University of Toronto

Faculty of Arts and Science
Quiz 4
MAT2371Y - Advanced Calculus
Duration - 50 minutes
No Aids Permitted

Surname: ______________________________

First Name: ______________________________

Student Number: ______________________________

Tutorial:

T0101	**T5101**	**T5102**
T4/R4	**T5/R5**	**T5/R5**
Chris	**Anne**	**Ivan**
SS1074	**SS1070**	**BA1240**

This exam contains 5 pages (including this cover page) and 3 problems. Check to see if any pages are missing and ensure that all required information at the top of this page has been filled in.

No aids are permitted on this examination. Examples of illegal aids include, but are not limited to textbooks, notes, calculators, or any electronic device.

Unless otherwise indicated, you are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	10	
2	10	
3	10	
Total:	30	

1. (a) (5 points) Find conditions on x and y which guarantee that one can locally solve the following for $u(x, y)$ and $v(x, y)$:

$$xu^2 + vy^2 = 9$$
$$xv^2 - yu^2 = 7$$

Solution: Define the function

$$F(x, y, u, v) = \begin{pmatrix} xu^2 + vy^2 - 9 \\ xv^2 - yu^2 - 7 \end{pmatrix}.$$

The determinant of $d_{(u,v)}F$ is given by

$$\det(d_{(u,v)}F) = \det \begin{pmatrix} 2xu & y^2 \\ -2yu & 2xv \end{pmatrix} = 4x^2uv + 2y^3u = 2u(2x^2v + y^3).$$

Now we will not be able to solve for (u, v) as functions of x and y if $2u(2x^2v + y^3) = 0$.

(b) (5 points) Define the set

$$M_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

to be the set of 2×2 matrices. Define a map $g : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ by $g(A) = A^2$. Determine whether g is invertible in a neighbourhood of $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Solution: In components, the map $g(A)$ looks like

$$g\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}.$$

However, this is the only part of the question that really requires that we use matrices. We can instead choose to identify $M_2(\mathbb{R})$ with $\mathbb{R}^4$ as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

With this transformation in mind, the map $g(A)$ now looks like

$$g(a, b, c, d) = (a^2 + bc,\ ab + bd,\ ac + cd,\ bc + d^2).$$

We want to determine whether this map is invertible around the identity I, which under the transformation has coordinates $(1, 0, 0, 1)$. Computing the derivative of g and evaluating at I we have

$$\begin{aligned} Dg(I) &= \left. \begin{pmatrix} 2a & c & b & 0 \\ b & a+d & 0 & b \\ c & 0 & a+d & c \\ 0 & c & b & 2d \end{pmatrix} \right|_{(1,0,0,1)} \\ &= \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}. \end{aligned}$$

This matrix is clearly invertible, so the Inverse Function Theorem applies and we conclude that the map g is invertible in a neighbourhood of the identity.

2. Determine whether the following spaces are smooth:

(a) (4 points) The surface S defined by the image of $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(s,t) = (3s, s^2 - 2t, s^3 + t^2)$.

Solution: Computing df we have

$$df = \begin{pmatrix} 3 & 0 \\ 2s & -2 \\ 3s^2 & 2t \end{pmatrix}.$$

This matrix clearly has full rank, or alternatively one could compute the cross product of its columns to get

$$d_s f \times d_t f = (6s^2 + 4, -6t, -6)$$

which will never be zero. Furthermore, the first component of f is $f_1(s,t) = 3s$ which is injective, making f globally injective. We conclude that the image of f is smooth.

(b) (6 points) The surface S defined by the image $g : \mathbb{R} \to \mathbb{R}^2$, $g(t) = (e^{2t}\cos^2(t), e^t \cos(t))$.

Solution: Trying to do this from the parametric viewpoint is very difficult. Instead, we recognize that if $x = e^{2t}\cos^2(t)$ and $y = e^t\cos(t)$ then $y^2 = x$. This is of course just a 'sideways' parabola, so we certainly expect it to be smooth.

We can define the image of g equivalently by the zero locus of $F(x,y) = y^2 - x$. The gradient of this function is $(-1, 2y)$, which is never zero, therefore S is smooth.

3. (a) (3 points) Define the Jordan measure in $\mathbb{R}$, what it means to be Jordan measurable, and what it means to have zero Jordan measure.

Solution: If $I = [a, b]$ let the length of I be $\ell(I) = b - a$. If $S \subseteq \mathbb{R}$ we define the *Jordan outer measure of S* as

$$m(S) = \inf \left\{ \sum_{k=1}^{n} \ell(I_k) : \begin{array}{c} I_k \text{ is an interval} \\ S \subseteq \bigcup_{k=1}^{n} I_k \end{array} \right\}.$$

If $m(S)$ exists and $m(\partial S) = 0$, we say that S is *Jordan measurable.* If $m(S) = 0$ we say that S has *Jordan measure zero.*

(b) (7 points) Let $(a_n)_{n=1}^{\infty}$ be a convergent sequence in $\mathbb{R}$. Show that as a set, (a_n) has Jordan measure zero.

Solution: Let $S = \{a_n : n \geq 1\}$, for which we will show that $m(S) < \epsilon$ for any $\epsilon > 0$.
Let $\epsilon > 0$ be given and let a be the limit of (a_n). Since (a_n) is convergent, there exists some N such that for every $k > N$, $|a_k - a_n| < \frac{\epsilon}{4}$. In particular, $I_0 = \left(a - \frac{\epsilon}{4}, a + \frac{\epsilon}{4}\right)$ has length $\epsilon/2$ and contains all elements of the sequence $(a_n)_{n=N+1}^{\infty}$, leaving only N elements outside of this set.
For $k = 1, \ldots, N$ define the interval $I_k = \left(a_k - \frac{\epsilon}{4N}, a_k + \frac{\epsilon}{4N}\right)$ which has length $\frac{\epsilon}{2N}$ and contains the point a_k. The collection of intervals $\{I_0, I_1, \ldots, I_n\}$ now covers S and has length

$$\sum_{k=0}^{N} \ell(I_k) = \frac{\epsilon}{2} + \sum_{k=0}^{N} \frac{\epsilon}{2N} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $m(S) < \epsilon$ as required.