Partial Big List Solutions

MAT 237 – Advanced Calculus– Summer 2015

Solutions

 # 1.6 Let $f : A \to B$ be a map of sets, and let $\{X_i\}_{i\in I}$ be an indexed collection of subsets of A.

(a) Prove that $f(\cup_{i\in I} X_i) = \cup_{i\in I} f(X_i)$

(b) Prove that $f(\cap_{i\in I} X_i) \subseteq \cap_{i\in I} f(X_i)$

(c) When does equality of sets hold in the b)

Proof (a) Let's begin with the forward containment. Take $y \in f(\cup_{i\in I} X_i)$, now there must be i such that $x \in X_i \subseteq \cup_{i\in I} X_i$ with the property $f(x) = y$. More specifically, we see $y \in f(X_i) \subseteq \cup_{i\in I} f(X_i)$. Thus $f(\cup_{i\in I} X_i) \subseteq \cup_{i\in I} f(X_i)$. The reserve containment is even easier since if $y \in \cup_{i\in I} f(X_i)$, then we know there must be an i such that $y \in f(X_i)$. This means that some $x \in X_i$ must get mapped to y, i.e. $f(x) = y$. Now since $x \in \cup_{i\in I} X_i$, we clearly have $y \in f(\cup_{i\in I} X_i)$. With both continments shown, we're done. $\square$

Proof (b) We only have to show the one containment, so take $y \in f(\cap_{i\in I} X_i)$. Using the same argument, we see that there must be $x \in \cap_{i\in I} X_i$ s.t. $f(x) = y$. Notice that since x is in the intersection of all $\{X_i\}'s$, we necessarily have that $x \in X_i$ for all $i \in I$. Thus $y \in f(X_i)$ for all $i \in I$, which implies $y \in \cap_{i\in I} f(X_i)$. $\square$

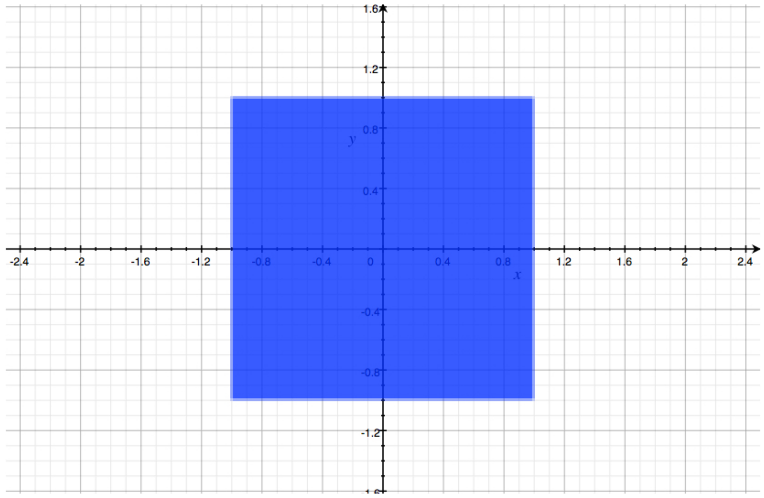
(c) One may check that injectivity of f is a necessary and sufficient condition for equality to hold. $\square$

 # 1.13 The intent of this exercise is to show that if we were to start the course over and use open squares instead of open balls in defining open sets, we would have actually had the same definition! Define a map

$$|| \cdot ||_\infty : \mathbb{R}^n \to \mathbb{R}, \quad ||x||_\infty = \max_i \{|x_i|\}$$

Where $x = (x_1, \ldots, x_i, \ldots, x_n)$. For any $a \in \mathbb{R}^n$, let $S(a, \epsilon) = \{x \in \mathbb{R}^n : ||x - a||_\infty < \epsilon\}$. We say that a set U is S-open (the S stands for square) if and only if for every $a \in U$, there exists $\epsilon > 0$ such that $S(a, \epsilon) \subseteq U$

(a) Make a sketch of $S(0, 1)$ when $n = 2$

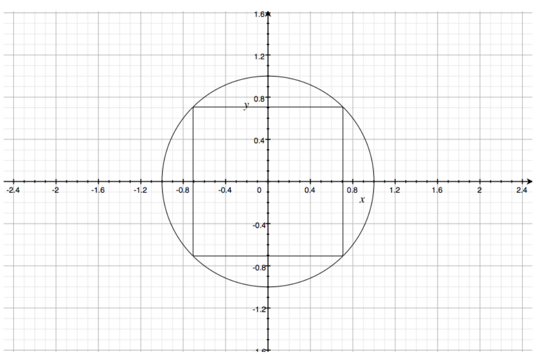
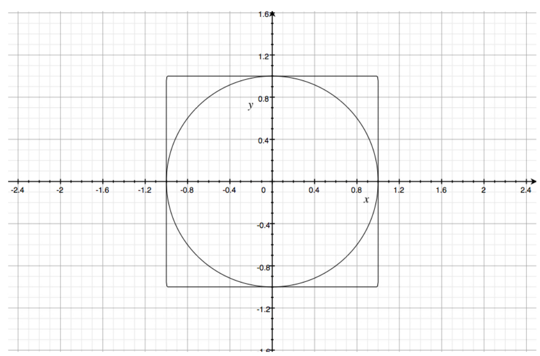


(b) Show that for any $x \in \mathbb{R}^n$, $||x||_\infty \leqslant ||x|| \leqslant \sqrt{n}||x||_\infty$, where $||x|| = \sqrt{x_1^2 + \ldots + x_n^2}$.

Proof The lower bound follows by dropping everything by the largest x_i^2, since $0 < x_i^2$ for all i. The upper bound is found by bounding every term by the largest x_i, i.e. $x_i^2 < \max_i x_i^2$.

(c) Prove that $U \subseteq \mathbb{R}^n$ is S-open if and only if U is open.

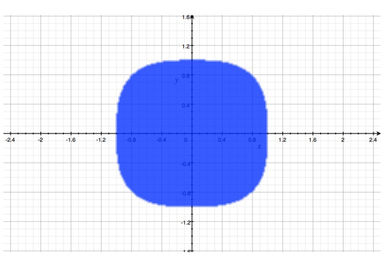
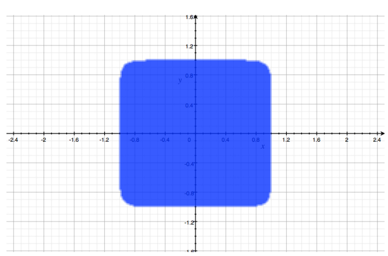
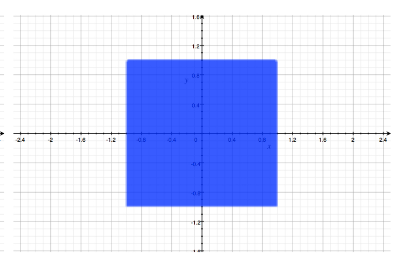
Proof By drawing the sets, we see

Thus we may always fit an open square in an open ball, and an open ball in an open square.

(d) Consider the functions $||\cdot||_p : \mathbb{R}^n \to \mathbb{R}$, $||x||_p = (|x_1|^p + \ldots + |x_n|^p)^{1/p}$, where $1 < p < \infty$. Plot the sets (by hand, or using a computer) $\{x \in \mathbb{R}^2 : ||x||_p < 1\}$. Do you expect the p-balls to define the same collection of open sets as the 2-balls? Explain.

Solution I've shown $p = 3, 10$ and 100.

It appears that as $p \to \infty$, we're approaching the same set as $||x||_\infty < 1$.

#2 .1 Let $\{x_k\}$ and $\{y_k\}$ be sequences such that $x_k \to a$ and $y_k \to b$. Show that $x_k + y_k \to a + b$ and $x_k y_k \to ab$.

Proof Recall by definition

$$x_k \to a \iff \forall \epsilon > 0, \exists N \in \mathbb{N} \quad such\, that \quad |x_k - a| < \epsilon \quad \forall k > N$$

Thus, if we fix $\epsilon > 0$, we may choose N large enough s.t. $|x_k - a| < \epsilon/2$ and $|y_k - b| < \epsilon/2$, so

$$|x_k + y_k - (a+b)| = |(x_k - a) + (y_k - b)| \leqslant |x_k - a| + |y_k - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

by the triangle inequality. Since ϵ was arbitrary, we conclude that $x_k + y_k \to a + b$. To handle the product, we see for $\epsilon > 0$, we have some N s.t. $x_k < a + \epsilon$ and $|y_k - b| < \epsilon$, so

$$x_k y_k - ab < (a+\epsilon)y_k - ab < a(y_k - b) + \epsilon y_k < |a|\epsilon + \epsilon(|b| + \epsilon) = \epsilon(|a| + |b|) + \epsilon^2 \leqslant \tilde{\epsilon}$$

The other inequality is found by using $a < x_k + \epsilon$ instead. Thus $x_k y_k \to ab$. □

#2.2 Let $x_k = \frac{3k+4}{k-5}$. Given $\epsilon > 0$, find an integer K such that $|x_k - 3| < \epsilon$ for all $k > K$.

Proof We compute

$$|x_k - 3| = \left|\frac{3k+4}{k-5} - 3\right| = \left|\frac{3k + 4 - 3k + 15}{k-5}\right| = \left|\frac{19}{k-5}\right| < \epsilon$$

If we simplify the expression, we see

$$\frac{19}{k-5} < \epsilon \implies \frac{19}{\epsilon} < k - 5 \implies \frac{19}{\epsilon} + 5 < k$$

So if we choose the the integer of the function of ϵ, i.e.

$$K = \left\lceil \frac{19}{\epsilon} + 5 \right\rceil$$

this will suffice. □

 # 2.4 Find an example of a sequence $\{x_k\}$ such that $|x_{k+1} - x_k| \to 0$, but $\{x_k\}$ isn't Cauchy.

Example A good example to have in mind is a partial sum of the Harmonic Series, namely

$$x_k = \sum_{n=1}^{k} \frac{1}{n}$$

Clearly

$$x_{k+1} - x_k = \frac{1}{k+1} \to 0 \quad \text{as} \quad k \to \infty$$

but the series doesn't converge.

$$\lim_{k\to\infty} x_k = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

so the sequence isn't Cauchy. □

 # 2.5 Let $S \subseteq \mathbb{R}$, and set $L = \inf S$. Show there exists a sequence $\{x_k\}$ converging to L.

Proof The easiest way to show the claim is to build such a sequence. This is easily accomplished by considering

$$A_n = S \cap B\left(L, \frac{1}{n}\right)$$

Take any sequence such that $x_n \in A_n$, by construction, we see that

$$x_n \to L$$

$\square$

 # 4.7 Let $f, g : \mathbb{R}^n \to \mathbb{R}^k$ be continuous functions and suppose that $D \subseteq \mathbb{R}^n$ is a dense set. If $f(x) = g(x)$ for every $x \in D$, then $f(x) = g(x)$ for every $x \in \mathbb{R}^n$.

Proof Consider $h(x) = f(x) - g(x)$, note that h is continuous on $\mathbb{R}^n$ and $h|_D = 0$. For the sake of contradiction, suppose that $h(x_0) = a \neq 0$ for some $x_0 \in \mathbb{R}^n$. Since h is continuous around x_0, we must have for any $\epsilon > 0$ that there is some $\delta > 0$ such that

$$||x - x_0|| < \delta \implies ||f(x) - f(x_0)|| < \epsilon$$

But we know that $h|_D = 0$ where D is dense in $\mathbb{R}^n$, thus we must have that $||f(x_0)|| = ||a|| < \epsilon$... This is a contradiction since the statement fails if $\epsilon = ||a||/2$ for example. Therefore no such a exists, which allows us to conclude that $h(x) = 0$ for all $x \in \mathbb{R}^n$. $\square$

 # 5.6 Use the Bolzano-Weierstrass theorem to prove that if $K_1 \supset K_2 \supset K_3 \supset \ldots$ is a chain of proper containments and each $K_i \subseteq \mathbb{R}^n$ is compact, then $\cap_{i=1}^{\infty} K_i \neq \emptyset$.

Proof Take any sequence such that $x_i \in K_i$. Then clearly $\{x_k\} \subset K_1$ since K_1 is the top containment. Now we're in a position to apply the Bolzano-Weierstras theorem since K_1 is compact. The theorem implies that there is a subsequence $\{x_{k_j}\}$ that converges to some $x \in K_1$. Now since $K_i \supset K_{i+1}$ and each K_i is compact (this removes the possible issue of the limit point x lying on the boundary and not in the set), we have that $x \in K_i$ for all i. Thus $x \in \cap_{i=1}^{\infty} K_i$, which means that the intersection isn't empty. $\square$

 # 6.7 Let $S \subseteq \mathbb{R}^n$. S is disconnected if and only if there exists a continuous function $f : S \to \mathbb{R}$ such that $f(S) = \{0, 1\}$.

Proof ($\implies$) Suppose that S is disconnected, so $S = S_1 \cup S_2$ and $S_1 \cap \bar{S}_2 = \emptyset$,$\bar{S}_1 \cap S_2 = \emptyset$ for some $S_1, S_2 \subseteq S$. Define

$$f(x) = \begin{cases} 0, & x \in S_1 \\ 1, & x \in S_2 \end{cases}$$

Clearly f is continuous and satisfies $f(S) = \{0, 1\}$. ($\impliedby$) Suppose we have $f(S) = \{0, 1\}$ and continuous. Since the image has only two points, define $S_1 = f^{-1}(0)$ and $S_2 = f^{-1}(1)$. Clearly $S = S_1 \cup S_2$, now we'll show these sets form a disconnection. Suppose that $S_1 \cap \bar{S}_2 \neq \emptyset$ i.e we have some $x_0 \in S_1 \cap \bar{S}_2$... but f is continuous, so for all $\epsilon > 0$ we have some $\delta > 0$ such that

$$||x - x_0|| < \delta \implies |f(x) - f(x_0)| = |f(x)| < \epsilon$$

We see if we choose $x \in S_2$, the inequality is violated with any $\epsilon < 1$, i.e. f isn't continuous. Thus we've reached a contradiction, which implies we must have $S_1 \cap \bar{S}_2 = \emptyset$. Since $\bar{S}_1 \cap S_2 = \emptyset$ follows with the same argument, we're done. $\square$

 # 7.8 Let $S \subseteq \mathbb{R}^3$ be the set of points, such that each point in the set has equal distance to $(-1, 0, 0)$ and to the plane $x = 1$. Find an equation $F(x, y, z)$ such that $S = \{(x, y, z) \in \mathbb{R}^3 : F(x, y, z) = 0\}$. Sketch this surface.

Proof Let's find a generic point in the set. i.e. suppose $x \in S$, then we must have

$$||x - (-1, 0, 0)|| = ||x - (1, a, b)||$$

where $x = (x, y, z)$ and $a, b, \in \mathbb{R}$. Expanding out this equality reveals

$$(x + 1)^2 + y^2 + z^2 = (x - 1)^2 + (y - a)^2 + (z - b)^2$$

Simplify the above by moving everything to the LHS to obtain

$$F(x, y, z) = 4x + a(2y - a) + b(2z - b) = 0$$

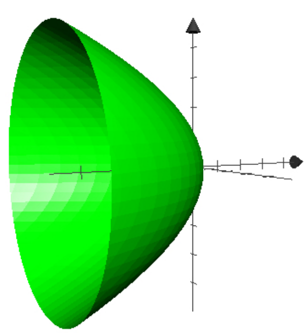
Now we just have to peg down a and b. You could match everything with normal vectors or something, but we just want to minimize the distance from x to the plane. i.e.

$$\min_{a,b\in\mathbb{R}} ||x - (1, a, b)||$$

clearly we may do this by choosing $y = a$ and $b = z$ since $(y - a)^2 = (z - b)^2 = 0 \iff y = a$ and $z = b$. Thus we see the function whose level set describes the surface is

$$F(x, y, z) = 4x + y^2 + z^2$$

Clearly this surface is a paraboloid, with a graph like



$\square$

 # 8.12 Let $C([0,1])$ be the collection of all continuous functions on the closed unit interval, given the sup norm norm. Consider the map:

$$F = \int_0^t : C([0,1]) \to C([0,1])$$

$$F(f) = \int_0^t f(x)dx$$

Compute DF_f.

Proof By definition of the directional derivative we need to estimate

$$DF_f(g) = \lim_{h \to 0} \frac{F(g+hf) - F(g)}{h}$$

for $h \in \mathbb{R}$ and $f, g \in C([0,1])$. By linearity of the integral, we see

$$F(g+hf) = \int_0^t [g(x) + hf(x)]dx = \int_0^t g(x)dx + h\int_0^t f(x)dx = F(g) + hF(f)$$

Thus

$$DF_f(g) = \lim_{h \to 0} \frac{hF(f)}{h} = F(f)$$

□

 # 10.8 Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Show that on the set $S = \{v \in \mathbb{R}^n : ||v|| = 1\}$, the maximum and the minimum of A are the largest and smallest eigenvalues of A, respectively.

Proof Using Lagrange multipliers, we see we're looking for $D(||Ax||) = \lambda Dg(x)$ where $g(x) = ||x|| = 1$. Note that $Dg(x) = x$ and

$$||Ax||^2 = \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}x_j \right)^2 \implies D(||Ax||) = \frac{1}{2||Ax||}2A^TAx$$

Thus we're trying to solve

$$\frac{A^TAx}{||Ax||} = \lambda x \implies A^TAx = \lambda||Ax||x$$

If we take the inner product (dot product) against x now, we see (since $||x|| = 1$)

$$||Ax||^2 = Ax \cdot Ax = x^TA^TAx = \lambda||Ax||x^Tx = \lambda||Ax||\,||x||^2 \implies ||Ax|| = \lambda||x||^2 = \lambda$$

Thus the Lagrange multiplier equation has now become

$$A^TAx = \lambda^2 x$$

i.e. x is an eigenvector of A^TA with eigenvalue λ^2. Now we simply have to take the largest eigenvalue to maximize $||Ax||$ and the smallest eigenvalue to minimize $||Ax||$. □

 # 11.8 A map $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be open if whenever U is open then $f(U)$ is open. Show that if f is C^1 and $Df(x_0)$ is invertible for all $x_0 \in \mathbb{R}^n$ then f is an open map.

Proof By the inverse function theorem we know that f is invertible around some neighbourhood of every $x_0 \in \mathbb{R}^n$ (since f is C^1 and $\det \partial_{ij} f \neq 0$). Defining $f^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ to be the piecewise continuous collection of the local inverse functions we have a continuous global inverse. Thus f is a continuous bijective map (a homeomorphism), i.e. if $U \subseteq \mathbb{R}^n$ is open then $f(U)$ is open. □

 # 11.2.6 Consider the curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(t) = 2e^{-t/2}(\cos(t), \sin(t))$.

(a) Show that $\gamma(t)$ defines a C^1 curve.

(b) Calculate the speed of this curve as a function of t.

(c) We define the unit tangent vector to the curve to be $T(t) = \gamma'(t)/||\gamma'(t)||$. Compute the unit tangent vector.

(d) Arc-length of a curve on the interval $[0, t]$ is given by

$$s(t) = \int_0^t ||\gamma'(u)||du$$

Compute the arc-length function $s(t)$ for the curve γ

(e) Inverting the arc-length formula gives a function $t(s)$ (time as a function of arc-length). The reparameterization of the curve $\gamma(t)$ using $t = t(s)$ is known as the arc length parameterization. Compute the arc-length parameterization of $\gamma(t)$.

Proof (a) Note that

$$\gamma'(t) = e^{-t/2}(-\cos(t) - 2\sin(t), -\sin(t) + 2\cos(t)) \neq 0 \quad \forall t \in \mathbb{R}$$

Thus $\gamma(t)$ is a smooth curve.

Proof (b) The speed of the curve is given by

$$||\gamma'(t)|| = 5e^{-t}(\cos^2(t) + \sin^2(t)) = 5e^{-t}$$

Proof (c) We compute the unit tangent vector

$$T(t) = \frac{\gamma'(t)}{||\gamma'(t)||} = \frac{e^{t/2}}{5}(-\cos(t) - 2\sin(t), -\sin(t) + 2\cos(t))$$

Proof (d) We compute arc-length

$$s(t) = \int_0^t ||\gamma'(u)||du = \int_0^t 5e^{-u}du = 5(1 - e^{-t})$$

Proof (e) The arc-length parameterization is given by

$$s = 5(1 - e^{-t}) \implies e^{-t} = \frac{5 - s}{5} \implies t(s) = \ln\left(\frac{5}{5 - s}\right)$$

□

 # 12.1.6 Show that if $f : [a,b] \to \mathbb{R}$ is Riemann integrable, then $F(x) = \int_a^x f(s)ds$ is uniformly continuous on $[a,b]$.

Proof Recall that a continuous function on a compact set A of a metric space is uniformly continuous on A. Since $A = [a,b] \subseteq \mathbb{R}$ is compact, it suffices to show that $F(x)$ is continuous. Therefore fix $\epsilon > 0$ and let's try to find δ such that

$$|x-y| < \delta \implies |F(x) - F(y)| < \epsilon$$

Notice that

$$|F(x) - F(y)| = \left| \int_a^x f(s)ds - \int_a^y f(s)ds \right| = \left| \int_x^y f(s)ds \right|$$

Since f is Riemann integrable, f is bounded. We see the easy upper bound of

$$|F(x) - F(y)| \leqslant \max_{z \in [a,b]} \left| f(z) \int_x^y ds \right| = \max_{z \in [a,b]} |f(z)||x-y| < \epsilon$$

Now the choice of δ is obvious, choose

$$\delta = \frac{\epsilon}{\max_{z \in [a,b]} |f(z)|}$$

Which shows $F(x)$ is continuous, i.e. $F(x)$ is uniformly continuous on $[a,b]$. □

 # 12.2.4 If $S = \{x_1, \ldots, x_n\}$ is a finite set consisting of precisely n-elements, show that S has zero Jordan measure.

Proof We already know a point has zero measure, and the finite union of measure zero sets has zero measure. So S clearly has zero measure. □

 # 13.4.10 Let $F(x) = (F_1(x), F_2(x), F_3(x))$ be a vector field in $\mathbb{R}^3$.

(a) For arbitrary $h > 0$, let $S_n = \{(x,y,z) : x^2 + y^2 + z^2 = h^2\}$ be the sphere of radius h. Parametrize S_h be a function $g : [a,b] \times [c,d] \to \mathbb{R}^3$. Compute $\partial_s g \times \partial_t g$

(b) Under the assumption that h is very small, we can use a first order approximation on the functions F_i. Write out the linear approximations for $F_i(x)$ at $(0,0,0)$ and evaluate these on the parameterization.

(c) Use parts a) and b) to determine $F(g(t)) \cdot (\partial_s g \times \partial_t g)$. [Ignore terms of order h^2].

(d) Compute

$$\lim_{h \to 0} \frac{3}{4\pi h^3} \oiint_{S_h} F \cdot n dS$$

Compare this to the divergence. Conclude that divergence is the infinitesimal flux. [Note that $4\pi h^3/3$ is the volume of the sphere, so we are normalizing by volume in the limit].

Proof (a) The most natural parametrization would be spherical coordinates, hence

$$g(\theta, \phi) = (x,y,z) = h(\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi) \quad \text{where} \quad (\theta, \phi) \in [0, 2\pi] \times [0, \pi]$$

is a good choice of parameterization. To compute the normal $\partial_\theta g \times \partial_\phi g$ you can go through the computation, or recall that the radial direction is the outward normal, i.e.

$$\frac{1}{h} \frac{\partial g}{\partial \theta} \times \frac{\partial g}{\partial \phi} = g(\theta, \phi)$$

Proof (b) We're asked to give the first order taylor expansion around 0. We compute

$$F(x) = F(0) + \partial_x F(0)x + \partial_y F(0)y + \partial_z F(0)z + \mathcal{O}(\text{2nd Order term})$$

Proof (c) Combining the above expressions together, we see

$$F(g(\theta,\phi)) \cdot n = F(g(\theta,\phi)) \cdot \frac{g(\theta,\phi)}{h} =$$

$$= \frac{1}{h}\left(F_1(0)g_1 + F_2(0)g_2 + F_3(0)g_3 + \partial_x F_1(0)g_1^2 + \partial_y F_2(0)g_2^2 + \partial_z F_3(0)g_3^2 + \text{odd terms}\right) + \mathcal{O}(h^2)$$

Proof (d) We compute

$$\lim_{h\to 0}\frac{3}{4\pi h^3}\oiint_{S_h} F\cdot n dS = \lim_{h\to 0}\frac{3}{4\pi h}\int_0^{2\pi}\int_0^{\pi} F\cdot n \sin\phi d\phi d\theta = \frac{3}{4\pi}\int_0^{2\pi}\int_0^{\pi}(\partial_x F_1 g_1^2\sin\phi + \partial_y F_2 g_2^2 \sin\phi + \partial_z F_z g_3^2\sin\phi)d\phi d\theta$$

Noting

$$\int_0^{2\pi}\int_0^{\pi}\cos^2\theta\sin^3\phi d\phi d\theta = \frac{4\pi}{3} \quad \& \quad \int_0^{2\pi}\int_0^{\pi}\cos^2\phi\sin\phi d\phi d\theta = \frac{4\pi}{3}$$

We see

$$\lim_{h\to 0}\frac{3}{4\pi h^3}\oiint_{S_h} F\cdot n dS = \partial_x F_1(0) + \partial_y F_2(0) + \partial_z F_3(0) = \text{div } F(0)$$

Re-centering the sphere around any $x \in \mathbb{R}^3$ shows we have that more generally that

$$\lim_{V\to\{x\}}\frac{1}{|V|}\oiint_{S(V)} F\cdot n dS = \text{div } F(x)$$

□

#13.5.4 Let S be a smooth oriented surface in $\mathbb{R}^3$ with piecewise smooth, compatible oriented boundary ∂S. Show that if $f \in C^1$ and $g \in C^2$ on S then

$$\int_{\partial S} f\nabla g \cdot dx = \iint_S (\nabla f \times \nabla g)\cdot n dA$$

Proof Recall Stoke's Theorem

$$\int_{\partial S} F\cdot dx = \int_S (\nabla \times F)\cdot n dA$$

and notice the curl of the quantity in question is

$$\nabla \times (f\nabla g) = \nabla f \times \nabla g + f \underbrace{(\nabla \times \nabla g)}_{\text{curl(grad}(g))=0} = \nabla f \times \nabla g$$

Applying Stoke's Theorem gives the claim. □

13.6.5 Let $f : \mathbb{R}^k \to \mathbb{R}$ be a 0-form and $F : \mathbb{R}^n \to \mathbb{R}^k$. Show that $F^*df = d(f\circ F)$.

Proof Let's perform the computation(with the convention that $x = F(y)$ under the pullback).

$$\begin{aligned}
F^* df =& F^*\left(\sum_{i=1}^{k} \frac{\partial f}{\partial x_i}(x) dx_i\right) && \text{expanding the differential of the 0-form} \\
=& \sum_{i=1}^{k} F^*\left(\frac{\partial f}{\partial x_i}(x) dx_i\right) && \text{pullback is linear on differential forms} \\
=& \sum_{i=1}^{k} \frac{\partial f}{\partial x_i}(F(y)) d(F_i(y)) && \text{applying the pullback} \\
=& \sum_{i=1}^{k} \left(\frac{\partial f}{\partial x_i}(F(y)) \sum_{j=1}^{n} \frac{\partial F_i}{\partial y_j} dy_j\right) && \text{expanding the differential of } F(y) \\
=& \sum_{j=1}^{n} \left(\sum_{i=1}^{k} \frac{\partial f}{\partial x_i}(F(y)) \frac{\partial F}{\partial y_i}(F(y))\right) dy_j && \text{rearranging the sum} \\
=& \sum_{j=1}^{n} \frac{\partial f}{\partial y_j}(F(y)) dy_j && \text{via chain rule} \\
=& d(f \circ F) && \text{this is the definition of the differential}
\end{aligned}$$

□

13.6.6 A k-form ω is said to be decomposable if $\omega = \omega_1 \wedge \ldots \wedge \omega_k$ where the ω_i are 1-forms. The form is said to be indecomposable otherwise.

(a) Show that $dx \wedge dy + dx \wedge dz + dy \wedge dz$ is decomposable in $\mathbb{R}^3$

(b) Show that $dx \wedge dy + dz \wedge d\omega$ is indecomposable.

Proof (a) We have that

$$dx \wedge dy + dx \wedge dz + dy \wedge dz = \underbrace{(dx + dy)}_{=\omega_1} \wedge \underbrace{(dy + dz)}_{=\omega_2}$$

Proof (b) Suppose the 2-form was decomposable, we'd therefore have

$$(a_1 dx + a_2 dy + a_3 dz + a_4 d\omega) \wedge (b_1 dx + b_2 dy + b_3 dz + b_4 d\omega) = dx \wedge dy + dz \wedge d\omega$$

i.e.

$$\begin{cases} a_1 b_2 - b_1 a_2 = 1 \\ a_1 b_3 - a_3 b_1 = 0 \\ a_1 b_4 - a_4 b_1 = 0 \\ a_2 b_3 - a_3 b_2 = 0 \\ a_2 b_4 - a_4 b_2 = 0 \\ a_3 b_4 - a_4 b_3 = 1 \end{cases}$$

But the above system has no solution, those the decomposition doesn't exist. □