Tutorial #10

MAT 188 – Linear Algebra I – Fall 2015

Solutions

Problems (Please note these are from Holt's Linear Algebra Text)

Question 1 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = Ax$ for the following matrices

$$a)A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad b)A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad c)A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

In each case find the eigenvalues and eigenvectors of A, if possible, and interpret your results in terms of the transformation T.

Solution For $a)$ we see the characteristic equation is given by

$$P(\lambda) = \det(A - 1\lambda) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1$$

Thus the eigenvalues for the equation are given by the roots $P(\lambda) = 0$, namely $\lambda_\pm = \pm 1$. Then eigenvectors are found by checking the kernel. We see

$$\lambda_+ = 1 \implies \ker \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \text{span} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \implies \vec{\lambda}_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_- = -1 \implies \ker \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \text{span} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \implies \vec{\lambda}_+ = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

We rinse and repeat for $b)$

$$P(\lambda) = \det(A - 1\lambda) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = (\lambda - 1)(\lambda + 1)$$

Thus the eigenvalues for the equation are given by the roots $P(\lambda) = 0$, namely $\lambda_\pm = \pm 1$. Then eigenvectors are found by checking the kernel. We see

$$\lambda_+ = 1 \implies \ker \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} = \text{span} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies \vec{\lambda}_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda_- = -1 \implies \ker \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \text{span} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \implies \vec{\lambda}_+ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We rinse and repeat for $c)$

$$P(\lambda) = \det(A - 1\lambda) = \frac{1}{2} \begin{vmatrix} 1-\sqrt{2}\lambda & -1 \\ 1 & 1-\sqrt{2}\lambda \end{vmatrix} = \lambda^2 - \sqrt{2}\lambda + 1$$

Thus the eigenvalues for the equation are given by the roots $P(\lambda) = 0$, namely $\lambda_{\pm} = (1 \pm i)/\sqrt{2}$. Then eigenvectors are found by checking the kernel. We see

$$\lambda_+ = \frac{1+i}{\sqrt{2}} \implies \ker \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} = \operatorname{span} \begin{pmatrix} 1 \\ i \end{pmatrix} \implies \vec{\lambda}_+ = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\lambda_- = \frac{1-i}{\sqrt{2}} \implies \ker \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} = \operatorname{span} \begin{pmatrix} 1 \\ -i \end{pmatrix} \implies \vec{\lambda}_+ = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

□

6.1 - # 52 Prove that if u is an eigenvector of A, then u is also an eigenvector of A^2.

Solution Assume u is an eigenvector with eigenvalue λ, then

$$A^2 u = A(Au) = A(\lambda u) = \lambda A u = \lambda^2 u$$

Thus we see it is an eigenvector of A^2 with eigenvalue λ^2. □

6.1 - # 57 Let A be an invertible matrix. Prove that if λ is an eigenvalue of A with associated eigenvector u, then λ^{-1} is an eigenvalue of A^{-1} with associated eigenvector u.

Solution Simply multiply Au by A^{-1}, we see

$$Au = \lambda u \implies A^{-1}Au = A^{-1}(\lambda u) \implies u = \lambda A^{-1} u \implies A^{-1} u = \lambda^{-1} u$$

Which is the claim. □

6.1 - #61 Suppose that the entries of each row of a square matrix A add to zero. Prove that $\lambda = 0$ is an eigenvalue of A.

Solution The entries of each row add to zero, thus we have that $u = (1, \dots, 1)$ is an eigenvector for 0, since

$$Au = 0u = 0$$

□

6.1 - # 63 Suppose that A is a square matrix. Prove that if λ is an eigenvalue of A, then λ is also an eigenvalue of A^T.

Solution Recall that $\det(B) = \det(B^T)$, thus we see the equation for the characteristic equation satisfies

$$P(\lambda) = \det(A - 1\lambda) = \det(A^T - 1\lambda)$$

Thus the eigenvalues for A and A^T are identical since the characteristic equation is identical. □