Tutorial #8

MAT 188 – Linear Algebra I – Fall 2015

SOLUTIONS

Fitting Curves to Experimental Data

Example 1 Suppose your experiment results in the following four data points

$$(x, y) = (0, 1), (1, 3), (2, 4), (3, 4)$$

1. Plot these points and try to draw a straight line that "best fits the data," that is, a line that is close to all the points without getting too far from any of them.

2. Now suppose you try to find a line $y = b + mx$ that actually passes through all four points at once: substitute each point into the equation to get a system of four equations in two variables, b and m. Write your system in the form
$$Ak = y$$
where A is a 4×2 matrix.
$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}, k = \begin{pmatrix} b \\ m \end{pmatrix}, y = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}$$
Is this system consistent or inconsistent?

3. As we will be able to show later, the "best" solution to such an inconsistent system can be found by simply multiplying both sides of your matrix equation by the transpose of A:
$$A^T Ax = A^T y$$
Solve for x and plot the line $y = b + mx$. How does it compare to the line you drew in?

4. In this case you should have noticed that $A^T A$ is invertible and that there is a unique solution for x. In general, if A is an $n \times m$ matrix, $A^T A$ need not be invertible. However it can be proved that $A^T A$ is invertible if and only if the rank of A is m. Can you prove this? (Hard)

Example 2 As a second example, consider the four data points

$$(x, y) = (-1, 0), (0, 2), (1, 1), (1, -1)$$

Plot these points and try to find a parabola $y = a + bx + cx^2$ that best fits the data.

Problems (Please note these are from Holt's Linear Algebra Text)

4.2 - #21 Suppose that A is a 6×11 matrix and that $T(x) = Ax$. If nullity$(A) = 7$, what is the dimension of the range of T?

Solution Using the rank-nullity theorem, we know if A is an $n \times m$ matrix, then

$$\text{rank}(A) + \text{nullity}(A) = m$$

Thus in our case, we see

$$\dim(\text{range}(T)) = \text{rank}(A) = m - \text{nullity}(A) = 11 - 7 = 4$$

$\square$

4.3 - #59 Suppose that A is a $n \times m$ matrix with $n \neq m$. Prove that either nullity$(A) > 0$ or nullity$(A^T) > 0$ (or both).

Solution The rank-nullity theorem we know

$$\text{rank}(A) + \text{nullity}(A) = m \implies \text{nullity}(A) = m - \text{rank}(A)$$

Using the fact that rank$(A) \leqslant \min(n, m)$, we see that

$$\text{nullity}(A) \geqslant m - \min(n, m)$$

For the transpose we have A^T is a $m \times n$ matrix, thus the same argument gives

$$\text{nullity}(A^T) \geqslant n - \min(n, m)$$

Since $n \neq m$, we have that

$$\text{nullity}(A) > 0 \quad \text{or} \quad \text{nullity}(A^T) > 0$$

and the case of both when rank$(A) < \min(n, m)$. $\square$

Question 3 Show that if A is an $n \times n$ matrix such that $A^2 = 0$, then rank$(A) \leqslant n/2$.

Solution Since $A^2 = 0$, we have that Range$(A) \subseteq$ Null(A) Therefore

$$\text{rank}(A) \leqslant \text{nullity}(A)$$

If we use the rank-nullity theorem, we see

$$n = \text{rank}(A) + \text{nullity}(A) \geqslant 2\,\text{rank}(A) \implies \text{rank}(A) \leqslant \frac{n}{2}$$

$\square$