MAT237 - Tutorial 2 - 26 May 2015

1 Coverage

Here's what they've covered in class between last Tuesday's tutorial and this one, which roughly comprises the material covered in this tutorial.

Sequences and Completeness: All of Chapter 4 of their lecture notes. (Recall that they had a question about sequences on their quiz last week.)

Continuity: Definition of a limit in $\mathbb{R}^n$ and a few examples of computations, definition of continuity, all results up to and including page 31 of the notes.

2 Problems

I suggest the following problems. Some are from the Big List, some not. I'll give you a quick idea of the solution if it seems necessary, and discuss what's worth stressing or notable connections to other things. Many of these problems are stated in very general terms, and you can choose to give them specific cases or versions of them as you see fit. I've commented a bit about that below. I don't think it's possible to do all of these, by the way.

1. (BL3.4, plus more) Find an example of a sequence $\{x_k\}$ in $\mathbb{R}^n$ such that $\|x_k - x_{k+1}\| \to 0$, but which is not Cauchy. Try to find a bounded such sequence as well.

2. (A generalization of BL3.5 Part (b) is left as an exercise from the notes.) Let $S \subseteq \mathbb{R}^n$.

   (a) Let $x \in \partial S$. Show that there is a sequence $\{x_n\} \subseteq S$ converging to x.

   (b) Let C be the set of all limit points of sequences in S. That is,

   $$C = \{\, x \in \mathbb{R}^n \,:\, \exists \{x_n\} \subseteq S \text{ such that } x_n \to x \,\}.$$

   Part (a) proves that $\partial S \subseteq C$, from which it easily follows that $\bar{S} \subseteq C$. Show that $\bar{S} = C$.

3. Evaluate the following limits, if they exist, or prove they don't exist.

   (a) (BL4.1(a)) $\lim_{(x,y)\to(0,0)} 5x^3 - x^2y^2$

   (b) $\lim_{(x,y)\to(0,0)} \frac{x^n y}{x^{2n} + y^2}$, where n is an integer greater than 1.

4. (Part of BL4.3) Let $f : \mathbb{R}^m \to \mathbb{R}^n$. Prove that the following are equivalent:

   (a) f is continuous

   (b) For every open $V \subseteq \mathbb{R}^n$, $f^{-1}(V)$ is open.

3 Solutions and Comments

Notably, there aren't any problems here of the form "does this sequence converge or not?". By now they've learned that a sequence in $\mathbb{R}^n$ converges if and only if all of the coordinate sequences converge, so any question of that type is essentially no longer interesting. I think you should actually mention this omission before you start. They may not have fully grasped the strength of that result. Any of the parts of problem 3.7 on the Big List, for example, should essentially be trivial for them at this point, with the only possible exception being the sequence $(\cos(Ck), \sin(Ck))$, where C/π is irrational.

1. ***Solution***: There are a lot of possible answers to this. My favourite is the partial sums of the harmonic series. $x_k = \sum_{n=1}^{k} \frac{1}{n}$. This sequence doesn't converge and therefore isn't Cauchy, while on the other hand $|x_k - x_{k+1}| = \frac{1}{k+1} \to 0$.

   As soon as you've found an unbounded answer $\{x_k\}$ in $\mathbb{R}$, for example, you can get a bounded answer by moving your sequence to the unit circle via the "wrap it around the circle" map: $p_k = (\cos(x_k), \sin(x_k))$.

   Comments: I'm actually pretty interested to see how people answer this. First of all, I suspect many people won't immediately grasp the difference between the property their sequence is supposed to have and the Cauchy property. If they can get past that, I imagine many of them will be able to describe a solution in vague terms, but won't be able to write one down. You can prod them a bit by reminding them of the "necessary condition test" for series. They have had the fact that showing a sequence goes to zero is not sufficient to show the corresponding series converges drilled into their heads by now, and that's a particular case of what we're asking for here.

   The extra part about finding a bounded example is good to get them thinking about how sequences and continuous functions interact.

2. ***Solution***: (a) By definition of the boundary of a set, every open ball centred at x intersects S. So let x_n be a point in $S \cap B(\frac{1}{n}, x)$. Then it's easy to see that $x_n \to x$.

   (b) To show $\bar{S} = C$, all that's left to do is show that $C \subseteq \bar{S}$. So fix $x \in C$. By definition of C there is a sequence $\{x_n\} \subseteq S$ converging to x. For any $\epsilon > 0$, by definition of convergence there is a tail of the sequence $\{x_n\}$ inside $B(\epsilon, x)$. In other words, it's not possible to find a ball centred at x disjoint from S. This means x cannot be in the exterior of S, and so is in either S or ∂S. In either case, we're done.

   Comments: All of these proofs are actually in their notes. The particular case of (a) where $S \subseteq \mathbb{R}$ and $x = \inf S$ is problem 3.5 on the List. Feel free to give them the less general version of the problem if you feel it's more appropriate, or feel free to give them the general version and remark that problem 3.5 is a special case. The proofs are exactly the same, but I suspect many students feel they understood the general proof in the notes (Proposition 6 on Page 22), but can't see how to do problem 3.5. Some people were confused about it on Piazza, for example.

   Any amount of picture drawing for part (a) should make the idea very clear for them, but I'm sure that being asked to construct a sequence makes many students uneasy. You actually have to have an idea for this proof, which makes it trickier than most proofs they've done themselves, most of which require no ideas and only require manipulating definitions.

3. ***Solution***: (a) It should be pretty obvious that this limit equals zero. They don't formally know that multivariable polynomials are continuous, but they should take it for granted. So to prove that it converges to 0: Fix $\epsilon > 0$. Then

$$|5x^3 - x^2y^2| = x^2|5x - y^2| \le x^2(5|x| + |y|^2).$$

If we're within 1 of the origin, then $|x|$ and $|y|$ are both bounded above by 1, so $5|x|+|y|^2 \le 6$. Then let $\delta = \min\{1, \sqrt{\frac{\epsilon}{6}}\}$. So if $\|(x,y)\| < \delta$, we have:

$$|5x^3 - x^2y^2| \le x^2(5|x| + |y|^2) \le 6x^2 < 6\sqrt{\tfrac{\epsilon}{6}} = \epsilon.$$

(b) This problem is great. It doesn't exist, but the limit equals zero along all curves of the form $y = mx^k$ where $k < n$. For example, if you check just along lines $y = mx$ through the origin:

$$\left|\frac{x^n y}{x^{2n}+y^2}\right| = \left|\frac{x^n \cdot mx}{x^{2n}+m^2x^2}\right| = \left|\frac{mx^{n+1}}{x^{2n}+m^2x^2}\right| = \left|\frac{mx^{n-1}}{x^{2n-2}+m^2}\right| \le \left|\frac{1}{m}\right| |x^{n-1}| \to 0$$

Or if you check along curves of the form $y = mx^{n-1}$:

$$\left|\frac{x^n y}{x^{2n}+y^2}\right| = \left|\frac{x^n \cdot mx^{n-1}}{x^{2n}+m^2x^{2(n-1)}}\right| = \left|\frac{mx^{2n-1}}{x^{2n}+m^2x^{2(n-1)}}\right| = \left|\frac{mx}{x^2+m^2}\right| \le \left|\frac{1}{m}\right| |x| \to 0$$

However, if you check along $y = mx^n$, you get:

$$\frac{x^n \cdot mx^n}{x^{2n}+m^2x^{2n}} = \frac{mx^{2n}}{x^{2n}+m^2x^{2n}} = \frac{m}{1+m^2}$$

whose limit depends on m, so the limit of the function doesn't exist.

Comments: These two problems have a nice contrast. The first shows how similar these limits are to what they already know. The second shows how completely different they are.

Part (a) is quite similar to the single variable proofs for limits of quadratics and the like, with the "take the minimum of two things" trick. That said, I'm sure this scares them much more than those do. I think the thing to stress here is exactly how similar the thought process is to single variable limits. All we're doing is bounding stuff. If we come to a thing we can't bound directly, we assume we're close to the origin and bound it there, and continue. Just the same as in one variable.

Part (b) should hopefully give them a feeling for just how much more subtle multivariable limits can be. If n is large, the paths along $y = x^{n-1}$ and $y = x^n$ *look* really similar to the eye if you draw them, but the function is extremely sensitive to these changes. If you do a few cases along curves of intermediate degree, you can see the bound we get above approaching 0 more and more slowly as the degree increases, until it finally breaks down completely at degree n.

If you think the general case is too hard for them to wrap their minds around, even just the case where $n = 2$ is worth doing.

4. ***Solution***: First assume f is continuous, and fix $V \subseteq \mathbb{R}^n$ open. We want to show that $f^{-1}(V)$ is open, so given $p \in f^{-1}(V)$ we want to find an open ball centred at p inside $f^{-1}(V)$. We know that $f(p) \in V$ and that V is open, so there is some $\epsilon > 0$ such that $f(p) \in B(\epsilon, f(p)) \subseteq V$. By definition of continuity there is a $\delta > 0$ such that $0 < \|x - p\| < \delta$ implies $\|f(x) - f(p)\| < \epsilon$. Then $p \in B(\delta, p) \subseteq f^{-1}(V)$, as required.

   Conversely, assume the statement in (b) is true. We want to show f is continuous, so we pick an arbitrary $p \in \mathbb{R}^m$ and show that f is continuous at p. Fix $\epsilon > 0$. Then $B(\epsilon, f(p))$ is an open subset of $\mathbb{R}^n$, and so by assumption $U := f^{-1}(B(\epsilon, f(p))) \subseteq \mathbb{R}^m$ is open, and p is in this set. By definition of openness there is a $\delta > 0$ such that $p \in B(\delta, p) \subseteq U$, and this δ satisfies the definition of continuity.

   Comments: This may be a bit ambitious to assign, but hopefully they can wrap their mind around the first direction at least. As with all proofs of this form, pictures are immensely helpful.

   Just like with some of the other questions I've written up, this is quite a general problem. It may be easier for them to picture things just in $\mathbb{R}$, then realise that the whole proof just works in any dimension so long as they talk about open balls in the appropriate way. That's a general trend with a lot of this $\mathbb{R}^n$ business, in fact. A lot of it is just like what happens in $\mathbb{R}$, but with open balls instead of intervals and norm bars instead of absolute value bars. The closer they get to an intuitive grasp of this, the happier we'll be. There will certainly be enough examples of screwy things in higher dimensions later in the course.