Tutorial #5

MAT 188 – Linear Algebra I – Fall 2015

SOLUTIONS

Problems (Please note these are from Holt's Linear Algebra Text)

3.1 - #31 Calculation Suppose that $T(x) = Ax$ for the given A. Sketch a graph of the image under T of the unit square in the first quadrant of $\mathbb{R}^2$.

$$A = \begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}$$

Solution We know the unit square is given by 4 lines, specifically: $y = 0$ with $0 \leqslant x \leqslant 1$, $x = 1$ with $0 \leqslant y \leqslant 1$, $y = 1$ with $0 \leqslant x \leqslant 1$ and $x = 0$ with $0 \leqslant y \leqslant 1$. We see the first line transforms like

$$\begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ 3x \end{pmatrix} \implies x' = x \text{ and } y' = 3x \implies y' = 3x'$$

The next line is given by

$$\begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ y \end{pmatrix} = \begin{pmatrix} 1 - 2y \\ 3 + y \end{pmatrix} \implies x' = 1 - 2y \text{ and } y' = 3 + y \implies y' = \frac{7 - x'}{2}$$

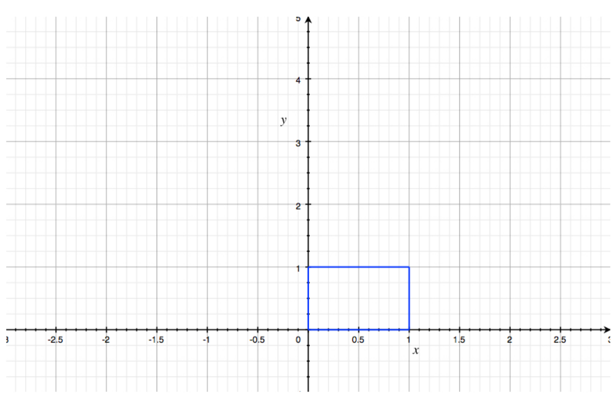
again...

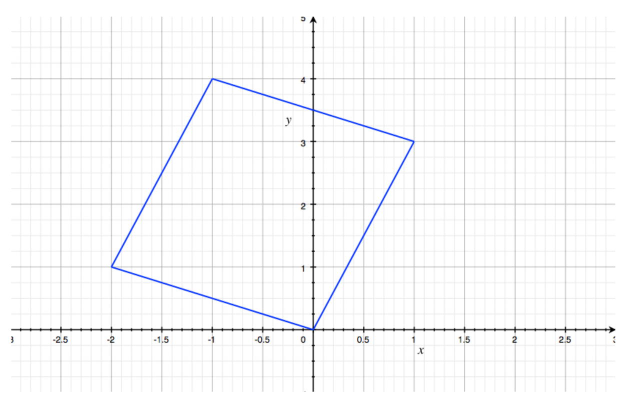
$$\begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} x - 2 \\ 3x + 1 \end{pmatrix} \implies x' = x - 2 \text{ and } y' = 3x + 1 \implies y' = 3x' + 7$$

last time

$$\begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} -2y \\ y \end{pmatrix} \implies x' = -2y \text{ and } y' = y \implies y' = -\frac{x'}{2}$$

If we draw these lines we see that the pre-image and image look something like





$\square$

3.1 #61 Proof Suppose that T is a linear transformation and that $T(u_1)$ and $T(u_2)$ are linearly independent. Prove that u_1 and u_2 must be linearly independent.

Solution Suppose for that sake of contradiction that u_1 and u_2 are linearly dependent i.e $u_1 = au_2$ with $a \in \mathbb{R}$. Since T is linear we see that

$$u_1 = au_2 \implies T(u_1) = aT(u_2)$$

but this means that $T(u_1)$ and $T(u_2)$ are linearly dependent, contradiction. Therefore u_1 and u_2 are linearly independent. □

3.2 # 13 Let T_1 and T_2 be linear transformations given by

$$T_1\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right] = \begin{pmatrix} 3x_1 + 5x_2 \\ -2x_1 + 7x_2 \end{pmatrix}$$

$$T_2\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right] = \begin{pmatrix} -2x_1 + 9x_2 \\ -5x_2 \end{pmatrix}$$

Find the matrix A such that

$$a)\quad T_1(T_2(x)) = Ax, \quad b)\quad T_2(T_1(x)) = Ax, \quad c)\quad T_1(T_1(x)) = Ax, \quad d)\quad T_2(T_2(x)) = Ax$$

Solution We see that

$$T_1(x) = A_1 x \quad \text{where} \quad A_1 = \begin{pmatrix} 3 & 5 \\ -2 & 7 \end{pmatrix}$$

$$T_2(x) = A_2 x \quad \text{where} \quad A_2 = \begin{pmatrix} -2 & 9 \\ 0 & -5 \end{pmatrix}$$

Thus

$$T_1(T_2(x)) = Ax \implies \boxed{A = A_1A_2 = \begin{pmatrix} -6 & 2 \\ 4 & -53 \end{pmatrix}}$$

$$T_2(T_1(x)) = Ax \implies \boxed{A = A_2A_1 = \begin{pmatrix} -24 & 53 \\ 10 & -35 \end{pmatrix}}$$

$$T_1(T_1(x)) = Ax \implies \boxed{A = A_1^2 = \begin{pmatrix} -1 & 50 \\ -20 & 39 \end{pmatrix}}$$

$$T_2(T_2(x)) = Ax \implies \boxed{A = A_2^2 = \begin{pmatrix} 4 & -63 \\ 0 & 25 \end{pmatrix}}$$

□