathbb{R}$ is continuous. We knew it was continuous at every point in S, and we've defined it to respect sequences converging to points in $\bar{S} \setminus S$, so it's continuous at those points as well.

   Second, this extension is unique. Let g be another such extension. By their definition, g and $\bar{f}$ agree on S. If $x \in \bar{S} \setminus S$, then fix a sequence $\{x_k\}$ in S converging to x. Then we have:
$$g(x) = g\left(\lim_{k\to\infty} x_k\right) = \lim_{k\to\infty} g(x_k) = \lim_{k\to\infty} f(x_k) = \bar{f}(x)$$
   where those equalities are by definition of $\{x_k\}$, continuity of g, definition of g, and definition of $\bar{f}$, respectively.

   An example that shows this need not be true for non-uniformly continuous function is the function $f(x) = \sin(\frac{1}{x}) : (0, \infty) \to \mathbb{R}$. This function has no continuous extension to $[0, \infty)$.

   Comments: First of all, the problem as it's currently stated in the Big List isn't possible. It asks for extensions of any continuous, bounded $f : S \to \mathbb{R}$. The $\sin(\frac{1}{x})$ example was born of thinking that seemed wrong.

   This is a pretty ambitious problem for a tutorial. I very much do not expect anyone to finish it, particularly with all the details (the $\epsilon/3$ proof, the stuff I left about sequences from $\bar{S}\setminus S$ which might converge to things, etc.). I would be very happy if people (a) realised that the previous question is useful, and (b) came up with the right idea for defining $\bar{f}$. Again, I think a picture of the situation makes things pretty easy to wrap one's head around.

   It might even be a good idea to present this problem with S being an open interval in $\mathbb{R}$. In that case, I think the problem is one that most students could solve easily. I've been stressing a lot with my students that much of this $\mathbb{R}^n$ business is the same as the stuff about $\mathbb{R}$ they know, just with a few minor changes like $|\cdot|$ turning to $\|\cdot\|$, or infima and suprema generalizing to boundaries, or whatever. This problem is no exception, really.

3. ***Solution***: The $(\Leftarrow)$ direction is obvious, since all finite sets are compact.

   Conversely, assume S is infinite, and let $\{x_k\}$ be a sequence of distinct points from S. This sequence cannot have a subsequence converging to a point in S, since for any point in S we can find a ball around it containing no other points of S, and in particular not containing a tail of this sequence.

   Comments: I expect this to be pretty straightforward if they have a good picture of what a discrete set looks like. The only think I might expect to be a little tricky is exactly how the argument at the end goes. The real fact at play here is that any convergent sequence in a discrete space is eventually constant.