Tutorial #7

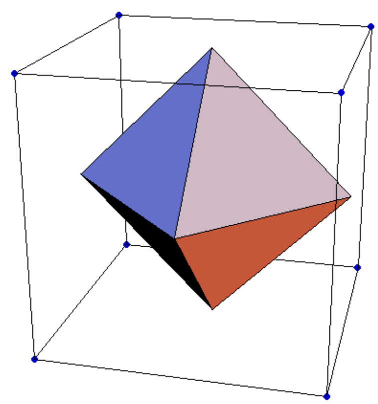
MAT 188 – Linear Algebra I – Fall 2015

Solutions

Problems

Question 1 Draw the cube with vertices $(\pm 1, \pm 1, \pm 1)$. The centres of the six faces of the cube are the vertices of an octahedron. Draw it. What are the vertices?

Solution We see the picture as



The six vertices are given by the points $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, and $(0, 0, \pm 1)$. □

Question 2 How many invertible transformations $T : \mathbb{R}^3 \to \mathbb{R}^3$ are there that fix the vertices of the octahedron? That is, if v is a vertex of the octahedron then $w = T(v)$ is also a vertex of the octahedron. To solve this problem, let the matrix of T be A and recall that

$$A = (T(e_1), T(e_2), T(e_3))$$

How many such invertible matrices A are there? Can you write the matrices down?

Solution Assume that $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$, note how these correspond to the vertices of the octahedron. Assuming

$$T(x) = Ax \quad \text{where} \quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

More specifically we have

$$T(e_1) = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} \quad \& \quad T(e_2) = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} \quad \& \quad T(e_3) = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}$$

Since $T(v) = w$ where v and w are both vertices of the octahedron, we see only one a_{ij} can be non-zero, and must be either ± 1. Now we count, assume $a_{11} = \pm 1$, then $a_{21} = a_{31} = 0$(i.e. 2 cases). Then this means that $a_{12} = 0$ or else the matrix is non-invertible. Thus either $a_{22} = \pm 1$ or $a_{32} = \pm 1$ (i.e. 4 cases), and lastly we have 2 cases for the final, $a_{i3} = \pm 1$. Thus we see $2 * 4 * 2 = 16$. We had 3 choices at the first step, so we have $3 \times 16 = 48$ Thus we should have 48 possible invertible transformation. We can obviously write the matrices down since thats how we counted.

Alternate Solution By symmetry of the octahedron, we see the only linear transformations will be rotations or reflections that preserve the vertices. Recall that orientation preserving rotation by the angle θ is characterized by

$$A_{rot} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Thus we see the transformations are rotations by $0, \frac{\pi}{2}$, π, and $\frac{3\pi}{2}$ or $-\frac{\pi}{2}$ in the xy, yz and xz planes. Thus there are $4 \times 3 = 12$ invertible orientation preserving transformation of the above form, if we include reflections and orientation preserving transformation, we see $12 \times 4 = 48$ invertible transformations. The matrices for the orientation preserving transformations with no reflection are given by

$$A_{xy} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \& \quad A_{yz} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \quad \& \quad A_{xz} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

evaluated at the 4 rotations. The other matrices are given by swapping the columns of ij and the sign of 1. □

Question 3 Explain why the invertible transformation that fix the vertices of the octahedron must be exactly the same as the invertible transformation that fix the vertices of the cube.

Solution This is because the vertices of the cube determined the vertices of the octahedron. You'll also notice, that the centres of the faces of the octahedron form the vertices of a cube!!! Thus, one might say these shapes are dual to one another. □