Contents

Chapter 1

Liouville Theorem On Integrable Systems

1.1 Hamiltonian Systems

Let $H(p,q)$ be a real function, $p = (p_1,\ldots,p_n), q = (q_1,\ldots,q_n)$, where $p,q \in \mathbb{R}^n$. A Hamiltonian vector-field is defined as

$$(-\partial_q H, \partial_p H) = (-\partial_{q_1} H, \ldots, -\partial_{q_n} H, \partial_{p_1} H, \ldots, \partial_{p_n} H)$$

The function H itself is called in this context the *Hamiltonian*
The ODE system

$$\dot{p} = -\partial_q H, \qquad \dot{q} = \partial_p H$$

is called a *Hamiltonian System* . The origin of Hamiltonian Mechanics goes back to Newtonian Mechanics

$$\ddot{x} = \mathbb{F}(x), \quad x \in \mathbb{R}^n$$

when the force $\mathbb{F}$ is generated via some potential $U(x)$, i.e.

$$\mathbb{F}(x) = -\nabla U(x)$$

Setting here

$$x = q, \quad \dot{x} = p, \quad H(p,q) = \frac{p^2}{2} + U(q)$$

one arrives at

$$\dot{p} = \ddot{x} = -\nabla U(x) = -\nabla U(q) = -\partial_q H$$

$$\dot{q} = p = \partial_p H$$

For instance, Newton's Gravitational Law for two bodies defines the force via

$$\mathbb{F} = -\frac{const}{|r|^2}\frac{r}{|r|}$$

where r stands for the displacement vector for the location of the second body in relation to the first. The potential here is

$$U(r) = \frac{const}{|r|}$$

This is a particular case of a *central* field which is

$$\mathbb{F}(r) = \phi(r)\frac{r}{|r|}$$

Where $\phi(r)$ is a scalar function.Central field motion has a remarkable feature: The vector

$$L = [r, \dot{r}]$$

called the *angular momentum* remains conserved along each trajectory, i.e.

$$\frac{d}{dt}[r(t), \dot{r}(t)] = 0$$

(For obvious reason). Since each component of L is conserved, one has here n scalar function which are conserved under the flow.

Definition 1.1. A real function which is conserved under the flow is called a *conservation law* or a *first integral* of the system.

1.2 Commuting Vector-Fields and Poisson Brackets

Consider two ODE systems

$$\dot{x} = A(x), \quad \dot{x} = B(x), \quad x \in \mathbb{R}^n$$

Denote by $g_A(x_0, t)$, respectively $g_B(x_0, t)$, the solution of the system with initial condition $g_A(x_0, 0) = x_0$, respectively $g_B(x_0, 0) = x_0$. One says the vector-fields A and B commute if for any $x_0, s, t > 0$ the following equality holds

$$g_B(g_A(x_0, s), t) = g_A(g_B(x_0, t), s)$$

We also say that the flows g_A, g_B commute. To measure "how large" is the *commutator* of two vector-fields A, B we use the *Poisson Bracket* which is defined as follows:

$$[A, B]_j = \sum_{i=1}^{n} B_i \partial_{x_i} A_j - A_i \partial_{x_i} B_j$$

$$[A, B] = ([A, B]_j)_{1 \leq j \leq n}$$

By direct calculation, one has the following

Theorem 1.1. *The vector fields A, B commute if and only if*

$$[A, B] = 0$$

Note also that the Poisson Bracket obeys the *Jacobi Identity*:

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$$

1.3 Poisson Bracket of Hamiltonians and First Integrals

Let $F(p, q), H(p, q)$ with $p, q \in \mathbb{R}^n$ be real functions. Consider the Hamiltonian vector-fields

$$A = (-\partial_q F, \partial_p F), \quad B = (-\partial_q H, \partial_p H)$$

By a direct calculation one has the following

Theorem 1.2.

$$[A, B] = (-\partial_q G, \partial_p G)$$

where

$$G(p, q) = \sum_{i=1}^{n} \partial_{p_i} H \partial_{q_i} F - \partial_{p_i} F \partial_{q_i} H$$

Thus, the poison bracket of Hamiltonian vector-fields is a Hamiltonian vector-field.

Definition 1.2. The function G is called the Poisson Bracket of the Hamiltonians F, H. It is denoted via

$$G = (F, H)$$

By direct calculation, one can verify the following formula

$$(F, H)(x) = \frac{d}{dt} F\left(g_H^t(x)\right)\Big|_{t=0}$$

where $g_H^t(x)$ stands for the follow of the Hamiltonian vector-field $A = (-\partial_q H, \partial_p H)$. This implies the following:

Corollary 1.1. *The function $F(p, q)$ is a first integral of the Hamiltonian H if and only if $(F, H) = 0$.*

Note that $(H, H) = 0$, thus H is a first integral. Of course, this can be checked via simple calculation:

$$\frac{d}{dt} H(p(t), q(t)) = \partial_p H \dot{p} + \partial_q H \dot{q} = -\partial_p H \partial_q H + \partial_q H \partial_p H = 0$$

1.4 Liouville Theorem

Definition 1.3. Two functions F_1, F_2 are *involutions* if $[F_1, F_2] = 0$

Consider the Hamiltonian system

$$\dot{p} = -\partial_q H, \quad \dot{q} = \partial_p H, \quad p, q \in \mathcal{D} \subset \mathbb{R}^n$$

Let $F_1 = H$ and assume that there are first integrals $F_2, \dots, F_n$ of this system such that the following conditions hold

- $(F_i, F_j) = 0$ for all i, j
- $F_1, \dots, F_n$ are independent in $\mathcal{D}$, i.e. the rank of their Jacobian matrix is equal to n.

It is convenient in this context to denote $(p, q) \in \mathbb{R}^{2n}$ via x and the flow with given Hamiltonian F via $g_F^t(x)$.
Take arbitrary $x^0 \in \mathcal{D}$. Set $f_j = F_j(x^{(0)})$, $f = (f_1, \dots, f_n)$,

$$M_f = \{x \in \mathcal{D} : F_j(x) = f_j, j = 1, \dots, n\}$$

Since the Jacobian matrix of $F_1, \dots, F_n$ has rank n, M_f has a structure of a n-dimensional manifold in $\mathbb{R}^{2n}$. Assume that

- M_f is compact and connected.

Note that since F_j's are first integrals M_f is invariant under the flow g^t. Liouville Theorem states that there is a diffeomorphism $\phi : M_f \to \mathbb{T}^n$ which conjugates the flow g^t with the linear flow on the torus $\mathbb{T}^n$

$$\dot{\varphi} = \omega$$

where $\omega \in \mathbb{R}^n$ is a fixed vector.
To prove this theorem note first of all that $M := M_f$ is invariant under $g_{F_j}^t$, $j = 1, \dots, n$. Moreover, the flow commute. Set

$$g^{(t_1, \dots, t_n)}(x) = g_1^{t_1} g_2^{t_2} \dots g_n^{t_n}(x), \quad \mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}^n$$

Then $\mathbf{t} \to g^{\mathbf{t}}$ is an *action* of $\mathbb{R}^n$ on M, i.e.

$$g^{\mathbf{t}+\mathbf{s}} = g^{\mathbf{t}} g^{\mathbf{s}}$$

due to the commutativity of $g_j^{t_j}$. Fix $x_0 \in M$ and set

$$g : \mathbb{R}^n \to M, \quad g(\mathbf{t}) = g^{\mathbf{t}}(x_0)$$

Definition 1.4. The stationary group of x_0 is defined via

$$\Gamma = \left\{\mathbf{t} \in \mathbb{R}^n : g^{\mathbf{t}}(x_0) = x_0\right\}$$

Clearly $\Gamma \subset \mathbb{R}^n$ is a subgroup.

1. Let N be a smooth submanifold in $\mathbb{R}^m$. That means for each $x_0 \in N$ there is a local chart $\phi : U_0 \to N$, where $U_0 = \{y \in \mathbb{R}^d : |y| < r_0\}$, is a smooth map from U_0 into $\mathbb{R}^m$. In this case the tangent space T_x can be identified with a linear subspace $\mathcal{J}_{x_0} \subset \mathbb{R}^m$, $\dim \mathcal{J}_{x_0} = d =$ the dimension of N.

2. Let $G(x)$ be a vector-field in $\mathbb{R}^m$ and let g^t be the flow defined via G. Let N be as in (1). Assume that N is invariant under the flow. In this case $G(x) \in \mathcal{J}_x$ for each $x \in L$.

3. Let us go back to the setting in the proof of Liouville Theorem. Let $x_0 \in M$ be arbitrary. Consider the map $g(\mathbf{t}) = g^{\mathbf{t}}(x_0)$, then

$$\partial_{t_j} g\Big|_{\mathbf{t}=0} = A_j(x_0)$$

where $A_j(x_0)$ is the Hamiltonian vector-field defined via F_j, i.e.

$$A_j(x_0) = (-\partial_q F_j, \partial_p F_j)\Big|_{x_0}$$

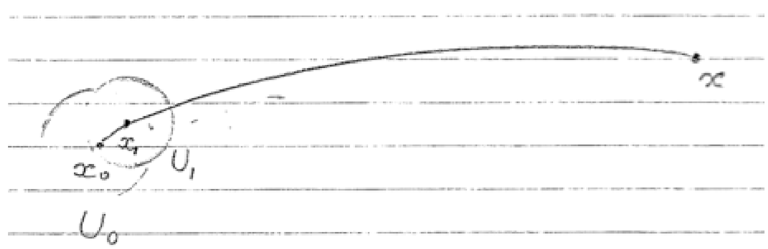
4. The rank of the system $A_1(x_0), \ldots, A_n(x_0)$ is equal to n. Indeed, we know that the rank of the system $(\partial_q F_j, \partial_q F_j), j = 1, \ldots, n$ is n. The linear map defined via the $2n \times 2n$ matrix

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

is invertible. J transforms the second system into the first one.

5. Due to (3),(4) the rank of the system $\partial_{t_1} g\Big|_{\mathbf{t}=0}, \ldots, \partial_{t_n} g\Big|_{\mathbf{t}=0}$ is equal to n. Thus, locally g is a diffeomorphism of a neighbourhood $\mathbf{t} = 0$ onto a neighbourhood $U_{x_0} \subset M$.

6. Since M is connected and g is onto. Indeed, since the map $g^{\mathbf{t}}(x)$ is an action. The statement follows from the following figure:



Here $x_0 \to x$ is a arbitrary path in M connecting x_0 with x,

$$x_1 = g^{t_1}(x_0), x_2 = g^{t_2}(x_1), \ldots, x = g^{t_r}(x_{r-1})$$

7. Due to (6), one concludes that the stationary group Γ does not depend on x_0, the group Γ is discrete, i.e. there exists a neighbourhood U_0 of $\mathbf{t} = 0$ such that $\Gamma \cap U_0 = \{0\}$.

8.

Lemma 1.5. *Let Γ be a discrete subgroup of $\mathbb{R}^n$. Then one can find linearly independent vectors $e_1, \ldots, e_k \in \Gamma$ such that*

$$\Gamma = \{y : y = \sum_{j=1}^{k} m_j e_j, m_j \in \mathbb{Z}\}$$

Proof. Let $e_0 \in \Gamma, e_0 \neq 0$. There exists $e_1 \in \Gamma$ such that $e_1 = \lambda_1 e_0$ where $\lambda_1 \in \mathbb{R}$, and

$$|e_1| = \min\{|e| : e \in \Gamma \cap \mathbb{R}e_0\}$$

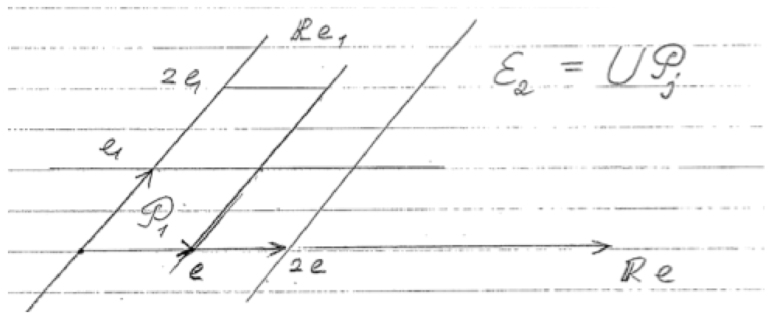
Moreover,

$$\Gamma \cap \mathbb{R}e_0 = \{m_1 e_1 : m_1 \in \mathbb{Z}\}$$

If $\Gamma = \Gamma \cap \mathbb{R}e_0$ then we are done. Let $e \in \Gamma \setminus \mathbb{R}e_1$. Consider

$$\mathcal{E}_2 = \mathbb{R}e_1 + \mathbb{R}e = \{y = \lambda_1 e_1 + \lambda e : \lambda_1, \lambda \in \mathbb{R}\}$$

Split $\mathcal{E}_2$ into the "fundamental parallelograms" as in the following figure



In $\bar{\mathcal{P}}_1$ find $e_2 \in \Gamma \setminus \mathbb{R}e_1$ which is the closest one to the line $\mathbb{R}e_1$. It may happen that $e_2 = e$. Note that

$$\operatorname{dist}(e_2, \mathbb{R}e_1) = \min\{\operatorname{dist}(y, \mathbb{R}e_1) : y \in \mathcal{E}_2 \cap \Gamma \setminus \mathbb{R}e_1\}$$

Indeed, let $\tilde{e} \in \mathcal{E}_2 \cap \Gamma \setminus \mathbb{R}e_1$, $\operatorname{dist}(\tilde{e}, \mathbb{R}e_1) < \operatorname{dist}(e_2, \mathbb{R}e_1)$. Let $\tilde{e} = \lambda_1 e_1 + \lambda e$. Let for instance $\lambda_1 \geq 1$. Let $m_1 = [\lambda_1]$, $\mu_1 = \{\lambda_1\}, \hat{e} = \mu_1 e_1 + \lambda e$. Then clearly

$$\operatorname{dist}(\hat{e}, \mathbb{R}e_1) = \operatorname{dist}(\tilde{e}, \mathbb{R}e_1)$$

$$\hat{e} = \tilde{e} - m_1 e_1 \in \Gamma$$

So, we can assume $\tilde{e} = \lambda_1 e_1 + \lambda e, 0 \leq \lambda_1 \leq 1$. One can see also that $-1 \leq \lambda \leq 1$. Using reflection one can assume $0 \leq \lambda \leq 1$. Thus, $\tilde{e} \in \mathcal{P}_1$. Hence

$$\operatorname{dist}(\tilde{e}, \mathbb{R}e_1) = \operatorname{dist}(e_2, \mathbb{R}e_1)$$

Note that in any event

$$\mathcal{E}_2 = \mathbb{R}e_1 + \mathbb{R}e_2 = \{y = \lambda_1 e_1 + \lambda_2 e_2 : \lambda_1, \lambda_2 \in \mathbb{R}\}$$

Once again using the fundamental domains with e_2 ini the role of e_1 we conclude that no point of Γ can fall into the interior of $\mathcal{P}_1$ and neither into the interior of any $\mathcal{P}_j$. That means

$$\Gamma \cap \mathcal{E}_2 = \{yLy = m_1 e_1 + m_2 e_2 : m_1, m_2 \in \mathbb{Z}\}$$

If $\Gamma = \Gamma \cap \mathcal{E}_2$ then we are done. Otherwise we proceed with a similar argument. $\square$

9. Let $\Gamma \subset \mathbb{R}^n$ be a discrete subgroup. Consider the quotient group $\mathbb{R}^n/\Gamma$. Let $\Gamma = \{m_1 e_1 + \ldots + m_n e_n : m_j \in \mathbb{Z}\}$. If $k = n$, then $\mathbb{R}^n/\Gamma$ is diffeomorphic to the torus $\mathbb{T}^n$. If $k < n$, then $\mathbb{R}^n/\Gamma$ is

diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k}$.

Proof. Assume $k < n$. Consider

$$\mathcal{E}_k = \{y = \lambda_1 e_1 + \ldots + \lambda_k e_k : \lambda_j \in \mathbb{R}\}$$

Clearly one can assume that

$$\mathcal{E}_k = \{y = (y_1, \ldots, y_k, 0, \ldots, 0) : y_j \in \mathbb{R}, \quad \mathbb{R}^n = \mathcal{E}_k \times \mathbb{R}^{n-k}$$

So, since $\Gamma \subset \mathcal{E}_k$

$$\mathbb{R}^n/\Gamma = (\mathcal{E}_k/\Gamma) \times \mathbb{R}^{n-k}$$

Assume for simplicity $k = 2$. Then

$$\mathcal{E}_2/\Gamma = \{\lambda_1 e_1 + \lambda_2 e_2 : 0 \leq \lambda_1, \lambda_2 \leq 1\}^x$$

where x stands for the identification of the points on the edges

$$\lambda_2 e_2 = e_1 + \lambda_2 e_2, \quad \lambda_1 e_1 = \lambda_1 e_1 + e_2$$

Clearly $\mathcal{E}_2/\Gamma$ is diffeomorphic to the torus $\mathbb{T}^2$. □

10. The map $\phi : [\mathbf{t}]_\Gamma \to g^{\mathbf{t}}(x_0)$ is a diffeomorphism from $\mathbb{R}^n/\Gamma$ onto M.

    The map is well-defined. Indeed, if $\mathbf{t} = \mathbf{s} \mod \Gamma$ then $g^{\mathbf{t}-\mathbf{s}}(x_0) = x_0, i.e g^{\mathbf{t}}(x_0) = g^{\mathbf{s}}(x_0)$. Clearly the map is smooth. If $g^{\mathbf{t}}(x_0) = g^{\mathbf{s}}(x_0)$ then $\mathbf{t} = \mathbf{s} \mod \Gamma$, i.e the map is injective. We know that $\mathbf{t} \to g^{\mathbf{t}}(x_0)$ is onto the manifold M. So, ϕ is one-to-one on M.

11. Since we assume that M is compact, $\mathbb{R}^n/\Gamma$ can not have a non-compact factor $\mathbb{R}^{n-k}$, so $k = n$, and $\mathbb{R}^n/\Gamma \simeq \mathbb{T}^n$. Thus, ϕ^{-1} is a diffeomorphism from M onto $\mathbb{T}^n$.

12. Finally, consider the H-flow which is $g^{(t,0,\ldots,0)}(x_0), t \in \mathbb{R}$. The diffeomorphism ϕ^{-1} conjugates the flow with the flow

    $$(t_1, \ldots, t_n) \mod \Gamma \to (t_1 + t, t_2, \ldots, t_n) \mod \Gamma$$

    Again for simplicity consider $n = 2$, $\mathbb{R}^2 = \mathcal{E}_2$. Let (ω_1, ω_2) be the components of the standard basic vector $(1, 0)$ with respect to the lattice basis $e_1, e_2 \in \Gamma$. Then in the angular coordinates φ_1, φ_2 on the torus the infinitesimal flow is

    $$(\varphi_1, \varphi_2) \to (\varphi_1, \varphi_2) + (\omega_1, \omega_2)dt$$

    $$\dot{\varphi_1} = \omega_1, \quad \dot{\varphi_2} = \omega_2$$

Chapter 2

Lax Theorem On The Korteweg-de-Vries Equation

The Korteweg-de-Vries Equation (KdV) is given by

$$u_t + uu_x + u_{xxx} = 0 \tag{2.1}$$

This is an evolution equation in the sense that

$$u_t = F(u, u_x, \ldots)$$

A first integral is a functional $I(u, u_x, \ldots)$ which value is conserved along the flow of the equation. In 1968, Gardner, Kruskal and Miura discovered that KDV has infinitely many conservation laws. Here are the first three integrals they discovered:

$$I_1(u) = \int_{\mathbb{R}} u^2 dx$$
$$I_2(u) = \int_{\mathbb{R}} \left(\frac{u^2}{3} - u_x^2 \right) dx$$
$$I_3(u) = \int_{\mathbb{R}} \left(\frac{1}{4}u^4 - 3uu_x^2 + \frac{9}{5}u_{xx}^2 \right) dx$$

In the same year Lax found the following fundamental mechanism built into the KdV equation. Consider the Sturm-Liouville Operator

$$\mathcal{L}(y) = -y'' + vy$$

Taking here $v = \frac{1}{6}u(t, x)$ where u is a solution of KdV one obtains a one parameter family $\mathcal{L}^{(t)}$ of linear operators. Lax theorem says that $\mathcal{L}^{(t)}$ is unitary conducted to $\mathcal{L}^{(0)}$. In particular the spectrum of $\mathcal{L}^{(t)}$ is the same as the spectrum of $\mathcal{L}^{(0)}$. Here is the derivation of Lax theorem.

Let $L(t)$ be a one-parameter family of self-adjoint operating acting in the Hilbert space. Lax suggests to find a condition which will allow to conjugate $L(t)$ and $L(0)$ via some unitary operator $U(y)$,i.e.

$$L(0) = U(t)^{-1}L(t)U(t)$$

Assuming differentiability, one obtains

$$\partial_t(U^{-1}LU) = -U^{-1}(\partial_t U)U^{-1}LU + U^{-1}\partial L_t U + U^{-1}L\partial_t U = 0$$

The idea is to set

$$U(t) = \exp(itA), \quad \text{with} \quad A^* = A$$

It is more convenient to set $B = iA$,

$$U(t) = \exp(tB), \quad \text{with} \quad B^* = -B$$

Then U obeys

$$\partial_t U = BU$$

which leads to

$$-BL + \partial_t L + BL = 0$$

Thus if $L(t)$ obeys

$$\partial_t L = BL - LB$$

with $B^* = -B$, then $L(t)$ indeed is unitary conjugated to $L(0)$. Take

$$L(t) = \partial_{xx}^2 + \frac{1}{6}u$$

Then

$$\partial_t L = \frac{1}{6}\partial_t u$$

Take $B_0 = \partial_x$. Then $B^* = B$,

$$[B_0, L] = \frac{1}{6}\partial_x u$$

Thus, if

$$\partial_t u = \frac{1}{6}\partial_x u$$

then $L(t)$ is unitary conjugate to $L(0)$. Next choose

$$B = 24\partial_{xxx}^3 + 3u\partial_x + 3\partial_x u$$

Then

$$[B, L] = -\partial_{xxx}^3 u - u\partial_x u$$

which leads to

$$\partial_t u = \partial_t L = [B, L] = -\partial_{xxx}^3 u - u\partial_x u$$

which is KdV. It turns out that KdV is a completely integrable infinite dimensional Hamiltonian system.

Chapter 3

Fundamental Solutions

3.1 The Variation of Parameters Method

Consider the Sturm-Liouville Equation

$$-y'' + qy = \lambda y \tag{3.1}$$

This is a linear ODE. It has two fundamental solutions y_1, y_2 defined via their initial data

$$\begin{aligned} y_1\Big|_{x=0} = 1 \quad y_1'\Big|_{x=0} = 0 \\ y_2\Big|_{x=0} = 0 \quad y_2'\Big|_{x=0} = 1 \end{aligned}$$

Any given solution y of is given via

$$y = C_1 y_1 + C_2 y_2, \quad \text{where} \quad C_1 = y\Big|_{x=0} \quad C_2 = y'\Big|_{x=0}$$

Here we want to view y_1, y_2 as functions of x, λ, q. We denote them as $y_1(x, \lambda, q), y_2(x, \lambda, q)$. For technical reasons it is convent to run q in the space $L^2_{\mathbb{C}}[0, 1]$, the space of all complex esquire integrable functions on $[0, 1]$. We want to develop series expansions of $y_1(x, \lambda, q), y_2(x, \lambda, q)$ following the Picard iteration method.

Theorem 3.1. *Let $f \in L^2_{\mathbb{C}}, a, b \in \mathbb{C}$. Set*

$$c_\lambda(x) = \cos(\sqrt{\lambda}x), \quad s_\lambda(x) = \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}, \quad y_f(x) = \int_0^x s_\lambda(x - t) f(t) dt$$

$$y(x) = a c_\lambda(x) + b s_\lambda(x) + y_f(x)$$

The function y is the unique solution of the equation

$$y'' = -\lambda y + f, \quad y(0) = a, \quad y'(0) = b \tag{3.2}$$

Proof. One has

$$y_f(x) = s_\lambda(x) \int_0^x c_\lambda(t) f(t) dt - c_\lambda(x) \int_0^x s_\lambda(t) f(t) dt$$

$$y_f'(x) = c_\lambda(x) \int_0^x c_\lambda(t) f(t) dt + \lambda s_\lambda(x) \int_0^x s_\lambda(t) f(t) dt$$

$$y_f''(x) = -\lambda s_\lambda(x) \int_0^x c_\lambda(t) f(t) dt + \lambda c_\lambda(x) \int_0^x s_\lambda(t) f(t) dt + \left(c_\lambda(x)^2 + \lambda s_\lambda(x)^2\right) f(x) = -\lambda y_f(x) + f(x)$$

$$y_f(0) = 0, \quad y_f'(0) = 0$$

Since

$$c_\lambda'' = -\lambda c_\lambda, \quad s_\lambda'' = -\lambda s_\lambda$$

$$c_\lambda(0) = 1, \quad c_\lambda'(0) = 0; \quad s_\lambda(0) = 0, \quad s_\lambda'(0) = 1$$

y obeys (3.1) . Let $\tilde{y}$ be another solution of (3.1). Then $v = y - \tilde{y}$ obeys

$$-v'' = \lambda v, \quad v(0) = 0, \quad v'(0) = 0$$

That implies $v(x) = 0$ for all x. Thus y is unique. □

3.2 The Volterra Integral Method

Recall the following simple fact on Volterra Integral Equations:
Let $K(x,t)$ be a function, $\alpha \leq x, t \leq \beta$, with

$$C \equiv \sup_x |K(x,t)| < +\infty$$

Consider the Volterra Integral Operator

$$[Ty](x) = \int_\alpha^x K(x,t) y(t) dt, \quad \alpha \leq x \leq \beta$$

Then

$$||T^n|| \leq \frac{C^n(\beta - \alpha)^n}{n!}$$

In particular, the integral equation

$$y - Ty = h$$

has a unique solution

$$y = \sum_{n=0}^{\infty} T^n h$$

Theorem 3.2. *Let y_1 be the unique solution of the integral equation:*

$$y(x) = c_\lambda(x) + \int_0^x s_\lambda(x-t) q(t) y(t) dt \tag{3.3}$$

Then y obeys the Sturm-Liouville Equation (3.1) *with $y_1(0) = 1, y_1'(0) = 0$. Similarly, let y_2 be the* *unique solution of the integral equation*

$$y(x) = s_\lambda(x) + \int_0^x s_\lambda(x-t)q(t)y(t)dt \tag{3.4}$$

Then y *obeys* (3.1) *with* $y_2(0) = 0, y_2'(0) = 1$

Proof. This is a straight forward calculation for (3.3)

$$y'(x) = -\lambda s_\lambda(x) + \int_0^x c_\lambda(x-t)q(t)y(t)dt$$

$$y''(x) = -\lambda c_\lambda(x) + q(x)y(x) - \int_0^x \lambda s_\lambda(x-t)q(t)dt = -\lambda y(x) + q(x)y(x)$$

$$y(0) = 1, \quad y'(0) = 0$$

The case for y_2 is similar. □

3.3 The Non-Homogeneous Solution

Given $u(x), v(x)$, the Wronskian $[u, v]$ is defined via

$$[u, v](x) = u(x)v'(x) - u'(x)v(x)$$

If u, v obey (3.1) then $[u, v]' = 0$, i.e. $[u, v]$ does not depend on x. In particular, $[y_1, y_2](x) = 1$ for any x.

Theorem 3.3. *Let* $f \in L^2_{\mathbb{C}}, a, b \in \mathbb{C}$. *The equation*

$$-y'' + qy = \lambda y - f, \quad y(0) = a, \quad y'(0) = b$$

has unique solution

$$y(x) = ay_1 + by_2 + \int_0^x (y_1(t)y_2(x) - y_1(x)y_2(t))f(t)dt$$

Proof. Just as in theorem 3.1, the formula comes from the Cauchy method of the coefficients variation. Instead of doing the Cauchy method, one can verify the identity directly like in theorem 3.1:

$$y' = ay_1' + by_2' + \int_0^x (y_1(t)y_2'(x) - y_1'(x)y_2(t))f(t)dt$$

$$y'' = ay_2'' + by_2'' + (y_1(x)y_2'(x) - y_1'(x)y_2(x))f(x) + \int_0^x ((y_1(t)y_2''(x) - y_1''(x)y_2(t))\, f(t)dt$$

Note that $y_1(x)y_2'(x) - y_1'(x)y_2(x) = 1$. Substituting here $y_j'' = (q-\lambda)y_j$, one obtains

$$y'' = (q-\lambda)(ay_1 + by_2) + f + (q-\lambda)\int_0^x ((y_1(t)y_2(x) - y_1(x)y_2(t))f(t)dt$$

□

3.4 Basic Estimates

Here we want to develop "series expansion" for y_1, y_2.

$$y_1(x) = c_\lambda(x) + \sum_{n\geq 1} C_n(x,\lambda,q), \quad x \geq 0$$

$$C_n(x,\lambda,q) = \int_{0\leq t_1\leq \ldots \leq t_n \leq x} c_\lambda(t_1) \prod_{i=1}^{n} \left[s_\lambda(t_{i+1}-t_i)q(t_i)\right] dt_1 \ldots dt_n, \quad t_{n+1} \equiv x$$

$$|C_n(x,\lambda,q)| \leq \exp\left(|\Im\lambda| x\right) \int_{0\leq t_1\leq \ldots \leq t_n \leq x} \prod_{i=1}^{n} |q(t_i)| dt_1 \ldots dt_n$$

$$\begin{aligned}\int_{0\leq t_1\leq \ldots \leq t_n \leq x} \prod_{i=1}^{n} |q(t_i)| dt_1 \ldots dt_n &= \frac{1}{n!} \int_{[0,x]^n} \prod_{i=1}^{n} |q(t_i)| dt_1 \ldots dt_n \\ &= \frac{1}{n!} \left(\int_0^x |q(t)| dt \right)^n \\ &\leq \frac{(||q||_{L^2}\sqrt{x})^n}{n!}\end{aligned}$$

Combining the above inequalities gives

$$|C_n(x,\lambda,q)| \leq \exp(|\Im\lambda| x) \frac{(||q||_{L^2}\sqrt{x})^n}{n!}$$

This estimate shows that the statements in Theorem 3.2 and the series expansion hold for $q \in L^2$. Similarly,

$$y_2(x,\lambda,q) = s_\lambda(x) + \sum_{n\geq 1} S_n(x,\lambda,q)$$

$$S_n(x,\lambda,q) = \int_{0\leq t_1\leq \ldots \leq t_n \leq x} s_\lambda(t_1) \prod_{i=1}^{n} \left[s_\lambda(t_{i+1}-t_i)q(t_i)\right] dt_1 \ldots dt_n, \quad t_{n+1} \equiv x$$

$$|S_n(x,\lambda,q)| \leq \exp(|\Im\lambda| x) \frac{(||q||_{L^2}\sqrt{x})^n}{n!}$$

Note also that

$$|y_1(x,\lambda,q)|, |y_2(x,\lambda,q)| \leq \exp(|\Im\sqrt{\lambda}| x + ||q||_{L^2}\sqrt{x})$$

Furthermore, one can see that the derivation works also for $\lambda = 0$ with $s_0(x) = x$, $c_0(x) = 1$. On the other hand $y_j(x,\lambda,q) = y_j(x,0,q-\lambda)$. Thus

$$C_n(x,\lambda,q) = \int_{0\leq t_1\leq \ldots \leq t_n \leq x} \prod_{i=1}^{n} \left[(t_{i+1}-t_i)(q(t_i)-\lambda)\right] dt_1 \ldots dt_n$$

$$S_n(x,\lambda,q) = \int_{0\leq t_1\leq \ldots \leq t_n \leq x} t_1 \prod_{i=1}^{n} \left[(t_{i+1}-t_i)(q(t_i)-\lambda)\right] dt_1 \ldots dt_n$$

We have the following basic estimates:

$$\left|y_1(x,\lambda,q)-\cos(\sqrt{\lambda}x)\right| \le \frac{\exp(|\Im\sqrt{\lambda}|x+||q||)}{\sqrt{|\lambda|}}$$

$$\left|y_2(x,\lambda,q)-\frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}\right| \le \frac{\exp(|\Im\sqrt{\lambda}|x+||q||)}{\sqrt{|\lambda|}}$$

$$\left|\partial_x y_1+\sqrt{\lambda}\sin(\sqrt{\lambda}x)\right| \le ||q||\exp(|\Im\sqrt{\lambda}|x+||q||)$$

$$\left|\partial_x y_2-\cos(\sqrt{\lambda}x)\right| \le \frac{||q||}{\sqrt{\lambda}}\exp(|\Im\sqrt{\lambda}|x+||q||)$$

Proof. Due to the series expansion

$$|y_1-\cos(\sqrt{\lambda}x)| \le \sum_{n\ge 1}|C_n(x,\lambda,q)|$$

This implies the first estimate. The derivation of the rest of the estimates is similar. □

3.5 Derivatives in λ and q

Let $\mathcal{H}$ be a Hilbert space. Let $u_0 \in \mathcal{H}$ and let $f(u)$ be a complex valued function defined in the ball $B(u_0,r_0)=\{u : ||u-u_0||<r_0\}$. Let $u_0\in\mathcal{H}$. If there exists $\phi(u_0)\in\mathcal{H}$ such that

$$f(u)-f(u_0)=\left(u-u_0,\overline{\phi(u_0)}\right)+o(||u-u_0||),$$

then $f(u)$ is called complex differentiable at $u=u_0$ and $\phi(u_0)$ is called the gradient, $\partial_u f\Big|_{u=u_0}=\phi(u_0)$. Real derivatives are defined similarly. If $\mathcal{B}$ is a Banach space then the gradient $\partial_u f\Big|_{u=u_0}$ defined as a vector in the dual space $\mathcal{B}^*$.

Example 3.1.

1. Let $\mathcal{H}=L^2[0,1]$,

   $$f(q)=\int_a^b K(t)q(t), \quad \sup K<+\infty$$

   Then f is complex differentiable

   $$\partial_q f = K(t), \quad 0\le t\le 1$$

2. Let $\mathcal{B}=C[0,1], 0\le x\le 1$,

   $$f(q)=q(x)$$

   Then f is complex differentiable,

   $$\partial_q f(x)=\delta_x\in\mathcal{B}^*$$

where

$$\delta_x(v) = v(x), \quad v \in \mathcal{B}$$

Note that f from (1) is well defined on $\mathcal{B}$ and

$$(\partial_q f)(v) = \int_0^1 K(t)v(t)dt, \quad v \in \mathcal{B}$$

In what follows we always work in $\mathcal{B} = C[0,1]$. Let $f(q), g(q)$ be functions on $\mathcal{B}$ then

$$\partial_q(f(q)g(q)) = g(q)\partial_q f + f(q)\partial_q g$$

provide the gradients $\partial_q f, \partial_q g$ exist.

Example 3.2. Let $y(x,q)$ be a function of $x \in [0,1]$ and $q \in \mathcal{B}$. Assume that $\partial_q y(x,q)$ exists. Consider

$$f(x,q) = q(x)y(x,q)$$

Then

$$\partial_q f = y(x,q)\delta_x + q(x)\partial_q y$$

Let $f(x,q)$ be a function of $x \in [0,1], q \in \mathcal{B}$. Assume $\partial_q f, \partial_x f$ exist. Assume also that $\partial_x \partial_q f$ and $\partial_q \partial_x f$ exist and continuous in x, q. Then

$$\partial_x \partial_q f = \partial_q \partial_x f$$

Similarly, provide that the gradients exist and continuous

$$\partial_{xx}^2 \partial_q f = \partial_q \partial_{xx}^2 f$$

Due to the series expansions previously, we have the following

Theorem 3.4. *For any fixed $x, y_j(x,\lambda,q)$ is complex differentiable in λ and $q, \lambda \in \mathbb{C}, q \in L^2$. The derivatives are continous*

One can easily calculate the derivatives

$$\frac{\partial C_n}{\partial q}, \quad \frac{\partial S_n}{\partial q}$$

It turns out that $(\partial y_j / \partial q)$ have nice formulas. To indicate here that the derivative is taken at fixed x we denote it as $(\partial y_j / \partial q(t))(x)$, where t is the variable for the $L^2[0,1]$ space.

Theorem 3.5.

$$\frac{\partial y_j}{\partial q(t)}(x) = y_j(t)[y_1(t)y_2(x) - y_1(x)y_2(t)]\square_{[0,x]}(t) \tag{3.5}$$

$$\frac{\partial y_j'}{\partial q(t)}(x) = y_j(t)[y_1(t)y_2'(x) - y_1'(x)y_2(t)]\square_{[0,x]}(t) \tag{3.6}$$

where $\square_{[0,x]}(t)$ stands for the indicator of $[0,x]$.Furthermore,

$$\frac{\partial y_j}{\partial \lambda}(x) = -\int_0^1 \frac{\partial y_j}{\partial q(t)} dt \tag{3.7}$$

$$\frac{\partial y_j'}{\partial \lambda}(x) = -\int_0^1 \frac{\partial y_j'}{\partial q(t)} dt \tag{3.8}$$

The gradients are continuous with respect to x, λ, q.

Proof. Though the theorem is stated in L^2, we do it in $\mathcal{B} = C[0,1]$ to make use of the derivatives

$$\partial_q(q(x)) = \delta_x$$

We differentiate

$$-y_j''(x) + q(x)y_j(x) = \lambda y_j(x)$$

with respect to q:

$$-(\partial_q y_{j_{|x}}(v))'' + y_j(x)\delta_x(v) + q(x)(\partial_q y_{j_{|x}}(v) = \lambda(\partial_q y_{j_{|x}}(v))$$

with $v \in \mathcal{B} = C[0,1]$. Now we can apply Theorem 3.12. Note that

$$(\partial_q y_{j_{|x=0}}) = 0, \quad (\partial_q y_{j_{|x}})_{|x=0} = \partial_q(\partial_x y_{j_{|x=0}}) = 0$$

Hence $a, b = 0$ in Theorem 3.12, i.e.

$$\begin{aligned}(\partial_q y_{j_{|x}})(v) &= \int_0^x (y_1(t)y_2(x) - y_1(x)y_2(t))y_j(t)\delta_t(v)dt \\ &= \int_0^x \big[(y_1(t)y_2(x) - y_1(x)y_2(t))y_j(t)\big]v(t)dt\end{aligned}$$

That verifies the first identity. Furthermore, differentiating the above with respect to x gives:

$$(\partial_q y'_{j_{|x}})(v) = \int_0^x \big[(y_1(t)y_2'(x) - y_1'(x)y_2(t))y_j(t)\big]v(t)$$

That verifies the second identity. Note that

$$y_j(x, \lambda + \xi, q) = y_j(x, \lambda, q - \xi)$$

Using the chain rule on $f(q(\cdot, \xi))$ gives:

$$\partial_\xi f(q(\cdot, \xi)) = \partial_q f(\partial_\xi q)$$

where $\partial_\xi q$ is viewed as a vector in $\mathcal{B}$. If $\partial_q f$ exists in L^2, then

$$\partial_q f(\partial_\xi q) = \int_0^1 (\partial_q f)(t)\partial_\xi q(t)dt$$

Thus,

$$\partial_\lambda y_j = -\int_0^1 (\partial_q y_j)(t)dt$$

i.e. the last 2 identities follow. □

Chapter 4

The Dirichlet Spectrum

4.1 Counting Eigenvalues

Consider the Sturm-Liouville equation

$$-y'' + qy = \lambda y, \quad x \geq 0 \tag{4.1}$$

λ is called a Dirichlet eigenvalue on $[0, 1]$ if (4.1) has a non-trivial solution y with $y(0) = 0, y(1) = 0$. Let $y_1(x, \lambda), y_2(x, \lambda)$ be fundamental solutions. Let y be a solution with $y(0) = 0$, then $y = ay_1 + by_2$. Since $y_1(0, \lambda = 1), y_2(0, \lambda) = 0$, one has $a = 0$. Thus the Dirichlet eigenvalues are the roots of the equation

$$y_2(1, \lambda) = 0 \tag{4.2}$$

Let $q = 0$. Then $y_2(x, \lambda) = \lambda^{-1/2} \sin(x\sqrt{\lambda})$ and the roots are as follows

$$\lambda_n = \pi^2 n^2, \quad n = 1, 2, 3, \ldots \tag{4.3}$$

All roots are simple. The collection of all Dirichlet eigenvalues is called the *Dirichlet Spectrum.*

Lemma 4.1 (Counting Lemma)**.** *Let $N > 2e^{||q||}$ be an integer. Equation* (4.3) *has exactly N roots in the half plane $\Re\lambda < (N + 1/2)^2\pi^2$*

Proof. Recall the estimate

$$\left| y_2(1, \lambda) - \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} \right| \leq \frac{\exp(||q|| + \Im\sqrt{\lambda})}{|\lambda|}$$

We want to invoke Rouche's Theorem. For that we want to compare $|\lambda|^{-1/2}\exp(|\Im\sqrt{\lambda}|)$ against $|\lambda|^{-1/2}|\sin\sqrt{\lambda}|$. This is done in Lemma (4.3) (see below): If $|z - m\pi| \geq \pi/4$ for all $m \in \mathbb{Z}$, then

$$4|\sin z| > \exp(|\Im z|)$$

So, provided that $|\lambda|^{-1/2}\exp(||q||) < 1/4$, one has

$$\left| y_2(1, \lambda) - \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} \right| < \frac{|\sin\sqrt{\lambda}|}{|\sqrt{\lambda}|}$$

Let N be an integer, $N > 2\exp(||q||)$. The function $\lambda^{-1/2}\sin\sqrt{\lambda}$ has exactly N roots in the half plane $\Re\lambda < (N+1/2)^2\pi^2$, see (4.3). For $\Re\lambda = (N+1/2)^2\pi^2$ one has

$$|\sqrt{\lambda} - m\pi| \geq \frac{\pi}{4}$$

for any $m \in \mathbb{Z}$.

$$|\sqrt{\lambda}| \geq \left(N + \frac{1}{2}\right)\pi > 2\pi\exp(||q||) > 4\exp(||q||)$$

This implies the statement due to Rouche's Theorem. □

Lemma 4.2. *Let $n > 2\exp(||q||+1)$, Equation (4.3) has exactly one root in the domain $|\sqrt{\lambda}-n\pi| > \pi/2$.*

Proof. If $|\sqrt{\lambda} - n\pi| = \pi/2$, then $|\sqrt{\lambda} - m\pi| \geq \pi/4$ for any $m \in \mathbb{Z}$.

$$|\sqrt{\lambda}| \geq \left(n - \frac{1}{2}\right)\pi > 2\pi\exp(||q||) > 4\exp(||q||)$$

and the statement follow from Rouche's Theorem. □

Lemma 4.3. *If $|z - m\pi| > \pi/4$ for any $m \in \mathbb{Z}$. Then*

$$4|\sin z| > \exp(|\Im z|)$$

Proof. Let $z = x + iy$. One cane assume $0 \leq x \leq \pi/2$. Recall

$$|\sin(x+iy)|^2 = \cosh^2 y - \cos^2 x$$

Let first $x \geq \pi/6$, so that $\cos^2 \geq (\sqrt{3}/2)^2 = 3/4$. Since $\cosh y \geq 1$ for any y, one has $\cosh^2 y \geq 4/3\cos^2 x$. For $0 \leq x \leq \pi/6$ we invoke the assumption $|z| > \pi/4$. So,

$$y^2 \geq (\pi/4)^2 - x^2 \geq (5\pi^2/144) \geq 1/3$$

Recall that

$$\cosh y = 1 + \frac{y^2}{2!} + \frac{y^4}{4!} + \ldots \geq 1 + \frac{y^2}{2}$$

So,

$$(\cosh^2 y) \geq 1 + y^2 \geq \frac{4}{3} \geq \frac{4}{3}\cos^2 x$$

Thus, in any event

$$|\sin(x+iy)|^2 \geq \frac{1}{4}\cosh^2 y > \frac{1}{16}\exp(2|y|)$$

□

Theorem 4.1. *If λ is a Dirichlet eigenvalue, then*

$$\partial_\lambda y_2(1,\lambda)\partial_x y_2(1,\lambda) = \int_0^1 y_2^2(t,\lambda)dt \tag{4.4}$$

If q is real then $\partial_\lambda y_2(1,\lambda) \neq 0$. In particular, in this case all the roots of $y_2(1,\lambda)$ are simple.

Proof. If one considers the ODE for y_2 and differentiate with respect to λ, one obtains

$$y_2\Big(-(\partial_\lambda y_2)'' + q\partial_\lambda y_2 = y_2 + \lambda\partial_\lambda y_2\Big) - \partial_\lambda y_2\Big(-y_2'' + qy_2 = \lambda y_2\Big)$$

$$\therefore y_2''\partial_\lambda y_2 - (\partial_\lambda y_2)'' y_2 = y_2^2$$

Note that

$$y_2''\partial_\lambda y_2 - (\partial_\lambda y_2)'' y_2 - (y_2'(\partial_\lambda y_2) - y_2(\partial_\lambda y_2)')'$$

Thus,

$$\int_0^1 y_2^2(t,\lambda)dt = y_2'(\partial_\lambda y_2) - y_2(\partial_\lambda y_2)'\Big|_{t=0}^{t=1}$$

Note that $y_2(0,\lambda) = 0$ implies

$$\partial_\lambda y_2(0,\lambda) = 0$$

Since λ is a Dirichlet eigenvalue $y_2(1,\lambda) = 0$. So,

$$\int_0^1 y_2(t,\lambda)dt = \partial_x y_2(1,\lambda)\partial_\lambda y_2(1,\lambda)$$

as claimed in (4.1). Theorem (4.9) below says that all Dirichlet eigenvalues are real. So, $y_2(x,\lambda)$ is real. That finishes the proof. □

Theorem 4.2. *If q is real then the Dirichlet eigenvalue are real.*

Proof. Let $y_2(1,\lambda) = 0$. Note that since q is real, one has

$$-y_2\Big(-\overline{y}_2'' + q\overline{y}_2 = \overline{\lambda}\overline{y}_2\Big) + \overline{y}_2\Big(-y_2'' + qy_2 = \lambda y_2\Big) \iff y_2\overline{y}_2'' - y_2''\overline{y}_2 = (\lambda - \overline{\lambda})|y_2|^2$$

Thus,

$$\begin{aligned}(\lambda - \overline{\lambda})\int_0^1 |y_2(t,\lambda)|^2 dt =& \int_0^1 \left(y_2(t,\lambda)\overline{y}_2'(t,\lambda) - y_2'(t,\lambda)\overline{y}_2(t,\lambda)\right)' dt \\ =& \left(y_2(t,\lambda)\overline{y}_2'(t,\lambda) - y_2'(t,\lambda)\overline{y}_2(t,\lambda)\right)\Big|_{t=0}^{t=1} \\ =& 0\end{aligned}$$

□

4.2 Eigenfunctions

We denote the Dirichlet eigenvalues via $\mu_j = \mu_j(q)$.

$$\mu_i < \mu_2 < \mu_3 < \dots \tag{4.5}$$

Due to Lemma (4.3)

$$|\sqrt{\mu_n} - n\pi| < \pi/2 \quad \text{for } n > 2\exp(||q||) + 1 \tag{4.6}$$

Set

$$g_n(x) = g_n(x,q) = \frac{y_2(x,\mu_n)}{||y_2(\cdot,\mu_n)||_{L^2}} \tag{4.7}$$

$g_n(x)$ is an eigenfunction

$$||g_n||_{L^2} = 1, \qquad \partial_x g_n\Big|_{x=0} = \frac{1}{||y_2(\cdot,\mu_n)||_{L^2}} \tag{4.8}$$

Due to Theorem (4.1) one has also

$$g_n(x) = \frac{y_2(x,\mu_n)}{\sqrt{\partial_\lambda y_2(1,\mu_n)\partial_x y_2(1,\mu_n)}}$$

$$\partial_x g_n\Big|_{x=0} = \frac{1}{\sqrt{\partial_\lambda y_2(1,\mu_n)\partial_x y_2(1,\mu_n)}}$$

Lemma 4.4. *If $q_m(x) \to q(x)$ pointwise and $q_m(x)$ are uniformly bounded, then $\mu_n(q_m) \to \mu_n(q)$*

Proof. It is easy to see that $y_2(1,\lambda,q_m) \to y_2(1,\lambda,q)$ uniformly for λ running in any bounded set. Since the roots $\mu_1(q) < \mu_2(q) < \ldots$ are simple the statement follows. □

To proceed we need to discuss briefly *analytic functions* defined on a Banach space $\mathcal{B}$. This is defined via *weak analyticity*:

$$z \to f(q_0 + zq)$$

is analytic in a small neighbourhood $|z| < \rho(q_0,q)$ for any q_0, q.

Lemma 4.5. *$\mu_n(q)$ is analytic around any real $q_0 \in L^2$*

Proof. We know due to Theorem (4.1) that

$$\partial_\lambda y_2\Big|_{\lambda=\mu_n(q)} \neq 0$$

Therefore the statement follows from the implicit function theorem. □

Theorem 4.3.

$$\partial_q \mu_n = g_n^2(t,q)$$

Proof. Differentiating

$$y_2(1,\mu_n(q),q) = 0$$

we obtain

$$(\partial_\lambda y_2(1,\mu_n)(\partial_q \mu_n) + (\partial_q y_2)\Big|_{\lambda=\mu_n(q)} = 0$$

By Theorem (3.5)

$$\partial_q y_2(1,\lambda,q) = y_2(t,\lambda,q)\,[y_1(t,\lambda,q)y_2(1,\lambda,q) - y_1(1,\lambda,q)y_2(t,\lambda,q)]\,\square_{[0,1]}(t) = -y_1(1,\lambda,q)y_2(t,\lambda,q)^2$$

Note that

$$1 = [y_1,y_2]_{x=1} = y_1(1)\partial_x y_2(1) - \partial_x y_1(1)y_2(1) = y_1(1)\partial_x y_2(1)$$

So,

$$\partial_q y_2(1,\lambda,q) = -\frac{y_2^2(t,\lambda,q)}{\partial_x y_2(t,\lambda,q)}$$

Thus

$$\partial_q \mu_n = \frac{y_2^2(t,\mu_n)}{\partial_\lambda y_2(1,\mu_n)\partial_x y_2(1,\mu_n)} = g_n^2(t)$$

□

Definition 4.6. $(\alpha_n)_{n\geq 1}$ belongs to l_k^2 if

$$\sum_{n\geq 1}(n^k\alpha_n)^2 < +\infty$$

Clearly l_k^2 is a Hilbert space. We write

$$\beta_n = \gamma_n + l_k^2(n), \quad n \geq 1$$

if

$$\beta_n = \gamma_n + \alpha_n, \quad (\alpha_n)_{n\geq 1} \in l_k^2$$

Theorem 4.4. *Let $q \in L^2$, then*

$$\mu_n(q) = n^2\pi^2 + \int_0^1 q(t)dt - \int_0^1 q(x)\cos(2\pi nx)dx + \mathcal{O}\left(\frac{1}{n}\right) = n^2\pi^2 + \int_0^1 q(t)dt + l^2(n) \tag{4.9}$$

$$g_n(x,q) = \sqrt{2}\sin(\pi nx) + \mathcal{O}\left(\frac{1}{n}\right) \tag{4.10}$$

$$\partial_x g_n(x,q) = \sqrt{2}\pi n\cos(\pi nx) + \mathcal{O}(1) \tag{4.11}$$

uniformly in x and on bounded sets in L^2.

Proof. We have

$$\sqrt{\mu_n} = n\pi + \mathcal{O}(1)$$

$$y_2(x,\mu_n) = \frac{\sin(\sqrt{\mu_n}x)}{\sqrt{\mu_n}} + \mathcal{O}\left(\frac{1}{|\mu_n|}\right) = \frac{\sin(\sqrt{\mu_n}x)}{\sqrt{\mu_n}} + \mathcal{O}\left(\frac{1}{n^2}\right)$$

Thus,

$$\int_0^1 y_2^2(x,\mu_n)dx = \int_0^1 \frac{\sin(\sqrt{\mu_n}x)}{\sqrt{\mu_n}}dx + \mathcal{O}\left(\frac{1}{n^3}\right) = \frac{1}{2\mu_n} + \mathcal{O}\left(\frac{1}{n^3}\right)$$

Now use this on g_n,

$$g_n(x) = \frac{y_2(x,\mu_n)}{||y_2(\cdot,\mu_n)||} = \sqrt{2}\sin(\sqrt{\mu_n}x) + \mathcal{O}\left(\frac{1}{n}\right)$$

Note that $\mu_n(0) = n^2\pi^2$,

$$\mu_n(q) - n^2\pi^2 = \int_0^1 \frac{d}{d\tau}\mu_n(\tau q)d\tau = \int_0^1 d\tau(\partial_q\mu_n, q) = \int_0^1 d\tau\left[\int_0^1 g_n^2(t,\tau q)q(t)dt\right] = \mathcal{O}(1)$$

Thus,

$$\mu_n = n^2\pi^2 + \mathcal{O}(1)$$

$$\sqrt{\mu_n} = n\pi + \mathcal{O}\left(\frac{1}{n}\right)$$

$$g_n(x) = \sqrt{2}\sin(\pi n x) + \mathcal{O}\left(\frac{1}{n}\right)$$

Once again,

$$\begin{aligned}\mu_n - n^2\pi^2 &= \int_0^1 d\tau \left[\int_0^1 2\sin^2(n\pi t)q(t)dt + \mathcal{O}\left(\frac{1}{n}\right)\right] \\ &= \int_0^1 d\tau \left[\int_0^1 (q(t) - \cos(2\pi n t)q(t))dt + \mathcal{O}\left(\frac{1}{n}\right)\right]\end{aligned}$$

which implies (4.9) since

$$\sum_{n\geq 1}\left(\int_0^1 \cos(2\pi n t)q(t)dt\right)^2 \leq ||q||_{L^2}^2 < +\infty$$

Let us estimate $\partial_x g_n$. One has due to the basic estimates

$$\partial_x y_2(x,\lambda) = \cos(\sqrt{\lambda}x) + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)$$

Since $\sqrt{\mu_n} = n\pi + \mathcal{O}(1/n)$, we have

$$\partial_x y_2(x,\mu_n) = \cos(n\pi x) + \mathcal{O}\left(\frac{1}{n}\right)$$

Since

$$\frac{1}{||y_2||} = \sqrt{2}\sqrt{\mu_n} + \mathcal{O}(1) = \sqrt{2}\pi n + \mathcal{O}(1)$$

one obtains

$$\partial_x g_2 = \frac{\partial_x y_2(x,\mu_n)}{||y_2||} = \sqrt{2}\pi n\cos(n\pi x) + \mathcal{O}(1)$$

□

Set

$$a_n = y_1(x,\mu_n)y_2(x,\mu_n)$$

Corollary 4.1.

$$g_n^2 = 1 - \cos(2\pi n x) + \mathcal{O}\left(\frac{1}{n}\right)$$

$$\partial_x g_n^2 = 2\pi n\sin(2\pi n x) + \mathcal{O}(1)$$

$$a_n = \frac{1}{2\pi n}\sin(2\pi n x) + \mathcal{O}\left(\frac{1}{n^2}\right)$$

$$\partial_x a_n = \cos(2\pi n x) + \mathcal{O}\left(\frac{1}{n}\right)$$

Proof. The first two estimates follow from Theorem (4.4). Due to the basic estimates

$$y_1(x,\lambda) = \cos(\sqrt{\lambda}x) + \mathcal{O}\left(\frac{1}{\sqrt{|\lambda|}}\right)$$

for $\sqrt{\lambda} = \sqrt{\mu_n} = \pi n + \mathcal{O}(1/n)$,

$$y_1(x,\mu_n) = \cos(\pi n x) + \mathcal{O}\left(\frac{1}{n}\right)$$

Furthermore,

$$y_2(x,\mu_n) = \frac{1}{\pi n}\sin(\pi n x) + \mathcal{O}\left(\frac{1}{n^2}\right)$$

$$\partial_x y_1(x,\mu_n) = -\pi n \sin(\pi n x) + \mathcal{O}(1)$$

$$\partial_x y_2(x,\mu_n) = \cos(\pi n x) + \mathcal{O}\left(\frac{1}{n}\right)$$

and the estimates for a_n follow. □

4.3 Product Expansions

Theorem 4.5.

$$y_2(1,\lambda,q) = \prod_{m\geq 1}\left(\frac{\mu_m(q)-\lambda}{m^2\pi^2}\right)$$

Proof.

$$\frac{\mu_m(q)-\lambda}{m^2\pi^2} = 1 + \mathcal{O}\left(\frac{1}{m^2}\right)$$

The product $p(\lambda)$ converges and defines an entire function of λ. The roots of p are $\lambda = \mu_n(q)$. So, p/y_2 is an entire function with no zeros. We invoke the expansion

$$\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} = \prod_{m\geq 1}\left(\frac{m^2\pi^2-\lambda}{m^2\pi^2}\right)$$

For $r_n = (n+1/2)^2\pi^2$, $n >> 1$ one concludes

$$\frac{p(\lambda)}{\left(\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\right)} = 1 + \mathcal{O}\left(\frac{\log n}{n}\right)$$

Recall that

$$y_2(1,\lambda) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} + \mathcal{O}\left(\frac{1}{|\lambda|}\right)$$

Thus

$$\frac{p(\lambda)}{y_2(1,\lambda)} = 1 + \mathcal{O}\left(\frac{\log n}{n}\right)$$

For $|\lambda| = r_n, n >> 1$. By Liouville's Theorem $p(\lambda)/y_2(1,\lambda) = 1$ everywhere. □

Lemma 4.7. *Let* $|a_{m,n}| = \mathcal{O}(|m^2 - n^2|^{-1}), m \neq n$. *Then*

$$\prod_{m\geq 1, m\neq n} (1 + a_{m,n}) = 1 + \mathcal{O}\left(\frac{\log n}{n}\right)$$

If $\sum |b_n|^2 < +\infty$ *then*

$$\prod_{m,n\geq 1, m\neq n} |1 + a_{m,n} b_n| < +\infty$$

Proof.

$$\sum_{m\geq 1, m\neq n} \frac{1}{|m^2 - n^2|} = \sum_{1\leq m\leq 2n, m\neq n} \frac{1}{|m^2 - n^2|} + \sum_{m>2n} \frac{1}{|m^2 - n^2|}$$

For the first sum we have

$$\sum_{1\leq m\leq 2n, m\neq n} \frac{1}{|m^2 - n^2|} \leq \frac{2}{n} \sum_{1\leq k\leq n} \frac{1}{k} \leq \frac{2}{n} \log n$$

For the second sum we have

$$\sum_{m>2n} \frac{1}{|m^2 - n^2|} \leq \sum_{k>n} \frac{1}{k^2} < \frac{1}{n}$$

For n large, $m \neq n, |a_{m,n}| < 1/2, |\log(1 + a_{m,n})| < 2|a_{m,n}|$,

$$\sum_{m\geq 1, m\neq n} |\log(1 + a_{m,n})| \leq 2 \sum |a_{m,n}| = \mathcal{O}\left(\frac{\log n}{n}\right)$$

The proof of the second part is similar. □

Lemma 4.8. *Let* $z_m = m^2\pi^2 + \mathcal{O}(1)$. *Then*

$$F(\lambda) = \prod_{m\geq 1} \frac{z_m - \lambda}{m^2\pi^2}$$

is an entire function with roots at z_m*,*

$$F(\lambda) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} \left(1 + \mathcal{O}\left(\frac{\log n}{n}\right)\right), \qquad \text{for } |\lambda| = (n + 1/2)^2\pi^2$$

Proof. Since

$$\frac{z_m - \lambda}{m^2\pi^2} = 1 + \mathcal{O}\left(\frac{1}{m^2}\right)$$

The product converges and $F(\lambda)$ is an entire function. Recall the product expansion

$$\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} = \prod_{m\geq 1} \frac{m^2\pi^2 - \lambda}{m^2\pi^2}$$

So,

$$F(\lambda) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} \prod_{m\geq 1} \frac{z_m - \lambda}{m^2\pi^2 - \lambda}$$

Let here $|\lambda| = (n+1/2)^2\pi^2$. Then, for $m \neq n$

$$\frac{z_m - \lambda}{m^2\pi^2 - \lambda} = 1 + \mathcal{O}\left(\frac{1}{|m^2\pi^2 - \lambda|}\right) = 1 + \mathcal{O}\left(\frac{1}{|m^2 - n^2|}\right)$$

and for $m = n$

$$\frac{z_m - \lambda}{m^2\pi^2 - \lambda} = 1 + \mathcal{O}\left(\frac{1}{n}\right)$$

□

Lemma 4.9. *Let $z_m = m^2\pi^2 + \mathcal{O}(1)$. Then*

$$F_n(\lambda) = \prod_{m\geq 1, m\neq n} \frac{z_m - \lambda}{m^2\pi^2}$$

is an entire function,

$$F_n(\lambda) = \frac{(-1)^{n+1}}{2}\left(1 + \mathcal{O}\left(\frac{\log n}{n}\right)\right)$$

Proof.

$$\partial_\lambda \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\Big|_{\lambda=n^2\pi^2} = \partial_\lambda \prod_{m\geq 1}\left(\frac{m^2\pi^2 - \lambda}{m^2\pi^2}\right) = \frac{1}{n^2\pi^2}\prod_{m\geq 1, m\neq n}\frac{m^2\pi^2 - n^2\pi^2}{m^2\pi^2}$$

$$\partial_\lambda \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\Big|_{\lambda=n^2\pi^2} = \frac{(-1)^n}{2n^2\pi^2}$$

Like in Lemma (4.8), for $\lambda = n^2\pi^2 + \mathcal{O}(1)$, one has

$$F_n(\lambda) = \partial_\lambda \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} \prod_{m\geq 1, m\neq n}\frac{z_m - \lambda}{m^2\pi^2 - n^2\pi^2} = \partial_\lambda \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\left(1 + \mathcal{O}\left(\frac{\log n}{n}\right)\right)$$

and the statement follows. □

Corollary 4.2.

$$\partial_\lambda y_2(1,\mu_n) = \prod_{m\geq 1, m\neq n}\frac{\mu_m - \mu_n}{m^2\pi^2} = \frac{(-1)^n}{2n^2\pi^2}\left(1 + \mathcal{O}\left(\frac{\log n}{n}\right)\right)$$

$$sgn(\partial_\lambda y_2(1,\mu_n)) = (-1)^n = sgn(\partial_x y_2(1,\mu_n))$$

Proof. The statement follows from Theorem (4.5) combined with Lemma (4.9) and Theorem (4.1) □

4.4 A Basis For L^2

Theorem 4.6. *a) g_n has exactly $(n+1)$ roots on $[0,1]$. The roots are simple, and*

$$sgn\partial_x g_n\Big|_{x=1} = (-1)^n$$

b) If q be even, then g_n is odd if n is even, g_n is even if n is odd.

To prove this theorem we use the following

Lemma 4.10 (Continuous Deformations)**.** *Let $h(t,x)$ be continuously differentiable in $t, x, t \in [0,1], x \in [a,b]$. Assume for each $t, h(t,\cdot)$ has a finite number of zeros and all zeros are simple. Suppose also that $h(t,a) = h(t,b) = 0$ for all t. Then $h(0,\cdot)$ and $h(1,\cdot)$ have the same number of zeros. Furthermore, if $a = \xi_1(t) < \ldots < \xi_n(t) = b$ are the roots then $sgn\partial_x h\Big|_{\xi_j(t)}$ does not deepen on t.*

Proof of (4.6). Part a) follows by applying the continuous deformation $h(t,x) = g_n(x,tq), 0 \le t, x \le 1$. Assume q is even, i.e. $q(1-x) = q(x)$. Then $g_n(1-x)$ is an eigenfunction for μ_n, $||g_n(1-\cdot)|| = 1$. Hence,

$$g_n(1-x) = \alpha_n g_n(x), \quad \alpha_n \in \{1,-1\}$$

Taking the derivatives at $x = 1$, one obtains

$$-\partial_x g_n\Big|_{x=0} = \alpha_n \partial_x g_n\Big|_{x=1}$$

Recall that $\partial_x g_n\Big|_{x=0} = 1/||g_n||$ just from the definition of $y_2(x,\lambda)$. Furthermore,

$$\text{sgn}\partial_x g_n\Big|_{x=1} = (-1)^n \implies \alpha_n = (-1)^{n+1}$$

□

Corollary 4.3.

$$sgn\partial_\lambda y_2(1,\lambda)\Big|_{\lambda=\mu_n} = (-1)^n$$

Proof. Due to Theorem (4.1)

$$\partial_\lambda y_2(1,\lambda)\Big|_{\lambda=\mu_n} \partial_x y_2(x,\mu_n)\Big|_{x=1} = \int_0^1 y(t,\mu_n)^2 dt > 0$$

□

Theorem 4.7. *g_n is an orthonormal basis in L^2.*

Proof. Orthogonality check:

$$(\mu_m - \mu_n)(g_m, g_n) = [g_m, g_n]\Big|_{x=0}^{x=1} = 0$$

To show that the system is complete, introduce

$$Af = \sum_{n\ge1} (f, e_n) g_n$$

where $e_n = \sqrt{2}\sin(\pi n x)$. Note that

$$||Af||^2 = \sum_{n\ge1} |(f,e_n)|^2 = ||f||^2$$

i.e. A is an isometry. Furthermore,

$$\sum_{n\ge1} ||(A-I)e_n||^2 = \sum_{n\ge1} ||g_n - e_n||^2 = \sum_{n\ge1} \mathcal{O}\left(\frac{1}{n^2}\right) < +\infty$$

Thus, $A - I$ is Hilbert-Schmidt. Due to the *Fredholm alternative* , A is also onto since $\ker A = 0$. □

Recall

$$a_n(x, q) = y_1(x, \mu_n) y_2(x, \mu_n)$$

Theorem 4.8. *1.*

$$(g_n^2, \partial_x g_n^2) = 0$$

2.

$$(a_m, \partial_x g_n^2) = \frac{1}{2}\delta_{m,n}$$

3.

$$(a_m, \partial_x a_n) = 0$$

Proof. Recall that $g_k\Big|_{x=0} = g_k\Big|_{x=1} = 0$. So,

$$\begin{aligned}(g_m^2, \partial_x g_n^2) &= \int_0^1 g_m^2(x)\partial_x g_n^2(x) dx \\ &= -\int_0^1 g_n^2(x)\partial_x g_m^2(x) \\ &= \frac{1}{2}\int_0^1 \left(g_m^2(x)\partial_x g_n^2(x) - g_n^2(x)\partial_x g_m^2(x)\right) dx \\ &= \int_0^1 g_m(x) g_n(x)[g_m, g_n](x) dx\end{aligned}$$

If $m = n$, then $[g_m, g_n] = 0$. Let $m \neq n$. Then

$$[g_m, g_n]' = (g_m g_n' - g_m' g_n)' = g_m g_n'' - g_m'' g_n = g_m(q - \mu_n) g_n - (q - \mu_m) g_m g_n = (\mu_m - \mu_n) g_m g_n$$

$$\therefore g_m g_n = \frac{1}{\mu_m - \mu_n}[g_m, g_n]'$$

Substituting this back into gives

$$\int_0^1 g_m g_n [g_m, g_n](x) dx = \frac{1}{\mu_m - \mu_n}\int_0^1 [g_m, g_n][g_m, g_n]' dx = \frac{[g_m, g_n]^2}{2(\mu_m - \mu_n)}\Big|_{x=0}^{x=1} = 0$$

That finishes 1. Now for 2.

$$\begin{aligned}2(a_m, \partial_x g_n^2 &= \int_0^1 (a_m \partial_x g_n^2 - \partial_x a_m g_n^2) dx \\ &= \int_0^1 (2y_1 y_2 g_n \partial_x g_n - \partial y_1 y_2 g_n^2 - \partial_x y_2 y_1 g_n^2 dx) dx \\ &= \int_0^1 (y_2 g_n [y_1, g_n] + y_1 g_n [y_2, g_n]) dx\end{aligned}$$

$y_j = y_j(x, \mu_m)$. If $m = n$, then $[y_2, g_n] = 0$, so

$$\int_0^1 y_2 g_n [y_1, g_n] dx = \int_0^1 \frac{y_2}{||y_2||} g_n [y_1, y_2] dx = \int_0^1 g_n^2 dx = 1$$

If $m \neq n$, then

$$(\mu_m - \mu_n) y_j g_n = \partial_x [y_j, g_n]$$

thus

$$\int_0^1 (y_2 g_n [y_1, g_n] + y_1 g_n [y_2, g_n]) dx = \frac{1}{\mu_m - \mu_n} \int_0^1 ([y_2, g_n] \partial_x [y_1, g_n] + [y_1, g_n] \partial_x [y_2, g_n]) \, dx = \frac{[y_1, g_n][y_2, g_n]}{\mu_m - \mu_n} \Big|_{x=0}^{x=1} = 0$$

That finishes 2. Part 2 is completely similar. □

For $q = 0$

$$g_n^2 - 1 = -\cos(2\pi n x)$$

$$\partial_x g_n^2 = 2\pi n \sin(2\pi n x)$$

These functions together with 1 are a basis in $L^2[0,1]$. We want to show that the same is true for $q \neq 0$. However, the basis is not orthogonal anymore.

Definition 4.11. A map $U : \mathcal{H}_1 \to \mathcal{H}_2$ is a linear isomorphism if it is a linear bijection and U, U^{-1} are bounded.

Definition 4.12. $d_n \in \mathcal{H}$ are linearly independent if for any $m, d_m \notin \operatorname{span}\{d_n\}_{n \neq m}$

Theorem 4.9.

$$U : (\xi, \eta) \to \sum \xi_n \partial_x g_n^2 + \eta_0 1 + \sum_{n \geq 1} \eta_n (g_n^2 - 1)$$

is an isomorphism , $U : l_1^2 \times \mathbb{R} \times l^2 \to L^2[0,1]$. The vectors $1, g_n^2 - 1$ are orthogonal to the vectors $\partial_x g_m^2$.

To prove this theorem, we prove the following first

Theorem 4.10. *Let e_n be an orthonormal basis of the Hilbert space $\mathcal{H}$. Let $d_n \in \mathcal{H}$ be linearly independent and obey*

$$\sum ||d_n - e_n||^2 < +\infty \tag{4.12}$$

Then $A : x \to \sum (x, e_n) d_n$ is an isomorphism, $\mathcal{H} \longleftrightarrow \mathcal{H}$. Furthermore, $U : x \to ((x, d_n))_{n \geq 1}$ is an isomorphism $\mathcal{H} \longleftrightarrow l^2$

Proof. Since $I(x) = x = \sum (x, e_n) e_n$ and (4.12) holds, $(I - A)$ is Hilbert-Schmidt. If $(\alpha_n) \in l^2, \sum_n \alpha_n d_n = 0$, then $\alpha_n = 0$ for all n, since otherwise there would be N such that

$$d_N = \sum_{n \neq N} \beta_n d_n \in \overline{\operatorname{span}\{d_n : n \neq N\}}$$

contrary to the linear independence of d_n, $n \geq 1$. Therefore, $\ker A = 0$ and the statement follows from the Fredholm alternative. □

Remark 4.13. Assume that d_n obeys (4.12) and $\overline{\{d_n\}} = \mathcal{H}$. Then U is an isomorphism. This is because of the Fredholm alternative: $\ker A = \{0\}$ if and only if $\overline{A\mathcal{H}} = \mathcal{H}$

We will show now that the vectors $1, g_n^2 - 1, n = 1, 2, \ldots$ are linearly independent. So are the vectors $\partial_x g_n^2, n = 1, 2, \ldots$. These sequences are mutually orthogonal and together constitute a basis in L^2. The map

$$U : (\xi, \eta) \to \sum_{n \geq 1} \xi_n \partial_x g_n^2 + \eta_0 + \sum_{n \geq 1} \eta_n (g_n^2 - 1)$$

is a linear isomorphism: $l_1^2 \times \mathbb{R} \times l^2 \to L^2$

Proof. By Theorem (4.8),

$$(a_m . \partial_x g_n^2) = \frac{1}{2} \delta_{m,n}$$

Recall that $g_n \Big|_{x=0} = g_n \Big|_{x=1} = 0$. So, integrating by parts gives

$$(\partial_x a_m, g_n^2) = -\delta_{m,n}$$

Recall also that $a_m \Big|_{x=1} = a_m \Big|_{x=1} = 0$. Hence

$$(\partial_x a_m, 1) = 0$$

Thus

$$(g_n^2 - 1, \partial_x a_m) = -\frac{1}{2} \delta_{m,n}$$

Furthermore

$$g_n^2 - 1, 1) = (g_n^2, 1) - 1 = ||g_n^2|| - 1 = 0$$

This implies

$$g_n^2 - 1 \notin \{1, g_m^2 - 1, m \neq n, m = 1, 2, \ldots\}$$

Furthermore, we have

$$\partial_x g_n^2 \notin \text{span}\{\partial_x g_m^2 : m \neq n, m = 1, 2, \ldots\}$$

Due to Theorem (4.8),

$$(g_m^2, \partial_x g_n^2) = 0, \quad m, n = 1, 2, \ldots$$

Since $g_n \Big|_{x=0} = g_n \Big|_{x=1} = 0$ one has

$$(1, \partial_x g_n^2) = 0$$

The statement regarding the linear independence and orthogonality follows from these relations. The invertibility of U follows from Theorem (4.10) □

Chapter 5

The Inverse Dirichlet Problem

Set

$$[q] = \int_0^1 q(x)dx, \quad \tilde{\mu}_n(q) = \mu_n(q) - n^2\pi^2 - [q], \quad \mu = ([q], (\tilde{\mu_n})_{n\geq 1}) \in \mathbb{R} \times l^2$$

Theorem 5.1. *$\tilde{\mu}$ is a real analytic map $\mu : L^2 \to \mathbb{R} \times l^2$*

$$\partial_q \mu(v) = ([v], (g_n^2 - 1, v)_{n\geq 1})$$

Proof. Let $p \in^2$. Given N, there exists $r_{p,N} > 0$ such that for $||q - p|| < r_{p,N}, \mu_1, \ldots, \mu_N$ are real analytic functions of q. Take $N > 2\exp(||p||)$. Then $N > 2\exp(||q||)$ for $||q - p|| < r_p$, provided r_o is small enough. It follows from the Counting Lemma that all μ_n's are real analytic in $||q - p|| < r_{p,N}$. Similarly,

$$g_n^2(x, q) = \frac{y_2(x, \mu_n)}{\partial_\lambda y_2(1, \mu_n)\partial_x y_2(1, \mu_n)}$$

is real analytic in $||q - p|| < r_p$. Furthermore,

$$\mu_n(q) = n^2\pi^2 + [q] - (\cos(2\pi n, x), q) + \mathcal{O}\left(\frac{1}{n}\right)$$

for complex $||q - p|| < r_p$. The map is analytic,

$$\frac{\partial \tilde{\mu}_n}{\partial q} = g_n^2 - 1$$

□

Remark 5.1. Let $q^*x(x) = q(1-x)$, then clearly $\mu_n(q^*) = \mu_n(q)$. So the map $q \to \mu(q)$ is not injective. We denote by E the set of all even functions $q \in L^2$, i.e.

$$E = \{q \ inL^2 : q^* = q\}$$

We denote by μ_E the restriction of μ on E.

Theorem 5.2. *μ_E is a local analytic diffeomorphism at each $p \in E$.*

Proof.

$$\partial_q \mu_E(v) = ([v], (g_n^2 - 1, v)_{n \geq 1})$$

Recall that by Theorem (4.9) $1, g_n^2 - 1, n = 1, 2, \ldots$ is a basis in E. Thus $\partial_q \mu_E(v)$ is invertible. □

Theorem 5.3 (Borg,1946)**.** *μ_E is injective on E.*

To prove this theorem we need the following:

Lemma 5.2. *Let f be meromorphic in $\mathbb{C}$. If*

$$\sup_{|\lambda|=r_n} |f(\lambda)| = o\left(\frac{1}{r_n}\right)$$

For $r_n \to \infty$, then

$$\sum Resf = 0$$

Proof.

$$\left| \int_{|\lambda|=r_n} f(\lambda) d\lambda \right| \leq o\left(\frac{1}{r_n}\right) 2\pi r_n \to 0$$

and the statement follows from the Cauchy Residue Theorem. □

Proof of (5.3). Assume $p, q \in E, \mu(p) = \mu(q)$. Consider

$$f(\lambda) = -\frac{(y_2(x,\lambda,q) - y_2(x,\lambda,p))(y_2(1-x,\lambda,q) - y_2(1-x,\lambda,p))}{y_2(1,\lambda,q)}$$

$f(\lambda)$ has simple poles at $\lambda = \mu_n$. Recall

$$y_2(1-x, \mu_n) = (-1)^n y_2(x, \mu_n)$$

So,

$$\text{Res} f\Big|_{\lambda=\mu_n} = \frac{(y_2(x,\mu_n,q) - y_2(x,\mu_n,p))^2}{\partial_\lambda y_2(1,\mu_n,q)} \geq 0$$

Furthermore,

$$|y_2(x,\lambda,q) - y_2(x,\lambda,p)||y_2(1-x,\lambda,q) - y_2(1-x,\lambda,p)| \leq \frac{\exp(|\Im\sqrt{\lambda}|x)}{\sqrt{|\lambda|}} \frac{\exp(|\Im\sqrt{\lambda}|(1-x))}{\sqrt{|\lambda|}} = \frac{\exp(|\Im\sqrt{\lambda}|)}{|\lambda|}$$

Since from the basic estimates we had

$$\left| y_2(1,\lambda) - \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} \right| \leq \frac{\exp(||q|| + |\Im\sqrt{\lambda}|)}{|\lambda|} = o\left(\frac{\exp(|\Im\sqrt{\lambda}|)}{\sqrt{|\lambda|}}\right)$$

Since

$$\left|\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\right| > \frac{\exp(|\Im\sqrt{\lambda}|)}{4\sqrt{|\lambda|}}, \quad \text{if } |\sqrt{\lambda} - m\pi| \geq \frac{\pi}{4}, \quad \forall m$$

$$|y_2(1,\lambda)| > \frac{\exp(|\Im\sqrt{\lambda}|)}{8\sqrt{|\lambda|}}, \quad \text{if } |\sqrt{\lambda} - m\pi| \geq \frac{\pi}{4}, \quad \forall m$$

Thus

$$|f(\lambda)| = o\left(\frac{1}{|\lambda|^{3/2}|}\right), \quad \text{for } |\lambda| = \left(n+\frac{1}{2}\right)^2 \pi^2$$

That implies that

$$\sum \text{Res} f = 0$$

Hence,

$$y_2(x,\mu_n,q) = y_2(x,\mu_n,p) \quad \forall x, n$$

Note that if $y(x,\lambda,q) = y(x,\lambda,p)$ for some particular λ and all x, then $p(x) = q(x)$ for almost all x. □

Set (Flaska, McLaughlin 1976)

$$\varkappa_n(q) = \log\Big((-1)^n \partial_x y_2(1,\mu_n)\Big) = \log\left|\frac{\partial_x g_n(1,q)}{\partial_x g_n(0,q)}\right|$$

Theorem 5.4.

$$\varkappa_n(q) = \frac{1}{2\pi n}(\sin(2\pi n x), q) + \mathcal{O}\left(\frac{1}{n^2}\right) = l_1^2(n)$$

$$\partial_q \varkappa = a_n(t,q) - [a_n] g_n^2(t,q) = \frac{1}{2\pi n}\sin(2\pi n t) + \mathcal{O}\left(\frac{1}{n^2}\right)$$

uniformly on bounded sets.

Proof. Since $\partial_x y_2(1,\mu_n(q),q) \neq 0$, $\varkappa_n(q)$ is a weakly continuous real analytic function. Furthermore,

$$\partial_q \varkappa_n = \frac{1}{\partial_x y_2(1,\mu_n,q)}\left(\partial_\lambda y_2\Big|_{x=1,\lambda=\mu_n}\partial_q\mu_n + \partial_q\partial_x y_2\Big|_{x=1,\lambda=\mu_n}\right)$$

Recall that due to Theorem (3.5)

$$\partial_q y_2'\Big|_{x=1} = y_2(t)(y_1(t)y_2'(x) - y_1'(x)y_2(t)]\square_{[0,x]}(t)\Big|_{x=1} = y_1(t)y_2(t)y_2'(t) - y_1'(1)y_2(t)^2$$

$$\partial_\lambda\partial_x y_2\Big|_{x=1} = -y_2'(1)\int_0^1 y_1 y_2 dt + y_1'(1)\int_0^1 y_2^2(t)dt$$

Recall also that by Theorem (4.3)

$$\partial_q\mu_n = g_n^2(t) = \frac{y_2^2(t,\mu_n)}{||y_2(\cdot,\mu_n)||^2}$$

One obtains

$$\begin{aligned}\partial_q\varkappa_n =& y_1(t,\mu_n)y_2(t,\mu_n) - \left(\int_0^1 y_1(t,\mu_n)y_2(t,\mu)dt\right) g_n^2(t) \\ =& a_n(t) - [a_n]g_n^2(t) \\ =& \frac{1}{2\pi n}\sin(2\pi n t) + \mathcal{O}\left(\frac{1}{n^2}\right)\end{aligned}$$

(See Corollary (4.1)) Furthermore, $\varkappa_n(0) = 0$,

$$\begin{aligned}\varkappa_n(q) &= \int_0^1 \frac{d}{dt}\varkappa(tq)dt \\ &= \int_0^1 dt \int_0^1 [(\partial_q \varkappa_n)(tq)](s)q(s)ds \\ &= \int_0^1 dt \int_0^1 \left(\frac{1}{2\pi n}\sin(2\pi ns) + \mathcal{O}\left(\frac{1}{n^2}\right)\right) q(s)ds \\ &= \frac{1}{2\pi n}(\sin(2\pi nx, q) + \mathcal{O}\left(\frac{1}{n^2}\right) \\ &= l_1^2(n)\end{aligned}$$

□

Lemma 5.3.

$$(\partial_q \varkappa_m, \partial_x \partial_q \varkappa_n) = 0$$

$$(\partial_q \varkappa_m, \partial_x \mu_n) = \frac{1}{2}\delta_{m,n}$$

$$(\partial_q \mu_m, \partial_x \partial_q \mu_n) = 0$$

Proof. Using Theorem (5.4), Theorem (**??**) and Theorem (4.8)

$$\partial_q \varkappa_m, \partial_x \partial_q \mu_n) = (a_m - [a_m]g_m^2, \partial_x g_n^2) = \frac{1}{2}\delta_{m,n}$$

The verification of the rest is similar. □

Theorem 5.5. *The map $q \to (\varkappa_m(q)), (\mu_m(q))$ is injective.*

Proof. Assume $\varkappa_m(q) = \varkappa_m(p), \mu_n(q) = \mu_n(p)$ for all m, n. Consider

$$f(\lambda) = -\frac{(y_2(x,\lambda,q) - y_2(x,\lambda,p))(y_2(1-x,\lambda,q^*) - y_2(1-x,\lambda,p^*))}{y_2(1,\lambda,q)}$$

since $\varkappa_m(q) = \varkappa_m(o), \mu_n(q) = \mu_n(p)$, we have

$$\partial_x y_2(1, \mu_n, q) = \partial_x y_2(1, \mu_n, p)$$

where $\mu_n \equiv \mu_n(p)$. Note also that

$$y_2(1-x, \mu_n, q^*) = -\frac{y_2(x, \mu_n, q)}{\partial_x y_2(x, \mu_n, q)\Big|_{x=1}}$$

Since both sides are solutions of $-y'' + qy = \mu_n y$ with the same initial conditions at $x = 1$. The same conclusion applies to p. Calculate:

$$\begin{aligned}
\mathrm{Res} f\Big|_{\lambda=\mu_n} &= \frac{(y_2(x,\mu_n,q)-y_2(x,\mu_n,p))}{y_2(1,\mu_n,q)}\left(\frac{y_2(x,\mu_n,q)}{\partial_x y_2(x,\mu_n,q)\Big|_{x=1}} - \frac{y_2(x,\mu_n,p)}{\partial_x y_2(x,\mu_n,p)\Big|_{x=1}}\right) \\
&= \frac{(y_2(x,\mu_n,q)-y_2(x,\mu_n,p))^2}{\partial_\lambda y_2(1,\mu_n,q)\partial_x y_2(x,\mu_n,q)\Big|_{x=1}}
\end{aligned}$$

Recall that by Theorem (4.1)

$$\partial_\lambda y_2(1,\lambda,q)\partial_x y_2(x,\lambda,q)\Big|_{x=1} = \int_0^1 y_2(x,\lambda,q)^2 dx > 0$$

Thus, $\mathrm{Res} f\Big|_{\lambda=\mu_n} > 0$ for all n. One can invoke Lemma (5.2) to conclude that

$$\sum \mathrm{Res} f\Big|_{\lambda=\mu_n} = 0$$

Thus,

$$y_2(x,\mu_n,q) = y_2(x,\mu_n,p), \quad \forall x$$

That implies $q = p$. □

Lemma 5.4.

$$\varkappa(q^*) = -\varkappa(q)$$

In particular, q is even if and only if $\varkappa(q) = 0$.

Proof. We know from the proof of Theorem (5.5) that

$$y_2(1-x,\mu_n,q^*) = -\frac{y_2(x,\mu_n,q)}{\partial_x y_2(x,\mu_n,q)\Big|_{x=1}}$$

Differentiating this identity at $x = 0$, we obtain

$$-\partial_\xi y_2(\xi,\mu_n(q),q^*)\Big|_{\xi=1} = -\frac{\partial_x y_2(x,\mu_n(q),q)\Big|_{x=0}}{\partial_x y_2(x,\mu_n(q),q)\Big|_{x=1}} = -\frac{1}{\partial_x y_2(x,\mu_n(q),q)\Big|_{x=1}}$$

We also know that $\mu_n(q^*) = \mu_n(q)$. Hence

$$\begin{aligned}
\varkappa_n(q^*) &= \ln(-1)^n \partial_\xi y_2(\xi,\mu_n(q^*),q^*)\Big|_{\xi=1} \\
&= \log \frac{(-1)^n}{\partial_x y_2(x,\mu_n(q),q)\Big|_{x=1}} \\
&= -\varkappa_n(q)
\end{aligned}$$

Furthermore, if q is even, then $q^* = q$, and $\varkappa_n(q) = 0$ for all n. Vice versa, assume $\varkappa_n(q) = 0$ for all n.

Then $(\varkappa(q^*), \mu(q^*)) = (\varkappa(q), \mu(q))$. By Theorem (5.5) $q^* = q$, i.e. , q is even. $\square$

Set

$$V_n(x,q) = 2\partial_x g_n^2 = 2\partial_x \partial_q \mu_n$$

$$W_n(x,q) = -2\partial_x(a_n - [a_n]g_n^2) = -2\partial x \partial_q \varkappa_n$$

Due to Corollary (4.1)

$$V_n = 4\pi n \sin(2\pi n x) + \mathcal{O}(1)$$

$$W_n = -2\cos(2\pi n x) + \mathcal{O}\left(\frac{1}{n}\right)$$

uniformly on bounded sets.

Theorem 5.6. *$(\varkappa, \mu)$ is a local real analytic diffeomorphism at each point $q \in L^2$. The inverse for $d_q(\varkappa, \mu)$ is a linear map from $l_1^2 \times \mathbb{R} \times l^2 \to L^2$ given by*

$$(d_q(\varkappa,\mu)^{-1})(\xi,\eta) = \sum \xi_n V_n + \eta_0 1 + \sum \eta_n W_n$$

Proof. μ is real analytic on L^2. Let us check that $\varkappa$ is real analytic. Let $p \in L^2_{\mathbb{R}}$. WE know that μ_n, g_n^2 are analytic for $||q-p|| < r_p$. Furthermore, $\partial_x y_n(1,\lambda,q)$ is analytic for $||\lambda - \mu_n(p)|| < \rho_p, ||q-p|| < r_p$ and does not vanish (since $\partial_x y_2(1, \mu_n(o), p) \neq 0$) by Theorem (5.4).

$$\partial_q \varkappa_n = a_n - [a_n]g_n^2$$

One can now repect the estimation from Theorem (5.4) to show that

$$\varkappa_n(q) = \frac{1}{2\pi n}(\sin(2\pi n x), q) + \mathcal{O}\left(\frac{1}{n^2}\right)$$

uniformly for $||q-p|| < r_p$. Thus $\varkappa$ is real analytic map with values in l_1^2. Let us now discuss the derivative of the map $(\varkappa, \mu)$:

$$v \to \Big(((\partial_q \varkappa), v); [v]; ((\partial_q \tilde{\mu}_n), v)\Big)$$

By Theorem (5.4)

$$2\pi n \partial_q \varkappa_n = (\sin(2\pi n x), q) + \mathcal{O}\left(\frac{1}{n}\right)$$

By Theorem (4.4)

$$\partial_q \tilde{\mu}_n = -2\cos(2\pi n x) + \mathcal{O}\left(\frac{1}{n}\right)$$

To show that the derivative map is invertible, we invoke Theorem (4.12) . For that we need to show that

$$\partial_q \varkappa_n, \quad n = 1,2,\ldots; 1; \partial_q \tilde{\mu}_n, \quad n = 1,2,\ldots$$

are linearly independent. Due to Lemma (5.3) for all m, n holds

$$(\partial_q \varkappa_m, \partial_x \partial_q \mu_n) = \frac{1}{2}\delta_{m,n}$$

$$(\partial_q \mu_m, \partial_x \partial_q \mu_n) = 0, \qquad \text{for all } m, n$$

Note that $\partial_q \tilde{\mu}_m = \partial_q \mu_m - 1, \partial_x(\partial_q \tilde{\mu}_m) = \partial_x \partial_q \mu_m$. Furthermore

$$\partial_q \mu_n \Big|_{x=0,1} = 0, \quad a_n \Big|_{x=0,1} - [a_n] g_n^2 \Big|_{x=0,1} = 0$$

Integrating by parts gives

$$(\partial_q \tilde{\mu}_m, \partial_x \partial_q \mu_n) = -(\partial_x(\partial_q \tilde{\mu}_m), \partial_q \mu_n) = -(\partial_x \partial_q \mu_n, \partial_q \mu_n) = 0$$

$$(1, \partial_x \partial_q \mu_n) = 0$$

It follows from (**??**) that $\partial_x \partial_q \mu_m$ is orthogonal to all vectors in (**??**) but $\partial_q \varkappa_m$. Therefore, if

$$\sum \xi_m \partial_q \varkappa_m + \eta_0 + \sum \eta_n \partial_q \tilde{\mu}_n = 0$$

For some $((\xi_m), \eta_0, (\eta_n)) \in l_1^2 \times \mathbb{R} \times l^2$, then $\xi_m = 0$ for all m (The series here converges in L^2). Recall that the vectors $1, \partial_q \tilde{\mu}_n$ are linearly independent (they form a basis of E by Theorem (4.9)). Hence, $\eta_n = 0, n = 0, 1, 2, \ldots$. We also have

$$\sum_m \xi_m \partial_q \varkappa_m - \sin(2\pi m x)||^2 + \sum || - \partial_q \tilde{\mu}_m - 2\cos(2\pi n x)||^2 < +\infty$$

Therefore the map

$$(\xi, \eta) \to \sum_m \xi_m m \partial_q \varkappa_m + \eta_0 + \sum_n \eta_n \partial_q \tilde{\mu}_n$$

is an isomorphism from $l^2 \times \mathbb{R} \times l^2$ onto L^2. Therefore the derivative map ($v \in L^2$)

$$v \to \Big((\partial_q \varkappa_m, v)_m, [v], (\partial_q \tilde{\mu}_n, v)_n \Big) \in l_1^2 \times \mathbb{R} \times l^2$$

is invertible. Therefore $(\varkappa, \mu)$ is a local real analytic diffeomorphism. Let us calculate the inverse. Given $(\xi_m) \in l_1^2, \eta_0, (\eta_n) \in l^2$ we are looking for v such that

$$(\partial_q \varkappa_m, v) = \xi_m, \quad [v] = \eta_0, \quad (\partial_q \tilde{\mu}_n, v) = \eta_n$$

We invoke Lemma (5.3), we have

$$\begin{aligned}
(\partial_q \varkappa_m, 2 \sum \xi_m \partial_x \partial_q \mu_n) &= \xi_m \\
(\partial_q \tilde{\mu}_n, 2 \sum_n \xi_n \partial_x \partial_q \mu_n) &= 0 \\
(\partial_q \tilde{\mu}_n, -2 \sum \eta_r \partial_x \partial_q \varkappa_r) &= 2(\partial_x \partial_q \mu_n, \sum_m \eta_r \partial_q \varkappa_r) \\
&= \eta_n \\
(\partial_q \varkappa_m, -2 \sum \eta_r \partial_x \partial_q \varkappa_r) &= 0 \\
(1, 2 \sum_n \xi_n \partial_x \partial_q \mu_n - 2 \sum_r \eta_r \partial_x \partial_q \varkappa_r) &= 0
\end{aligned}$$

That verifies the formula for the inverse. □

Chapter 6

Isospectral Sets, The $\varkappa$-Flow

Given $p \in L^2[0,1]$ set

$$M(p) = \mu^{-1}(\mu(p))$$

$$M_n(p) = \{q : \mu_n(q) = \mu_n(p)\}$$

Note that since $\partial_q \mu_n = g_n^2 > 0$. $M_n(p)$ is a smooth manifold. We know also that g_n^2's are linearly independent, so

$$M_1(p) \cap \ldots \cap M_n(p)$$

is a smooth manifold. Set

$$U_0 = 1, \quad U_n = g_n^2 - 1, \quad V_n = 2\partial_x g_n^2$$

$$U_\eta = \sum \eta_n U_n$$

$$V_\xi = \sum \xi_n V_n$$

By Theorem (4.8), (**??**)

$$\{U_\eta : \eta \in \mathbb{R} \times l^2\} \perp \{V_\xi : \xi \in l_1^2\}$$

$$\mathbb{R} \oplus \{U_\eta : \eta \in \mathbb{R} \times l^2\} \oplus \{V_\xi : \xi \in l_1^2\} = L^2$$

Theorem 6.1. *a) For any p, $M(p)$ is a real analytic submanifold of L^2.*

$$M(p) \subset \{q : [q] = [p]\}$$

b)

$$T_q M(p) = \{V_\xi(q) : \xi \in l_1^2\}$$

$$N_q M(p) = \{U_\eta : \eta \in l^2\}$$

Proof.

$$(d_q \mu)(w) = \Big((U_n, w) : n \geq 0\Big)$$

set $\ker_q = \ker d_q$. We want to show that $d_q\mu$ restricted to $\ker_q^{\perp}$ is invertible. Clearly

$$\ker_q^{\perp} = \{U_\eta : \eta \in \mathbb{R} \times l^2\}$$

Recall that

$$(U_m, U_n) = \delta_{m,n} + \left(\cos(2\pi mx), \mathcal{O}\left(\frac{1}{n}\right)\right) + \left(\cos(2\pi nx), \mathcal{O}\left(\frac{1}{m}\right)\right) + \mathcal{O}\left(\frac{1}{mn}\right)$$

So,

$$\sum_{m,n} \left((U_m, U_n) - \delta_{m,n}\right)^2 = \sum_{m,n} \left(\cos(2\pi mx), \mathcal{O}\left(\frac{1}{n}\right)\right)^2 + \sum_{m,n} \left(\cos(2\pi nx), \mathcal{O}\left(\frac{1}{m}\right)\right)^2 + \sum \mathcal{O}\left(\frac{1}{m^2}\right)\mathcal{O}\left(\frac{1}{n^2}\right)$$

Due to Bessel inequality

$$\sum_n \sum_m \left(\cos(2\pi mx), \mathcal{O}\left(\frac{1}{n}\right)\right)^2 \leq \sum_n \mathcal{O}\left(\frac{1}{n^2}\right) < +\infty$$

Thus,

$$\Big((U_m, U_n)\Big)_{m,n} - I = \text{Hilbert-Schmidt}$$

Since U_m is a basis in $\ker_q^{\perp}$, $((U_m, U_n))_{m,n}$ is one-to-one. By the Fredholm alternative $(U_m, U_n))$ is invertible. That implies the statement. □

Corollary 6.1. *$\varkappa(q)$ defines "global" coordinates on M_p.*

$$d_q\varkappa(V_\xi) = \xi$$

Proof. Recall that $q \to (\mu(q), \varkappa(q))$ is an analytical embedding. The identity $d_q\varkappa(V_\xi) = \xi$ follows from Lemma (5.3) □

Let $\phi^t(q, \xi)$ be the flow of the vector-field V_ξ. One has

$$\phi^{dt}(q, \xi) = q + V_\xi dt + \mathcal{O}(dt^2),$$

$$\varkappa_n\Big(\phi^{dt}(q, \xi)\Big) = \varkappa_n(q) + \left(\partial_q \varkappa_n, dt \sum_m \xi_m V_m\right) + \mathcal{O}(dt^2) = \varkappa_m(q) + \xi_n dt + \mathcal{O}(dt^2),$$

$$\varkappa\Big(\phi^t(q, \xi)\Big) = \varkappa(q) + t\xi$$

The flow is defined as long as there is no blow up i.e. $||\phi^t(q, \xi)||$ does not how to $+\infty$ when $t \to t_0$

Lemma 6.1.

$$(q, V_n) = (-1)^n \frac{4\sinh(\varkappa_n(q))}{\partial_\lambda y_2(1, \mu_n(q), q)}$$

Proof.

$$(q, \partial_x g_n^2) = \int_0^1 q 2 g_n \partial_x g_n dx$$

Recall that

$$-\partial_{xx}^2 g_n + q g_n = \mu_n g_n$$

so

$$\begin{aligned}(q, \partial_x g_n^2) =& 2\int_0^1 (\partial_{xx}^2 g_n + \mu_n g_n)\partial_x g_n dx \\ =& \int_0^1 \partial_x\Big((\partial_x g_n)^2 + \mu_n g_n^2\Big) = \Big((\partial_x g_n)^2 + \mu_n g_n^2\Big)\Big|_{x=0}^{x=1} \\ =& \frac{(\partial_x y_2(x, \mu_n(q), q))^2}{\partial_\lambda y_2(1, \mu_n(q), q)\partial_x y_2(1, \mu_n(q), q)}\Big|_{x=0}^{x=1} \\ =& \frac{1}{\partial_\lambda y_2(1, \mu_n(q), q)}\left(\partial_x y_2(1, \mu_n(q), q) - \frac{1}{\partial_x y_2(1, \mu_n(q), q)}\right)\end{aligned}$$

We used $\partial_x y_2(x, \lambda, q)\big|_{x=0} = 1$ for any λ. Recall that

$$\varkappa_n(q) = \log\Big((-1)^n \partial_x y_2(1, \mu_n(q), q)\Big)$$

That implies the identity. □

It is convenient to introduce the notation

$$\gamma_n = \frac{(-1)^n}{\partial_\lambda y_2(1, \mu_n(q), q)}$$

By Corollary (**??**)

$$\partial_\lambda y_2(1, \mu_n) = \frac{(-1)^n}{2n^2\pi^2}\left(1 + \mathcal{O}\left(\frac{\log n}{n}\right)\right)$$

So,

$$\gamma_n = n^2\pi^2\left(1 + \mathcal{O}\left(\frac{\log n}{n}\right)\right) > 0, \quad n >> 1$$

Furthermore, by Corollary (**??**)

$$\partial_\lambda y_2(1, \mu_n) = -\frac{1}{n^2\pi^2}\prod_{m\geq 1, m\neq n}\frac{\mu_m - \mu_n}{m^2\pi^2}$$

In particular, $\partial_\lambda y_2(1, \mu_n)$ depends only on the Dirichlet spectrum. In other words we have the following statement.

Lemma 6.2.

$$\gamma_n(\phi^t(q)) = \gamma_n(q), \quad n \geq 1$$

We prove now

Lemma 6.3.

$$||\phi^t(q, V_\xi)||^2 = ||q||^2 + 8\sum_{n\geq 1}\gamma_n(q)\big(\cosh(\varkappa_n(q) + t\xi_n) - \cosh\varkappa_n(q)\big)$$

Proof.

$$\frac{1}{2}\partial_s||\phi^s(q)||^2_{s=0} = (q, V_\xi(q)) = \sum_{n\geq 1}\xi_n(q, V_n(q)) = 4\sum_{n\geq 1}\xi_n\gamma_n(q)\sinh(\varkappa_n(q))$$

Thus, for general t we have the following

$$\frac{1}{2}\partial_t||\phi^t(q)||^2 = 4\sum_{n\geq 1}\xi_n\gamma_n(\phi^t(q))\sinh(\varkappa(\phi^t(q))) = 4\sum_{n\geq 1}\xi_n\gamma_n(q)\sinh(\varkappa_n(q)+t\xi_n)$$

Note that here

$$|\xi_n\gamma_n\sinh(\varkappa_n+t\xi_n)| = \mathcal{O}\left((\xi_n^2+|\xi_n\varkappa_n|)\gamma_n\right)$$

provided $|t| = \mathcal{O}(1)$. Since $\gamma_n = o(n^2)$ and $\xi_n, \varkappa_n \in l_1^2$, the series here converges. Therefore

$$||\phi^t(q)||^2 - ||q||^2 = 8\sum_{n\geq 1}\gamma_n\int_0^t\xi_n\sinh(\varkappa_n+s\xi_n)ds = 8\sum_{n\geq 1}\gamma_n\big(\cosh(\varkappa_n(q)+t\xi_n)-\cosh\varkappa_n\big)$$

□

Theorem 6.2. *The Flow $\phi^t(q, V_\xi)$ is well defined for all t.*

Proof. Due to Theorem (5.5) the map $q \to (\mu(q), \varkappa(q))$ is an injective diffeomorphism from L^2 to $l^2\times l_1^2$. Since $\sup||\phi^t(q, V_\xi)|| < +\infty$, the statement follows. □

Remark 6.4. Since $\varkappa(\phi^t(q)) = \varkappa(q)+\xi t$, $t\in\mathbb{R}$ the set $M(q)$ is unbounded.

For $q\in M(o)$ and $\xi\in l_1^2$, set

$$\exp_q(V_\xi) = \phi^t(q, V_\xi)\Big|_{t=1}$$

Note that

$$\varkappa\big(\exp_q(V_\xi)\big) = \varkappa(q)+\xi$$

Clearly we have the following statement

Theorem 6.3. *For fixed q,$\exp_q(V_\xi)$ is a real analytic isomorphism between $T_qM(p)\simeq l_1^2$ and $M(p)$.*

Corollary 6.2. *There is a unique even point $q_0\in M(p)$. Moreover*

$$||q_0|| < ||q||, \quad \textit{for any } q\in M(p)$$

Proof. Set $q_0 = \exp_p(V_{-\varkappa(p)})$. Then $\varkappa(q_0) = \varkappa(p)-\varkappa(p) = 0$. By Lemma (**??**) q_0 is even. By Theorem (5.5) the map $q\to(\mu(q), \varkappa(q))$ is injective. Since $\mu(q) = \mu(p)$ for $q\in M(0)$, $\varkappa(q)\neq 0$ for any $q\neq q_0$. Again by Lemma (**??**) no $q\neq q_0$ is even. Furthermore, for any $\xi\neq 0$, one has due to Lemma (6.3)

$$\begin{aligned}||\exp_{q_0}(V_\xi)||^2 =& ||q_0||^2 + 8\sum_{n\geq 1}\gamma_n(q_0)\big(\cosh(\varkappa_n(q_0)+\xi)-\cosh\varkappa_n(q_0)\big)\\ =& ||q_0||^2 + 8\sum_{n\geq 1}\gamma_n(q_0)\big(\cosh\xi - 1\big) > ||q_0||^2\end{aligned}$$

Since the range of $\xi \to \exp_{q_0}(V_\xi)$ is $M(p)$, one has

$$||q_0|| < ||q||, \text{ for any } q \in M(p)$$

□

Chapter 7

The Spectral Map Range, The μ-Flow

Let $\mu_n(q)$ be the Dirichlet eigenvalues. In this section $q \in L^2$ as usual, and

$$\tilde{\mu}_n(q) = \mu_n(q) - \pi^2 n^2 - [q], \quad \mu(q) = ([q], \tilde{\mu}_n(q), n \geq 1) \in \mathbb{R} \times l^2.$$

Our first goal is to show that the map is onto

$$S = \{s, (\gamma_n) \in l^2 : \pi^2 n^2 + \gamma_n < \pi^2 (n+1)^2 + \gamma_{n+1}\}$$

Let $\mathbb{I}$ be the constant vector-field on $\mathbb{R} \times l^2 \times l_1^2$

$$\mathbb{I}_n = \{o, \delta_{m,n}; m \geq 1, o \in l_1^2\} \tag{7.1}$$

Consider its pull back via the map μ. By

$$(d_q \mu)^{-1} \mathbb{I}_n = -2\partial_x (a_n - [a_n] g_n^2) = W_n(x, q) \tag{7.2}$$

Set

$$W_\eta = \eta_0 + \sum_{n \geq 1} \eta_n W_n, \quad \eta \in \mathbb{R} \times l^2 \tag{7.3}$$

Let $\phi^t(q, W_\eta)$ be the W_η-flow. Clearly

$$\mu\big(\phi^t(q, W_\eta)\big) = \mu(q) + t\eta \tag{7.4}$$

consider $\phi^t(q, W_n)$. Then

$$\mu\big(\phi^t(q, W_n)\big) = \begin{cases} \mu_m(q), & m \neq n \\ \mu_n(q) + t, & m = n \end{cases} \tag{7.5}$$

As we know $\mu_{n-1}(\tilde{q}) < \mu_n(\tilde{q}) < \mu_{n+1}(\tilde{q})$ for any $\tilde{q} \in L^2$ and any n. So

$$\mu_{n-1}(q) < \mu_n(q) + t < \mu_{n+1}(q) \tag{7.6}$$

That defines the interval of t where there is a chance to define the flow. We want to show that for all t in (7.6) the flow is indeed on defined. First of all we need some auxiliary lemmas.

Lemma 7.1. *Let f be a nontrivial solution of*

$$-y'' + qy = \lambda y \tag{7.7}$$

and let g be a nontrivial solution of

$$-y'' + qy = \mu y, \quad \mu \neq \lambda \tag{7.8}$$

Then

$$\frac{[g, f]}{g} \tag{7.9}$$

is a non trivial solution of

$$-y'' + \left(q - 2\partial^2_{xx} \log|g|\right)y = \lambda y \tag{7.10}$$

For $\lambda = \mu$ the general solution of (7.10) is as follows:

$$\frac{1}{g}\left(a + b\int_0^x g^2(s)ds\right) \tag{7.11}$$

Here if g has roots, then the equation is considered between them.

Proof. The proof can be done just by direction calculation. Here is a slightly nicer way to verify the claim. Set

$$A = g(\partial_x)\frac{1}{g}, \quad A^* = -\frac{1}{g}(\partial_x)g$$

Using the equation

$$-g'' + qg = \mu g$$

one obtains

$$A^*A = -\frac{d^2}{dx^2} + q - \mu$$

So, f obeys

$$A^*Ay = (\lambda - \mu)y \tag{7.12}$$

Similarly, using the equation $(g''/g) = \mu - q$, one obtains

$$AA^* = -\partial^2_{xx} - \frac{g''}{g} + 2\left(\frac{g'}{g}\right)^2 = -\partial^2_{xx} + \left(q - 2\partial^2_{xx}\log|g|\right) - \mu$$

Applying A to both sides to (7.12) one obtains

$$AA^*Ay = (\lambda - \mu)Ay$$

So, f is a solution of (7.12) then Af is a solution of

$$\big(-\partial_{xx}^2 + (q - 2\partial_{xx}^2 \log|g|)\big) y = \lambda y$$

Let us calculate Af:

$$Af = g(\partial_x)\frac{f}{g} = \partial_x f - f\frac{\partial_x g}{g} = \frac{[g,f]}{g}$$

Note that $[g,f]$ can not be identically zero. This is because

$$[g,f]' = g\partial_{xx}^2 f - f\partial_{xx}^2 g = g(\mu - q)f - f(\lambda - q)g = (\mu - \lambda)gf$$

Let h obey

$$-\partial_{xx}^2 h + (q - 2\partial_{xx}^2 \log|g|)h = \mu h$$

Then

$$AA^* h = \big(-\partial_{xx}^2 + (q - 2\partial_{xx}^2 \log|g|) - \mu \big) h = 0$$

on the other hand

$$AA^* h = -g\partial_x \left(\frac{1}{g^2}\partial_x(gh) \right)$$

Thus, between any two roots of g

$$\partial_x(gh) = bg^2$$

with b depending on these roots. That implies the second statement. □

Lemma 7.2. *Let g, h, f be non-trivial solutions:*

$$-\partial_{xx}^2 g + qg = \mu g, \quad -\partial_{xx}^2 h + qh = \nu h, \quad -\partial_{xx}^2 f + qf = \lambda f, \quad \lambda \neq, \mu, \nu$$

Then

$$\frac{1}{h}\left[h, \frac{1}{g}[g,f]\right] = (\mu - \lambda) f - \frac{1}{g}[g,f]\partial_x \log|gh|$$

is a nontrivial solution of

$$-y'' + \left(q - 2\partial_{xx}^2 \log|gh|\right) y = \lambda y$$

Proof. This is just an iteration of the previous lemma. □

Remark 7.3. Lemma 1 was discovered by Gaston Darboux in 1882.

Set

$$w_n(x,\lambda,q) = y_1(x,\lambda) + \frac{y_1(1,\mu_n) - y_1(1,\lambda)}{y_2(1,\lambda)} y_2(x,\lambda)$$

w_n is a unique solution of

$$-y'' + qy = \lambda y$$

with $w_n(0,\lambda) = 1, w_n(1,\lambda) = y_1(1,\mu_n)$ provided $\lambda \neq \mu_m, m = 1, 2, \ldots$. At $\lambda = \mu_m$ with $m \neq n$, w_n has a pole. There is no singularity at $\lambda = \mu_n$ for $\partial_\lambda y_2(1,\mu_n) \neq 0$. Set

$$z_n(x,q) = y_2(x,\mu_n(q),q)$$

consider

$$\omega_n(x,\lambda,q)=[w_n,z_n],\quad x\in[0,1],\lambda\in(\mu_{n-1}(q),\mu_{n+1}(q))$$

Note that

$$\omega_n\Big|_{x=0}=w_n\Big|_{x=0}\partial_x z_n\Big|_{x=0}-\partial_x w_n\Big|_{x=0}z_n\Big|_{x=0}=1-\partial_x w_n\Big|_{x=0}0=1$$

$$\omega_n\Big|_{x=1}=w_n\Big|_{x=1}\partial_x z_n\Big|_{x=1}=y_1(1,\mu_n,q)\partial_x y_2(1,\mu_n,q)=1,\quad \text{for all } \lambda$$

$$\omega_n\Big|_{\lambda=\mu_n}=[y_1,y_2]=1,\ \text{for all } x$$

Lemma 7.4. *The function ω_n is strictly positive for $x\in[0,1],\lambda\in(\mu_{n-1}(q),\mu_{n+1}(q))$.*

Proof. Assume the statement fails. Then there exists $\lambda_0\in(\mu_{n-1}(q),\mu_n(q))\cup(\mu_n(q),\mu_{n+1}(q))$ and $0<x_0<1$ such that $\omega_n(x_0,\lambda_0,q)=0$ and $\omega_n(\cdot,\lambda_0,q)$ has a local minimum at $x=x_0$. One has

$$\begin{aligned}0=&\partial_x\omega_n(x_0,\lambda_0,q)\\=&w_n(x_0,\lambda_0,q)\partial^2_{xx}y_2(x_0,\mu_n(q),q)-\partial^2_{xx}w_n(x_0,\lambda_0,q)y_2(x_0,\mu_n(q),q)\\=&-w_n(x_0,\lambda_0,q)\big(\mu_n-q\big)y_2(x_0,\mu_n(q),q)+\big(\lambda_0-q\big)w_n(x_0,\lambda_0,q)y_1(x_0,\mu_n(q),q)\end{aligned}$$

Since $\omega_n(x_0,\lambda_0,q)=0$ we have also

$$0=w_n(x_0,\lambda_0,q)\partial_x y_2(x_0,\lambda_0,q)=\partial_x w_n(x_0,\lambda_0,q)y_2(x_0,\lambda_0,q)$$

If $w_n(x_0,\lambda_0,q)=0$ then $\partial_x w_n(x_0,\lambda_0,q)\neq 0$ and $y_2(x_0,\lambda_0,q)$ must vanish. Similarly, if $y_2(x_0,\lambda_0,q)=0$, then $w_n(x_0,\lambda_0,q)$ must vanish. Thus $w_n(x_0,\lambda_0,q)=0$ and $y_2(x_0,\lambda_0,q)=0$, so

$$w_n(x,\lambda_0,q)=\partial_x w_n(\cdot,\lambda_0,q)\Big|_{x=x_0}(x-x_0)+\mathcal{O}\left((x-x_0)^2\right)$$

$$y_2(x,\mu_n(q),q)=\partial_x y_2(\cdot,\mu_n(q),q)(x-x_0)+\mathcal{O}\left((x-x_0)^2\right)$$

Here

$$\partial_x w_n(\cdot,\lambda_0,q)\Big|_{x=x_0}\neq 0,\quad \partial_x y_2(\cdot,\mu_n(q),q)\Big|_{x=x_0}\neq 0$$

Hence,

$$\partial\omega_n(x,\lambda_0,q)=(\lambda_0-\mu_n)\partial_x w_n(\cdot,\lambda_0,q)\Big|_{x=x_0}\partial_x y_2(\cdot,\mu_n(q),q)\Big|_{x=x_0}(x-x_0)^2+\mathcal{O}\left((x-x_0)^3\right)$$

This contradicts the assumption that $\omega_n(\cdot,\lambda_0,q)$ has a local minimum at $x=x_0$. □

Theorem 7.1.

$$\phi^t(q,W_n)=q-\partial^2_{xx}\log\omega_n(x,\mu_n+t,q)$$

for all $\mu_{n-1}<\mu_n(q)+t<\mu_n(q)$.

Proof. Let $w_{n,t}=w_n(x,\mu_n+t,q),\omega_{n,t}=\omega_n(x,\mu_n+t,q)$. By Lemma (**??**)

$$h=\frac{1}{z_n}[w_{n,t},z_n]=\frac{\omega_{n,t}}{z_n}$$

obeys

$$-y'' + (q - \partial_{xx}^2 \log z_n)y = (\mu_n + t)t$$

By Lemma (6.1) $\omega_{n,t}$ is strictly positive for $x \in [0,1], \mu_{n-1} < \mu_n + t < \mu_{n+1}$. Therefore

$$q^t = q - 2\partial_{xx}^2 \log \omega_{n,t}$$

is in L^2. Let $z_{n,t} = 1/h = z_n/\omega_{n,t}$. this function also belongs to L^2. For $j \neq n$ consider

$$z_{j,t} \equiv z_j - \frac{1}{\mu_n - \mu_j} \frac{[z_n, z_j]}{z_n} \partial_x \log \omega_{n,t}$$

By Lemma (**??**), $z_{j,t}$ obeys

$$-y'' + q^t y = (\mu_j + \delta_{j,n} t)t$$

Note also that $z_{j,t}\Big|_{x=0} = z_{j,t}\Big|_{x=1} = 0$. Recall that $(\mu_n + t)$ does not recover q^t. For that we have to verify that $\varkappa_j(q^t) = \varkappa_j(q)$ for all j. That is exactly what is needed to see that $q^t = \phi^t(q, W_n)$. Recall that

$$\varkappa_j(q^t) = \log \left| \frac{\partial_x z_{j,t}\big|_{x=1}}{\partial_x z_{j,t}\big|_{x=0}} \right| = \log \left| \frac{\partial_x z_j(1)}{\partial_x z_j(0)} \right| = \varkappa_j(q)$$

□

Theorem 7.2. *The range of the map μ is $S \subset \mathbb{R} \times l^2$.*

Proof. Let $\sigma \in S$ be arbitrary. Clearly we can assume $\sigma = (0, \tilde{\sigma})$ where $\tilde{\sigma} \in l^2$. Consider

$$\sigma^N = (\mu_1^{(0)}, \ldots, \mu_n^{(0)}, \ldots), \quad \mu_j^{(0)} = \pi^2 j^2$$

Clearly $\sigma^N \to \mu(0)$. Since μ is a local diffeomorphism there exists N large enough such that $\sigma^N = \mu(q)$. The vector fields $\mathbb{I}_k$, $k = 1, 2, \ldots, N$ define flows which act transitively on $\mathbb{R}^n \subset l^2$. Since S is defined via

$$\pi^2 n^2 + \gamma_n < \pi^2 (n+1)^2 + \gamma_{n+1}$$

The flows $\phi_n^t(q)$, $n = 1, \ldots, N$ allow one to transform q into $\tilde{q}$ with $(\tilde{\mu}_n(\tilde{q}))_{n=1}^N$ begin arbitrary, as long as

$$\tilde{\mu}_n(\tilde{q}) < \tilde{\mu}_{n+1}(\tilde{q})$$

□

Corollary 7.1. *The sequence $\mu_1 < \mu_2 < \ldots < \mu_n < \ldots$ is a dirichlet spectrum if some $q \in L^2[0,1]$ if and only if*

$$\mu_n = \pi^2 n^2 + s + l^2(n)$$

Remark 7.5. This result was discovered in Gelfand-Levitan's 1951 paper which appeared in AMST,1,253-304 1955.

Chapter 8

Interpolation Formula for Hill Discriminants

Let $y_1(x, \lambda, q), y_2(x, \lambda, q)$ be the fundamental solutions. The following function

$$\Delta(\lambda, q) = y_1(1, \lambda, q) + \partial_x y_2(1, \lambda, q) \tag{8.1}$$

is called the Hill discriminant. It is the trace of the fundamental matrix and it plays a very important role in the periodic spectrum which we study in Part 9. Here we are concerned with the following problem. Assume $\mu_n(p) = \mu_n(q), n = 1, \ldots$. Assume also

$$\Delta(\mu_n, p) = \Delta(\mu_n, q), \quad n = 1, 2, \ldots \tag{8.2}$$

where $\mu_n = \mu_n(p)$. We want to show that in this case

$$\Delta(\lambda, p) = \Delta(\lambda, q) \tag{8.3}$$

for all $\lambda \in \mathbb{C}$. Since $\Delta(\lambda, p), \Delta(\lambda, q)$ are entire functions this is a problem of uniqueness and interpolation. If we could interpolate $\Delta(\lambda, p)$ from $\lambda = \mu_n$, $n = 1, \ldots$ to all $\lambda \in \mathbb{C}$ this would resolve the problem. To do the interpolation one needs "good" asymptotic for the function at $|\lambda| \to \infty$. It turns out that the function $\Delta(\lambda, p)$ does not obey the needed estimates at $|\lambda| \to \infty$. First we will consider some other important functions which do obey the needed estimates. That allows us to develop partial fraction expansions for these functions. In regard of $\Delta(\lambda, p) = \Delta(\lambda, q)$ we just consider $\Delta(\lambda, p) - \Delta(\lambda, q)$ and show that this function also obeys the needed estimates. Therefore it vanishes everywhere.

Lemma 8.1. *Let $f \in L^1[0, 1]$. Then for any $\epsilon > 0$ there exists a constant $C(\epsilon)$ such that for any $\xi \in \mathbb{C}$, we have*

$$\left| \int_0^1 f(x) \cos(\xi x) dx \right| \leq \exp(|\Im \xi|) \left(\epsilon + \frac{C(\epsilon)}{|\xi|} \right)$$

$$\left| \int_0^1 f(x) \sin(\xi x) dx \right| \leq \exp(|\Im \xi|) \left(\epsilon + \frac{C(\epsilon)}{|\xi|} \right)$$

Proof. Given $\epsilon > 0$, we find $g \in C^1[0,1]$ such that

$$\int_0^1 |f(x) - g(x)|dx < \epsilon$$

Then,

$$\left|\int_0^1 f(x)\cos(\xi x)dx\right| \le \left|\int_0^1 g(x)\cos(\xi x)dx\right| + \max_x |\cos \xi x| \int_0^1 |f(x) - g(x)|dx$$

Note that $|\cos \xi x|, |\sin \xi x| \le \exp(|\Im \xi|)$, and

$$\int_0^1 g(x)\cos(\xi x)dx = \frac{g(x)}{\xi}\sin(\xi x)\Big|_{x=0}^{x=1} - \frac{1}{\xi}\int_0^1 g'(x)\sin(\xi x)dx$$

and the estimate follows.

□

This version of the Riemann-Lebesgue Lemma allows us to improve a bit on the basic estimates and this is exactly what we need. Now we introduce an important function which allows interpolation:

$$u(\lambda, p) = y_1(1, \lambda, q) - \partial_x y_2(1, \lambda, q)$$

We want to estimate $|u(\lambda, q)|$ using Lemma (8.1). Recall that

$$y_1(x, \lambda) = c_\lambda(x) + \int_0^x s_\lambda(x - t)q(t)y_1(t, \lambda)dt$$

$$y_2(x, \lambda) = s_\lambda(x) + \int_0^x s_\lambda(x - t)q(t)y_2(t, \lambda)dt$$

where

$$c_\lambda(x) = \cos(\sqrt{\lambda}x), \quad s_\lambda(x) = \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}$$

Furthermore,

$$\partial_x y_2 = c_\lambda(x) + \int_0^x c_\lambda(x - t)q(t)y_2(t, \lambda)dt$$

Thus,

$$u(\lambda) = \int_0^1 s_\lambda(1 - t)q(t)y_1(t, \lambda)dt - \int_0^1 c_\lambda(1 - t)q(t)y_2(t, \lambda)dt$$

Lemma 8.2. *Given $\epsilon > 0$ there exists a constant $C(\epsilon, q)$ such that*

$$|u(\lambda)| \le \frac{1}{\sqrt{|\lambda|}}\left(\epsilon + \frac{C(\epsilon, q)}{\sqrt{|\lambda|}}\right)\exp(|\Im\sqrt{\xi}|)$$

Proof. Using the integral equation for $y_1(x, \lambda), y_2(x, \lambda)$, once again, one obtains

$$u(\lambda) = \int_0^1 s_\lambda(1 - t)q(t)c_\lambda(t)dt + \int_0^1 s_\lambda(1 - t)q(t)\int_0^1 s_\lambda(t - \tau)q(\tau)y_1(\tau)d\tau dt -$$
$$- \int_0^1 c_\lambda(1 - t)q(t)s_\lambda(t)dt - \int_0^1 c_\lambda(1 - t)q(t)\int_0^t s\lambda(t - \tau)q(\tau)y_2(\tau)d\tau dt$$

One has

$$|s_\lambda(1-t)s_\lambda(t-\tau)y_1(\tau)| \leq \frac{\exp(|\Im\sqrt{\lambda}|(1-t))}{\sqrt{|\lambda|}}\frac{\exp(|\Im\sqrt{\lambda}|(1-\tau))}{\sqrt{|\lambda|}}\mathcal{O}\left(\exp(|\Im\sqrt{\lambda}|\tau)\right)$$
$$=\mathcal{O}\left(\frac{\exp(|\Im\sqrt{\lambda}|)}{|\lambda|}\right)$$

Also,

$$|c_\lambda(1-t)s_\lambda(t-\tau)y_2(\tau)| \leq \exp(|\Im\sqrt{\lambda}|(1-t))\frac{\exp(|\Im\sqrt{\lambda}|(1-\tau))}{\sqrt{|\lambda|}}\mathcal{O}\left(\frac{\exp(|\Im\sqrt{\lambda}|\tau)}{\sqrt{|\lambda|}}\right)$$
$$=\mathcal{O}\left(\frac{\exp(|\Im\sqrt{\lambda}|)}{|\lambda|}\right)$$

Note that

$$s_\lambda(1-t)c_\lambda(t) - c_\lambda(1-t)s_\lambda(t) = \frac{\sin(\sqrt{\lambda}(1-2t))}{\sqrt{\lambda}} \tag{8.4}$$

Thus

$$u(\lambda) = \frac{1}{\sqrt{\lambda}}\int_0^1 \sin(\sqrt{\lambda}(1-2t)q(t)dt + \mathcal{O}\left(\frac{\exp(|\Im\sqrt{\lambda}|)}{|\lambda|}\right)$$

Applying the estimate of Lemma (8.1) one obtains the statement.

□

We turn now to the interpolation formula for $u(\lambda), \lambda \in \mathbb{C}$. Usually the derivation is done for the function

$$u_-(z) = u(z^2)$$

which plays an important role in the context of the periodic spectral problem.

Lemma 8.3. *Let $\Gamma_n = \{|z| = \pi(n+1/2)\}$. Given $\epsilon > 0$, there exists $C(\epsilon)$ such that for $n > N_0$ and $|z| = \mathcal{O}(1)$*

$$\left|\int_{\Gamma_n} \frac{u_-(\zeta)}{y_2(1,\zeta^2)(\zeta - z)}d\zeta\right| \leq C_0\left(\epsilon + \frac{C(\epsilon,q)}{n}\right)$$

where $C_0 = C_0(q)$.

Proof. Recall that for $|\lambda| >> 1$,

$$\left|y_2(x,\lambda) - \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}\right| = \mathcal{O}\left(\frac{\exp(|\Im\sqrt{\lambda}|x)}{|\lambda|}\right)$$

Recall also that if $\min_n |z - m\pi| \geq \pi/2$ then

$$|\sin(z)| \geq \frac{1}{4}\exp(|\Im z|)$$

Hence, for $|\zeta| = \pi(n+1/2)$ with $n >> 1$ one has

$$|y_2(1,\zeta)^2| \gtrsim \frac{\exp(|\Im\zeta|)}{|\zeta|} \gtrsim \frac{\exp(|\Im\zeta|)}{n}$$

By Lemma (8.2)

$$|u_(\zeta)| \leq \frac{1}{|\zeta|}\left(\epsilon + \frac{C(\epsilon,q)}{|\zeta|}\right)\exp(|\Im\zeta|)$$

Setting $\zeta = R_n \exp(i\theta), 0 \leq \theta \leq 2\pi$ where $R_n = \pi(n+1/2)$, noting that $d\zeta = R_n i \exp(i\theta)d\theta$, and supposing $|\zeta - z| \sim R_n$ one obtains

$$\int_{\Gamma_n} \frac{|u_-(\zeta)|}{|y_2(1,\zeta)||\zeta - z|}|d\xi| \lesssim \left(\epsilon + \frac{C(\epsilon,q))}{|\xi|}\right)$$

as claimed.

□

Corollary 8.1. *For $z^2 \neq \mu_k$,*

$$u_-(z) = y_2(1,z^2)\sum_{k=1}^{\infty} \frac{\sqrt{\mu_k}u_-(\sqrt{\mu_k})}{\partial_\lambda y_2(1,\mu_k)(\mu_k - z^2)}$$

Proof. By the Cauchy Reside Theorem for $z \neq \mu_k$

$$\frac{1}{2\pi i}\int_{\Gamma_n} \frac{u_-(\zeta)}{y_2(1,\zeta^2)(\zeta - z)}d\zeta = \frac{u_-(z)}{y_2(1,z^2)} + \sum_{\sqrt{\mu_k}<\pi(n+1/2)} \frac{u_-(\sqrt{\mu_k})}{\pm\sqrt{\mu_k} - z}\mathrm{Res}\frac{1}{y_2(1,\zeta^2)}\Big|_{\zeta=\pm\sqrt{\mu_k}}$$

Note that

$$\mathrm{Res}\frac{1}{y_2(1,\zeta^2)}\Big|_{\zeta=\pm\sqrt{\mu_k}} = \frac{1}{\partial_\zeta y_2(1,\zeta^2)}\Big|_{\zeta=\pm\sqrt{\mu_k}} = \frac{\pm 1}{2\partial_\lambda y_2(1,\mu_k)\sqrt{\mu_k}}$$

and the statement follows □

We turn now to the main question: Does (8.2) imply (8.3), i.e. $\Delta(\lambda,p) = \Delta(\lambda,q)$? Just like in (8.3) - (**??**) one obtains

$$\Delta(\lambda) = 2c_\lambda(1) + \int_0^1 s_\lambda(1-t)q(t)c_\lambda(t)dt + \int_0^1 c_\lambda(1-t)q(t)s_\lambda(t)dt + \mathcal{O}\left(\frac{\exp(|\Im\sqrt{\lambda}|)}{|\lambda|}\right)$$

Instead of (8.4) this time we have

$$s_\lambda(1-t)c_\lambda(t) + c_\lambda(1-t)s_\lambda(t) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}$$

Thus,

$$\Delta(\lambda,q) = 2\cos\sqrt{\lambda} + \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\int_0^1 q(t)dt + \mathcal{O}\left(\frac{\exp(|\Im\sqrt{\lambda}|)}{|\lambda|}\right) \tag{8.5}$$

Set

$$S(\lambda, q) = \Delta(\lambda, q) - 2\cos\sqrt{\lambda} - [q]\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}$$

Theorem 8.1. *For* $\lambda \neq \mu_k$,

$$S(\lambda) = y_2(1, \lambda)\sum_{k=1}^{\infty}\frac{S(\mu_k)}{\partial_\lambda y_2(1, \mu_k)(\mu_k - \lambda)}$$

Proof. Goes the same way as Corollary (8.1) □

Chapter 9

Periodic Spectrum

Consider the Sturm-Liouville Equation

$$-y'' + qy = \lambda y \tag{9.1}$$

with periodic boundary conditions (P)

$$y(1) = y(0), \quad y'(1) = y'(0)$$

and anti-periodic conditions (AP)

$$y(1) = -y(0), \quad y'(1) = -y'(0)$$

The Floquet matrix is as follows,

$$F(\lambda, q) = \begin{pmatrix} m_1 & m_2 \\ m_1' & m_2' \end{pmatrix}, \quad m_j = y_j(1, \lambda, q), \quad m_j' = \partial_x y_j(1, \lambda, q)$$

Note that

$$\det F(\lambda, q) = [y_1, y_2]\Big|_{x=1} = 1 \tag{9.2}$$

Hill's Discriminant is as follows

$$\Delta = \text{trace}(F) = m_1 + m_2'$$

λ called a periodic (respectively anti-periodic) eigenvalue if there exists a non-trivial solution of (9.1) which obey the condition (P) (respectively (AP)). Recall that if y is a solution of (9.1) then

$$\begin{pmatrix} y(1) \\ y'(1) \end{pmatrix} = F \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix}$$

Thus, λ is a (P) (resp. (AP)) eigenvalue if and only if the matrix $F(\lambda, q)$ has an eigenvalue 1 (resp. -1). Since $\det F$ =1, the eigenvalues of F are as follows

$$\frac{\Delta \pm \sqrt{\Delta^2 - 4}}{2}$$

Thus the (P) (resp. (AP)) eigenvalue equation is as follows

$$\Delta = 2 \qquad (\text{resp. } \Delta = -2) \tag{9.3}$$

Recall the basic estimates

$$\left| y_1(x,\lambda,q) - \cos(\sqrt{\lambda}x) \right| \leq \frac{\exp(|\Im\sqrt{\lambda}|x + ||q||\sqrt{x})}{\sqrt{|\lambda|}}$$

$$\left| \partial_x y_2(x,\lambda,q) - \cos(\sqrt{\lambda}x) \right| \leq ||q|| \frac{\exp(|\Im\sqrt{\lambda}|x + ||q||\sqrt{x})}{\sqrt{|\lambda|}}$$

Thus,

$$\left| \Delta(\lambda) - 2\cos\sqrt{\lambda} \right| \leq \frac{(1 + ||q||)}{\sqrt{|\lambda|}} \exp(|\Im\sqrt{\lambda}|x + ||q||) \tag{9.4}$$

The function $2\cos\sqrt{\lambda}$ is the Hill discriminate for $q = 0$. It is analytic and the roots of $(2\cos\sqrt{\lambda} \mp 2)$ are $\pi^2 n^2$, n-even (respectively n-odd). Now, just as in Part 4, one has the following

Lemma 9.1. *Let N be an integer $N > N(q)$. Then $(\Delta(\lambda) \mp 2)$ has exactly $2n-1$ (respectively $2n$) roots in the half-plane $\Re\sqrt{\lambda} < \pi^2(2n+1/2)^2$. The function $y_2(1,\lambda)$ has exactly $2n+1$ roots in this half-plane.*

- For real q, the roots of $\Delta \mp 2 = 0$ are real (again due to self-adjointness)
- $m_1(\lambda,q)m_2'(\lambda,q) - m_1'(\lambda,q)m_2(\lambda,q) = 1$

If λ is a Dirichlet eigenvalue then $m_2(\lambda,q) = 0$. So

$$m_1(\mu_n(q),q)m_2'(\mu_n(q),q) = 1$$

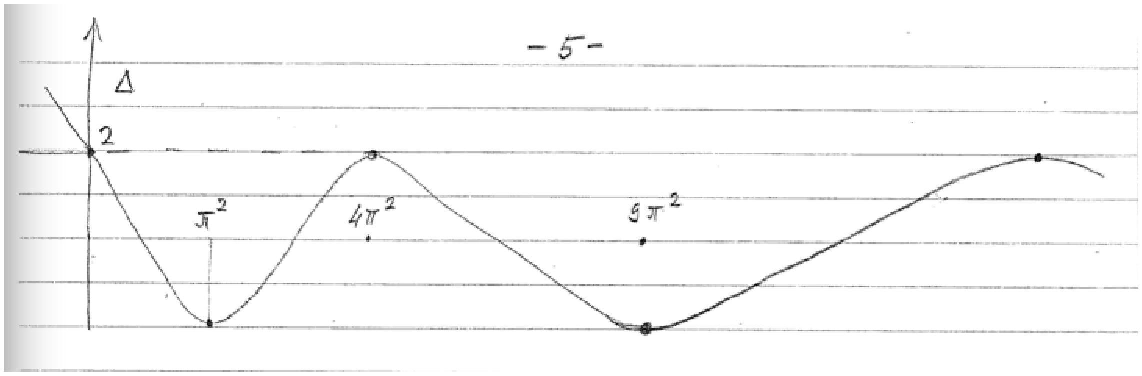
For real q we have : $\operatorname{sgn} m_2'(\mu_n(q),q) = (-1)^n$.

$$\Delta(\mu_n) = m_1(\mu_n) + m_2'(\mu_n) = \frac{1}{m_2'(\mu_n)} + m_2'(\mu_n) \begin{cases} \geq 2 & \text{if } n \text{ is even} \\ \leq -2 & \text{if } n \text{ is odd} \end{cases}$$

Due to the basic estimates, one has

$$\partial_\lambda \Delta - \partial_\lambda(2\cos\sqrt{\lambda}) = o\left(\frac{|\Im\sqrt{\lambda}|}{\sqrt{|\lambda|}} \right) \tag{9.5}$$

- Once again, one obtains the bouncing lemma for ∂_Δ. The function $\partial_\lambda(2\cos\sqrt{\lambda})$ has simple zeros at $\lambda = \pi^2 k^2, k = 1,2,\dots$ (There is no zero at $\lambda = 0$). That implies the following statement: The function $\partial_\lambda \Delta$ has exactly $(2n-1)$ zeros in the Half-Plane $\Re\sqrt{\lambda} \leq \pi^2(2n+1/2)^2$.
- Since $\Delta(\lambda) + 2$ has exactly $2n$ roots on the interval $(-\infty, \pi^2(2n+1/2)^2)$, its derivative $\partial_\lambda \Delta$ has $(2n-1)$ roots interlacing the roots of the function (Rolle's Theorem). Thus, all the roots of $\partial_\lambda \Delta$ are real and they interlace the roots of $\Delta(\lambda) + 2$ (Some may coincide).
- For $q = 0$, $\Delta(\lambda) = 2\cos\sqrt{\lambda}$, which has the following graph



In the general case we know that

$$\Delta(\lambda) \to \infty \quad \text{as} \quad \lambda \to -\infty$$

$$\Delta(\lambda) - 2\cos\sqrt{\lambda} \to \quad \text{as} \quad \lambda \to \infty$$

$\Delta(\lambda) \mp 2$ has (2n-1) (Respectively $2n$) roots on $(-\infty, \pi^2(2n+1/2)^2$.

$$\Delta(\mu_k) \begin{cases} \geq 2, & \text{if } k \text{ is even} \\ \leq 2, & \text{if } k \text{ is odd} \end{cases}$$

$$\mu_1 < \mu_2 < \ldots < \mu_{2n} < \pi^2(2n+1/2)^2 < \mu_{2n+1}$$

Let $\Delta(\lambda_0) = 2, \Delta(\lambda) > 2$ for $\lambda < \lambda_0$. Between λ_0 and the next root of $\Delta(\lambda) = 2$ seats a roots of $\Delta(\lambda) = -2$. Indeed, assume $\Delta(\lambda_0) = 2, \Delta(\tilde{\lambda}) = 2$ and all $2n$ roots of $\Delta(\lambda) + 2, \lambda \in (-\infty, \pi^2(n+1/2)^2)$ belong to $(\tilde{\lambda}, \pi^2(n+1/2)^2)$. Then by Rolle's Theorem $\partial_\lambda \Delta(\lambda)$ would have a roots on $(\lambda_0, \tilde{\lambda})$ and at least $(2n-1)$ roots on $(\tilde{\lambda}, \pi^2(2n+1/2)^2)$. Thus there exists: $\lambda_0 < \lambda_1 < \pi^2(2n+1/2)^2$ such that

$$\Delta(\lambda_0) = 2, \quad \Delta(\lambda_1) = -2$$

$\Delta(\lambda) - 2$ has $(2n-2)$ roots on $(\lambda_1, \pi^2(2n+1/2)^2), \Delta(\lambda) + 2$ has $(2n-1)$ roots on $(\lambda_1, \pi^2(2n+3/2)^2)$. Note that $\partial_\lambda \Delta \neq 0$ for $\lambda \in (\lambda_0, \lambda_1)$ since otherwise $\partial_\lambda \Delta$ would have, $2n$ roots on $(-\infty, \pi^2(2n+1/2)^2)$. In particular :

$$\Delta(\lambda) \text{ strictly decreases on } (\lambda_0, \lambda_1)$$

Obviously, there exists $\lambda_1 \leq \lambda_2 < \pi^2(2n+1/2)^2$ such that $\Delta(\lambda_2) = -2, \partial_\lambda \Delta$ has a simple roots on $[\lambda_1, \lambda_2]$. Note that $\lambda_2 = \lambda_1$ is possible, as it is the case for $q = 0$. Just as above, using a counting argument one concludes that there exists $\lambda_2 < \lambda_3 < \pi^2(2n+1/2)^2$ such that $\Delta(\lambda_3) = 2$ and

$$\Delta(\lambda) \text{ strictly increases on } (\lambda_2, \lambda_3)$$

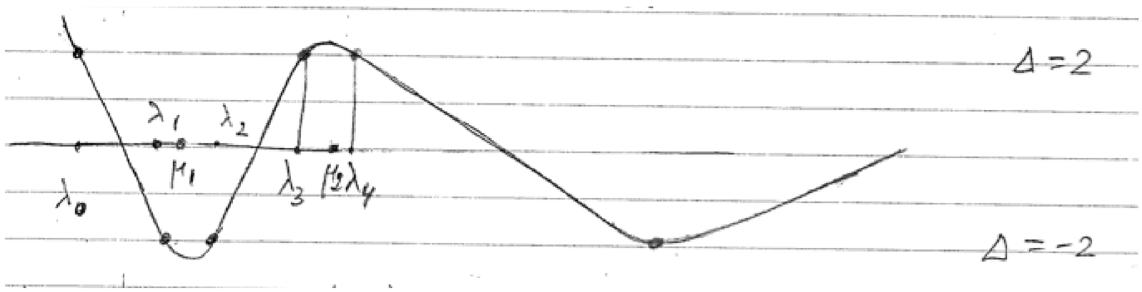
Finally, one obtains

$$\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \ldots \leq \lambda_{4n} < \lambda_{4n+1} \leq \lambda_{4n+2} < \pi^2(2n+1/2)^2$$

where

$$\Delta(\lambda_0) = \Delta(\lambda_{4k+3}) = \Delta(\lambda_{4k+4}) = 2 \quad \& \quad \Delta(\lambda_{4k+1}) = \Delta(\lambda_{4k+2})$$

The graph of $\Delta(\lambda)$ looks as follows



Note that since $\Delta(\mu_n) \geq 2$ if n is even, and $\Delta(\mu_n) \leq -2$ if n is odd and $\mu_1 < \mu_2 < \ldots < \mu_{2n} < \pi^2(2n+1/2)^2 < \mu_{2n+1} < \ldots$. The μ_k are situated as follows

$$\lambda_1 \leq \mu_1 \leq \lambda_2, \quad \lambda_3 \leq \mu_2 \leq \lambda_4, \ldots$$

Note also that μ_k mat lie on the edge of $[\lambda_{2k-1}, \lambda_{2k}]$

Lemma 9.2. *For any $t \in \mathbb{R}$ the periodic spectra of $q(t+x)$ is the same as for $q(x)$.*

Proof. Let $y(x)$ be an eigenfunction of

$$-\partial_{xx}^2 y + q(x)y(x) = \lambda y(x)$$

with $y(1) = y(0), \partial_x y(1) = \partial_x y(0)$. Since $y(x)$ is a solution of a linear differential equation it is defined for all x. Since the initial conditions at $x = 1$ are the same as for $x = 0$, one have

$$y(1+x) = y(x)$$

i.e. $y(x)$ is a 1-periodic function. Given $t \in \mathbb{R}, y(t+x)$ obeys

$$-\partial_{xx} y(t+x) + q(t+x)y(t+x) = \lambda y(t+x)$$

$$y(t+1) = y(t), \partial_x y(t+x)\Big|_{x=1} = \partial_x y(t+x)\Big|_{x=0}$$

That proves the statement. □

- The Dirichlet eigenvalues for $q(t+x)$ are different from the dirichlet eigenvalues for $q(x)$. Let us denote them as

$$\mu_1(t) < \mu_2(t) < \ldots$$

- The following identities are important

$$\begin{aligned}\Delta^2(\mu_n) - 4 =& \big(y_1(1,\mu_n) + \partial_x y_2(1,\mu_n)\big)^2 - 4\\ =& \big(y_1(1,\mu_n) - \partial_x y_2(1,\mu_n)\big)^2 + 4y_1(1,\mu_n)\partial_x y_2(\mu_n) - 4\\ =& \big(y_1(1,\mu_n) - \partial_x y_2(1,\mu_n)\big)^2 + 4[y_1,y_2]\Big|_{x=\mu_n} - 4 = \qquad \big(y_1(1,\mu_n) - \partial_x y_2(1,\mu_n)\big)^2\end{aligned}$$

i.e.

$$\sqrt{\Delta(\mu_n)^2 - 4} = \pm\big(y_1(1,\mu_n) - \partial_x y_2(1,\mu_n)\big) \tag{9.6}$$

since $y_1(1,\mu_n)\partial_x y_2(1,\mu_n) = [y_1,y_2]_{x=\mu_n} = 1$, one has

$$y_1(1,\mu) = \frac{1}{\partial_x y_2(1,\mu_n)}$$

thus

$$\sqrt{\Delta(\mu_n)^2 - 4} = \pm\left(\frac{1 - (\partial_x y_2(1,\mu_n))^2}{\partial_x y_2(1,\mu_n)}\right) \tag{9.7}$$

- Now we will derive a system of differential equation for $\mu_n(t)$.

Lemma 9.3.
$$\frac{d}{dt}\mu_n(t) = \pm\frac{\sqrt{\Delta(\mu_n(t), q(t+\cdot))^2 - 4}}{\partial_\lambda y_2(1,\mu_n(t), q(t+\cdot)}$$

Proof. using the notation $\mu_n(p)$ one has

$$\frac{d}{dt}\mu_n(t) = \left(\partial_p\mu_n\Big|_{p=q(t+\cdot)}, q'(t+\cdot)\right)$$

recall that

$$\partial_p\mu_n(p) = g_n^2(x,p)$$

So,
$$\frac{d}{dt}\mu_n = \int_0^1 g_n^2(x)q'(t+x)dx = -\int_0^1 2g_n g_n' q(t+x)dx$$

since $g_n(0) = g_n(1) = 0$. Note that

$$g_n q(t+x) = \mu_n g_n + g_n''$$

hence,

$$\begin{aligned}\frac{d}{dt}\mu_n &= -2\int_0^1 g_n'(\mu_n g_n + g_n'')dx \\ &= -\int_0^1 (\mu_n(g_n^2)' + (g_n'^2)'dx \\ &= -\mu_n g_n^2\Big|_{x=0}^{x=1} - g_n'^2\Big|_{x=0}^{x=1} \\ &= -g_n'^2(1) + g_n'^2(0) \\ &= -\frac{(\partial_x y_2(1,\mu_n))^2}{||y_2(\cdot,\mu_n)||^2} + \frac{1}{||y_2(\cdot,\mu_n)||^2} \\ &= \frac{1-(\partial_x y_2(1,\mu_n))^2}{||y_2(\cdot,\mu_n)||^2}\end{aligned}$$

Recall that $\partial_\lambda y_2(1,\mu_n)\partial_x y_2(1,\mu_n) = ||y_2(\cdot,\mu_n)||^2$. Thus

$$\frac{d}{dt}\mu_n = \frac{1-(\partial_x y_2(1,\mu_n))^2}{\partial_\lambda y_2(1,\mu_n)\partial_x y_2(1,\mu_n)} = \pm\frac{\sqrt{\Delta(\mu_n(t),q(t+\cdot))^2-4}}{\partial_\lambda y_2(1,\mu_n(t),q(t+\cdot)}$$

see (9.6). □

Corollary 9.1. *Let $q \in L^2$ be arbitrary. Let $\lambda_0 < \lambda_1 \le \lambda_2 \ldots$ be the periodic and snit-periodic eigenvalues of q. Let $\mu_n(t)$ be the Dirichlet eigenvalues of $q(t+x), \lambda_{2n-1} \le \mu(t) \le \lambda_{2n}(t)$. For any n there exists t_n', t_n'' such that $\mu_n t_n') = \lambda_{2n-1}$ and $\mu_n(t_n'') = \lambda_{2n}$*

Proof. If $\lambda_{2n-1} = \lambda_{2n}$ then $\mu_n = \lambda_{2n-1}$. Let $\lambda_{2n-1} < \lambda_{2n}$. We have

$$\frac{d\mu_n}{dt} = \sigma_n\frac{\sqrt{\Delta(\mu_n(t),q(t+\cdot))^2-4}}{\partial_\lambda y_2(1,\mu_n(t),q(t+\cdot)}, \quad \sigma_n = \pm$$

Note that since $d\mu_n/dt$ is continuous, σ_n can not change unless $\mu_n(t)$ hits one of the edges, since $\mathrm{sgn}\partial_\lambda y_2(1,\mu_n(t)) = (-1)^n$. Furthermore, $\partial_\lambda y_2 = \mathcal{O}(1)$. Therefore,

$$\left|\frac{d\mu_n}{dt}\right| \ge p > 0 \text{ as long as } \mu_n(t) \in [\lambda_{2n-1}+\delta, \lambda_{2n}-\delta]$$

That implies the statement. □

Corollary 9.2.

$$\lambda_{2n-1}, \lambda_{2n} = n^2\pi^2 + [q] + l^2(n)$$

Proof. We have that $\mu_n(t) = n^2\pi^2 + [q] + l^2(n)$ uniformly in t. Therefore the statement follows from Corollary (9.1) □

Corollary 9.3.

$$\Delta^2(\lambda) - 4 = 4(\lambda_0 - \lambda)\prod_{n=1}^{\infty}\frac{(\lambda_{2n-1}-\lambda)(\lambda_{2n}-\lambda)}{n^4\pi^4}$$

Proof. Since $\lambda_{2n-1}, \lambda_{2n} = n^2\pi^2 + \mathcal{O}(1)$, the produce converges and defines an entire function $P(\lambda)$ which zeros are exactly $z = \lambda_k, k = 0, 1, 2, \ldots$. Just like in Lemma (4.8) one obtains the identity. □

Isospectral set. Let $L_0^2 = \{q \in L^2[0,1] : [q] = 0\}$

$$\mathrm{Iso}(q) = \{p \in L_0^2 : \lambda_k(p) = \lambda_k(q), k = 1, 2, \ldots\}$$

For $a < b$, denote $[|a,b|]$ the following set

$$[|a,b|] = \{(a,0) \cup (b,0)\} \cup \big((a,b) \times \{-1,1\}\big)$$

Clearly $[|a,b|]$ can be identified with the circle. For convenience we identify if with the circle centred at $(a+b)/2$ and radius $(b-a)/2$.

Given $p \in \mathrm{Iso}(q)$ set $\sigma_n(p) = \mathrm{sgn}(y_1(1,\mu(p),p) - \partial_x y_2(1,\mu_n(p),p))$. Note that since $\Delta^2(\mu_n) - 4 = (y_1(1,\mu_n) - \partial_x y_2(1,\mu_n))^2$, one has

$$\sigma_n(p) = 0 \iff \mu_n(p) \in \{\lambda_{2n-1}(p), \lambda_{2n}(p)\} \tag{9.8}$$

Consider the map

$$\Phi : p \to (\mu_n(p), \sigma_n(p)) \in \prod_{\lambda_{2n-1}<\lambda_{2n}, n\geq 1} [|\lambda_{2n-1}, \lambda_{2n}|]$$

Theorem 9.1. *Φ is a diffeomorphism from Iso(q) onto the torus $\prod_{\lambda_{2n-1}<\lambda_{2n},n\geq 1}[|\lambda_{2n-1},\lambda_{2n}|]$.*

Proof. We know that $p \to (\mu_n(p))$ is real analytic. We know also that $y_1(1,\mu_n(p),p) - \partial_x y_2(1,\mu_n(p),p)$ are real analytic. Using (9.8) one can easily verify that Φ is smooth. To show that Φ is injective we prove the following formula.

$$\partial_x y_2(1,\mu_n(p),p) = \frac{1}{2}\left(\Delta(\mu_n(p),p) - \sigma_n(p)\sqrt{\Delta^2(\mu_n(p),p) - 4}\right) \tag{9.9}$$

for all n, including the cases of $\sigma_n(p) = 0$. To verify the above, we invoke the identities

$$\Delta^2(\mu_n(p),p) - 4 = \big(y_1(1,\mu_n) - \partial_x y_2(1,\mu_n)\big)^2 \tag{9.10}$$

$$y_1(1,\mu_n(p),p) = \frac{1}{\partial_x y_2(1,\mu_n(p),p)} \tag{9.11}$$

Since $\Delta^2(\mu_n) - 4 \geq 0$, we have

$$\sqrt{\Delta^2(\mu_n(p),p) - 4} = \sigma_n(p)\left(\frac{1}{\partial_x y_2(1,\mu_n(p),p)} - \partial_x y_2(1,\mu_n(p),p)\right) \tag{9.12}$$

Solving this quadratic equation one obtains

$$\partial_x y_2(1,\mu_n) = \frac{1}{2}\left(-\sigma_n(p)\sqrt{\Delta^2(\mu_n(p),p) - 4} \pm \Delta(\mu_n(p),p)\right) \tag{9.13}$$

To determine the $\pm$ sign here, consider for instance n odd, i.e

$$\Delta(\lambda_{2n-1}) = \Delta(\lambda_{2n}) = -2, \quad \Delta(\mu_n) \leq -2$$

Recall that $\partial_x y_2(1,\mu_n) = (-1)^n$. So, $\partial_x y_2(1,\mu_n) < 0$ if $\sigma_n(p) = 1$ then

$$\frac{1}{\partial_x y_2(1,\mu_n)} - \partial_x y_2(1,\mu_n) > 0$$

That implies $\partial_x y_2(1,\mu_n) \leq -1$, Clearly,

$$|\Delta(\mu_n)| < \sqrt{\Delta^2(\mu_n) - 4}$$

That implies we need the + sign in (9.13). One can verify that it is + in all possible cases. That validates (9.9). Since $\lambda_k(p) = \lambda_k(q)$ for $p \in \mathrm{Iso}(q)$ it follows from corollary (9.2) that

$$\Delta^2(\lambda,p) - 4 = \Delta^2(\lambda,q) - 4 \quad \text{for } p \in \mathrm{Iso}(q)$$

Thus

$$\partial_x y_2(1,\mu_n(p),p) = \left(-\sigma_n(p)\sqrt{\Delta^2(\mu_n(p),q) - 4} + \Delta(\mu_n(p),q)\right) \tag{9.14}$$

So, if $\Phi(p) = \Phi(r)$ then $\mu_n(p) = \mu_n(r), \sigma_n(p) = \sigma_n(r)$ and $\partial_x y_2(1,\mu_n(p),p) = \partial_x y_2(1,\mu_n(r),r)$, for all n with $\lambda_{2n-1} < \lambda_{2n}$. If $\lambda_{2n-1} = \lambda_{2n}$ then $\sigma_n(p) = 0, \mu_n(p) = p$ for all $p \in \mathrm{Iso}(q)$. That implies $\partial_x y_2(1,\mu_n(p),p) = \partial_x y_2(1,\mu_n(q),q)$ for all such n. Thus,

$$\partial_x y_2(1,\mu_n(p),p) = \partial_x y_2(1,\mu_n(r),r) \quad \text{for all } n \geq 1$$

Recall that

$$\varkappa_n(p) = \log(-1)^n \partial_x y_2(1,\mu_n(p),p)$$

Thus $\varkappa_n(p) = \varkappa_n(r)$ for all n. By Theorem (5.5) one concludes $p = r$. So, Φ is indeed injective. Let $(\mu_n,\sigma_n) \in \prod_{\lambda_{2n-1}<\lambda_{2n}, n\geq 1} [|\lambda_{2n-1},\lambda_{2n}|]$ be arbitrary. Recall that

$$\lambda_{2n-1},\lambda_{2n} = n^2\pi^2 + l^2(n), \quad \lambda_{2n-1} \leq \mu_n \leq \lambda_{2n} \tag{9.15}$$

Therefore

$$\tilde{\mu}_n = \mu_n - n^2\pi^2 \in l^2(n)$$

Set

$$\varkappa_n = \log\left(\frac{(-1)^n}{2}\left(\Delta(\mu_n,q) - \sigma_n\sqrt{\Delta^2(\mu_n,q) - 4}\right)\right)$$

We want to estimate $\varkappa_n$. For that we use corollary (9.2):

$$|\Delta^2(\mu_n) - 4| = 4(\mu_n - \lambda_0)\frac{(\lambda_{2n} - \mu_n)(\mu_n - \lambda_{2n-1})}{n^4\pi^4}\left|\prod_{m\neq n, m\geq 1}\frac{(\lambda_{2m-1} - \mu_n)((\lambda_{2m} - \mu_n)}{m^4\pi^4}\right| \tag{9.16}$$

Due to Lemma (4.9) the product here is $\mathcal{O}(1)$ (due to (9.15) Lemma (4.9) applies). Together with (9.15) this implies

$$\Delta^2(\mu_n) - 4 = \frac{l^2(n) \times l^2(n)}{n^2}$$

Thus

$$\sqrt{\Delta^2(\mu_n) - 4} = l_1^2(n) \tag{9.17}$$

$$\Delta(\mu_n) = 2(-1)^n + \mathcal{O}\left(\frac{l^2(n) \times l^2(n)}{n^2}\right)$$

$$\Longrightarrow \varkappa_n = l_1^2(n)$$

By Theorem (**??**) there exists a unique $p \in L^2[0,1]$ such that

$$\mu_n(p) = \mu_n, \quad \varkappa_n(p) = \varkappa_n$$

We need to show that $\lambda_k(p) = \lambda_k$ for all k. We have

$$\log(-1)^n \partial_x y_2(1, \mu_n(p), p) = \varkappa_n = \log\left(\frac{(-1)^n}{2}\left(\Delta(\mu_n, q) - \sigma_n\sqrt{\Delta^2(\mu_n, q) - 4}\right)\right)$$

i.e.

$$\partial_x y_2(1, \mu_n(p), p) = \frac{1}{2}\left(\Delta(\mu_n, q) - \sigma_n\sqrt{\Delta^2(\mu_n, q) - 4}\right)$$

Recall that

$$\Delta(\mu_n(p), p) = \frac{1}{\partial_x y_2(1, \mu_n(p), p)} + \partial_x y_2(1, \mu_n(p), p)$$

Hence,

$$\Delta(\mu_n(p), p) = \frac{2}{\Delta(\mu_n(q), q) - \sigma_n\sqrt{\Delta^2(\mu_n(q), q) - 4}} + \frac{1}{2}\left(\Delta(\mu_n(q), q) - \sigma_n\sqrt{\Delta^2(\mu_n(q), q) - 4}\right) = \Delta(\mu_n(q), q)$$

i.e. $\Delta(\mu_n, p) = \Delta(\mu_n, q)$, $n = 1, 2, \ldots$. Due to Theorem (4.5)

$$y_2(1, \lambda, p) = \prod_{n \geq 1}\left(\frac{\mu_n - \lambda}{n^2\pi^2}\right) = y_2(1, \lambda, q), \quad \text{for all } \lambda \in \mathbb{C}$$

Since $[p] = [q]$, Theorem (**??**) implies that $\Delta(\lambda, p) = \Delta(\lambda, q)$ for all $\lambda \in \mathbb{C}$. In particular $\lambda_k(p) = \lambda_k(q)$ for all k. Thus $p \in \mathrm{Iso}(q)$. We have $\mu_n(p) = \mu_n(q)$, $[p] = [q]$,

$$\varkappa_n(p) = \log\left(\frac{(-1)^n}{2}\left(\Delta(\mu_n, q) - \sigma_n\sqrt{\Delta^2(\mu_n, q) - 4}\right)\right)$$

$$\partial_x y_2(1, \mu_n(p), p) = \frac{1}{2}\left(\Delta(\mu_n, q) - \sigma_n\sqrt{\Delta^2(\mu_n, q) - 4}\right)$$

The last equation implies $\sigma_n(p) = \sigma_n$. Thus, $\Phi(p) = (\mu_n, \sigma_n)$. □

Chapter 10

Description of the Periodic Spectrum

Let

$$u(\lambda) = \frac{1}{2}\left(y_1(\pi,\lambda) + \partial_x y_2(\pi,\lambda)\right)$$

Let $\lambda_0 < \lambda_1^- \leq \lambda_1^+ < \lambda_2^- \leq \lambda_2^+ < \dots$ be the periodic and anti-periodic eigenvalues. The λ_j are the roots of $1 - u(\lambda)^2 =$. Replacing q by $q - \lambda_0$ we assume in this section that $\lambda_0 = 0$. Set $u_+ = u(z^2)$. Consider the roots of the equation

$$1 - u_+(z)^2 = 0 \tag{10.1}$$

and enumerate them as follows

$$\alpha_{2k-1}^{\mp} = \sqrt{\lambda_{2k-1}^{\mp}}, \quad u_+(\alpha_{2k-1}^{\mp}) = -1$$

$$\alpha_{2k}^{\mp} = \sqrt{\lambda_{2k}^{\mp}}, \quad u_+(\alpha_{2k}^{\mp}) = 1$$

and denote

$$\alpha_{-(2k-1)}^{\mp} = -\alpha_{2k-1}^{\mp} \quad \& \quad \alpha_{-2k}^{\mp} = -\alpha_{2k}^{\mp}$$

Let $\sqrt{1 - u_+(z)^2}$ be the branch of the square roots with $\Im z > 0$ which has a continuation of $(0, \alpha_1^-)$ and $\sqrt{1 - u_+^2(x)} > 0$, for $x \in (0, \alpha_1^-)$. Set

$$\theta(z) = \int_0^z \frac{u_t'(\zeta)}{\sqrt{1 - u^2(\zeta)}} d\zeta, \quad \Im z > 0 \tag{10.2}$$

Lemma 10.1. *For* $\Im z > 0, \cos\theta(z) = u_+(z)$,

Proof. The function $\arccos w$ is analytic in the domain $\mathbb{C} \setminus ((-\infty, -1) \cup (1, \infty))$ and obeys $(\arccos w)' = -(1 - w^2)^{-1/2}$. Thus $\theta' = (\arccos u_+(z))'$ provided $u_+(z)$ belongs to this domain. Since $u_+(z)$ is a non-constant analytic function $u^{-1}\left(-\infty, -1\right] \cup [1, \infty))$ consists of a countable union of analytic curves and points. Therefore the upper half plane splits into a union of domains and curves such that in each domain

$$\theta' = (\arccos u_+(z))', \quad \theta(z) = \arccos u_+(z) + 2\pi l_j, \quad l_j \in \mathbb{Z}$$

holds. Thus $\cos\theta(z) = u_+(z)$ everywhere except a union of some curves. Since both functions are analytic in the upper half plane. $\cos\theta(z) = u(z), \Im z > 0$ □

Lemma 10.2. *The function $\theta(z)$ can be extended analytically via the reflection principle $\theta(\bar{z}) = \overline{\theta(z)}$, into the domain $\mathbb{C} \setminus \cup_{k\in\mathbb{Z}\setminus\{0\}}[\alpha_k^+, \alpha_k^+]$. The identity $\cos\theta(z) = u_t(z)$ holds.*

Proof. We verify first that θ can be extended continuously to the real axis, $\Im z = 0$, i.e. the limit

$$\lim_{z\to x_0, \Im z>0} \theta(z)$$

exists for any $x_0 \in \mathbb{R}$. For $x_0 \neq a_j^{\pm}$ this is clear since integrand in (10.2) is continuous in the neighbourhood of x_0. Take $x_0 = \alpha_j^-$. Assume first that α_j^- is a simple root of (10.1), i.e $\lambda_j^- < \lambda_j^+, \alpha_j^- < \alpha_j^+$. Then

$$1 - u_t^2(z) = (z - x_0)\varphi(x_0, z)$$

where $\varphi(x_0, z)$ is analytic for z in a neighbourhood of x_0, $\varphi(x_0, x_0) \neq 0$, then

$$\left|\frac{1}{\sqrt{1 - u_+^2(z)}}\right| \leq \frac{C(x_0)}{\sqrt{|z - x_0|}}$$

The integral

$$\int_{x_0}^{z} \left|\frac{u_+'(\zeta)}{\sqrt{1 - u_+^2(\zeta)}}\right| |d\zeta| \tag{10.3}$$

converges and continuities follows. If α_j^- is a double root, then $u_+'(\alpha_j) = 0$ and the estimation of the integral is even better. Note that this argument also verifies the correctness of the definition of (10.2). Thus $\theta(z)$ can be extended continuously to the real axis.

Recall that $\theta(z) = \arccos u_+(z) + 2\pi l_j, z \in D_j$, and the D_j's are domains which together with part of their boundaries partition the upper half-plane, $\Im z > 0$. Recall also that $-1 \leq u_+(x) \leq 1$ for $\mathbb{R} \setminus \cup_{k\in\mathbb{Z}\setminus\{0\}}[\alpha_k^-, \alpha_k^+]$. So, $\arccos u_t(x)$ assumes real values on this set. Die to the continuity one conclude that for x in this set, the following holds

$$\Im\theta(x) = 0$$

Therefore the reflection principle applies and the statement follows □

Lemma 10.3. *$\theta(z)$ conformally maps the upper half-plane onto*

$$\Theta_t\{h_k\} = \{\Im\theta > \theta\} \setminus \bigcup_{k=-\infty}^{\infty} \{\theta : \Re\theta = k\pi, 0 \leq \Im\theta \leq h_k\} \tag{10.4}$$

where $h_0 = 0$, and $h_k = h_{-k}$, and $\sum k^2 h_k^2 < +\infty$. Furthermore, $\theta(0) = 0$ and

$$\lim_{y\to+\infty} \frac{\theta(iy)}{iy} = \pi$$

Proof. We will identify the image of the real axis under θ. We use $\theta(x) = \arccos u_+(x)$ and continuity. We have $\theta(0) = 0$. Since $u(x)$ decreases from $u(0) = 1$ to $u(\alpha_1^-) = -1$, we have $\theta(x) = \arccos u(x)$, where $\arccos(1) = 0, \arccos(-1) = \pi$. Thus $\theta(x)$ increase from $\theta(0) = 0$ to $\theta(\alpha_1^-) = \pi$, when $0 \le x \le \alpha_1^-$. For $x \in (\alpha_1^-, \alpha_1^+)$, $u(x) < -1$. For $t \in (-\infty, -1)$, we use

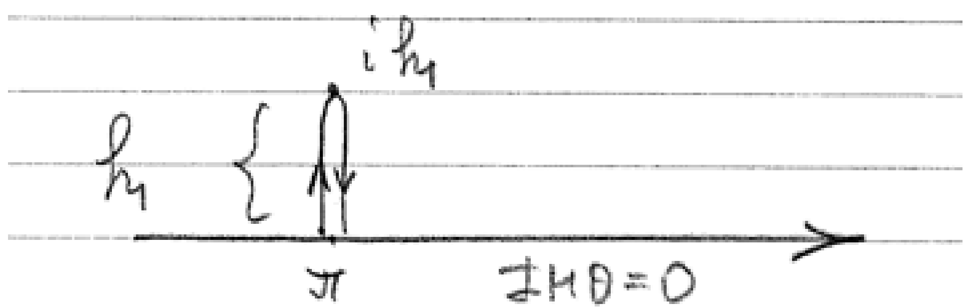
$$\arccos t = \pi + i \log(-t - \sqrt{t^2 - 1})$$

since

$$\begin{aligned}\cos(\pi + i\log(-t - \sqrt{t^2-1})) &= -\frac{1}{2}\left(\exp(-\log(-t - \sqrt{t^2-1})) + \log(\log(-t - \sqrt{t^2-1}))\right)\\ &= -\frac{1}{2}\left(\frac{1}{(-t - \sqrt{t^2-1})} + (-t - \sqrt{t^2-1})\right)\\ &= t\end{aligned}$$

$$\pi + i\log(-t - \sqrt{t^2-1})\Big|_{t=-1} = \pi = \theta(\alpha_1^-)$$

On the interval (α_1^-, α_1^+) the function $u_+(x)$ has two monotonicity intervals, (α_1^-, γ_1) and (γ_1, α_1^+), where $\partial_x u_t\big|_{x=\gamma_1} = 0$. Therefore for $x \in (\alpha_1^-, \alpha_1^+), \theta(x) = \pi + i\log(-u_+(x) - \sqrt{u_+(x)^2 - 1}), \Re\theta(x) = \pi, \Im\theta(x) = \log(-u_+(x) - \sqrt{u_+^2(x) - 1})$, $\Im\theta(x)$ increases from 0 to some value h_1 when $\alpha_1^- \le x \le \gamma_1$ and then decreases from h_1 to 0 when $\gamma_1 \le x \le \alpha_1^+$.



e.t.c. Thus θ indeed maps the real axis onto the boundary of $\Theta_+\{h_k\}$. Moreover when x runs $(-\infty, \infty), \theta(x)$ runs the boundary of Θ_+ from left to right. By the argument principle $\theta(z)$ conformally maps $\Im z > 0$ onto $\Theta_+\{h_k\}$.
By construction, $h_0 = 0$, since u is even we have $h_{-k} = h_k$. We need to estimate h_k. Recall that due to Corollary (9.2) we have

$$|\Delta^2(\lambda) - 4| = 4|\lambda - \lambda_0|\frac{|\lambda_n^- - \lambda||\lambda_n^+ - \lambda|}{n^4}\left|\prod_{m\ne n, m\ge 1}\frac{\lambda_m^- - \lambda)(\lambda_m^+ - \lambda)}{m^4}\right| \tag{10.5}$$

We have $\lambda_m^\pm = m^2 + [q] + l^2(m)$. For $\lambda_m^- \le \lambda \le \lambda_m^+$, Lemma (4.9) says that the product here is $\mathcal{O}(1)$. Hence

$$0 \le \Delta^2(\lambda) - 4 = \frac{l^2(n) \times l^2(n)}{n^2}, \text{ for } \lambda_n^- \le \lambda \le \lambda_n^+$$

thus

$$|\Delta(\lambda)| \le 2 + \frac{l^2(n) \times l^2(n)}{n^2}, \text{ for } \lambda_n^- \le \lambda \le \lambda_n^+ \tag{10.6}$$

So

$$\max_{\alpha_n^- \le x \le \alpha_n^+} |u_+(x)| \le 1 + \frac{l^2(n) \times l^2(n)}{n^2}$$

We go back to the formula for $\arccos u_+(x)$ for $\alpha_k^- \le x \le \alpha_k^+$:

$$\arccos t = k\pi + i \log(|t| + \sqrt{t^2 - 1})$$

here $t = u_+(x)$, and

$$|t| \le \max_{\alpha_k^- \le x \le \alpha_k^+} |u_+(x)| \le 1 + \frac{l^2(k) \times l^2(k)}{k^2} \implies \log(|t| + \sqrt{t^2 - 1}) \le \frac{l^2(k)}{k} = l_1^2(k)$$

□

Lemma 10.4. *Let θ be a conformal map from the upper half-plane $\Im z > 0$ onto the Domain $\Theta\{h_k\}$ with $H \equiv \sup h_k < +\infty$, $\theta(0) = 0$,*

$$\lim_{y \to +\infty} \frac{\theta(iy)}{iy} = \pi$$

The following statements hold:

- $u(z) = \cos\theta(z)$ *is an entire function,*

$$\max_{|z| \le R} \log|u(z)| \le \pi R + H$$

$$\sup_{x \in \mathbb{R}} |u(x)| = \cosh(H)$$

- *let* $\alpha_k^\pm = \theta^{-1}(k\pi \pm 0)$. *Then*

$$1 \ge \alpha_k^- - \alpha_{k-1}^+ \ge \frac{2}{\pi \cosh H}$$

$$\frac{2h_k}{\pi} \ge \alpha_k^+ - \alpha_k^- \ge \frac{h_k}{\pi \cosh H}$$

$$\frac{2|k|}{\pi \cosh H} \le |\alpha_k^\pm| \le |k| \left(1 + \frac{2H}{\pi}\right)$$

- *For* $x \in (\alpha_k^-, \alpha_k^+)$,

$$0 < \Im\theta(x) \le \pi\sqrt{\cosh H}\sqrt{(x - \alpha_k^-)(\alpha_k^+ - x)}$$

Proof. $u(z) = \cos\theta(z)$ is analytic in $\Im z > 0$, continuous on $\Im z = 0$ for $x \in (\alpha_{k-1}^+, \alpha_k^-), \Im\theta(x) = 0$. For $x \in [\alpha_k^-, \alpha_k^+], \theta(x) = \pi k + i\eta(x), 0 \le \eta \le h_k$,

$$\cos\theta(x) = (-1)^k \cosh\eta(x)$$

$$\Im u(x) = 0$$

By the symmetry principle, u has an extension to the entire plane $\mathbb{C}$. Since $\Im\theta(z)$ is harmonic in the upper half plane, $\Im z > 0$, and continuous on $\Im z \ge 0$, one has

$$\Im\theta(z) = ay + \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\Im\theta(t)}{(x - t)^2 + y^2} dt, \quad z = x + iy, \text{ with } a > 0 \tag{10.7}$$

Since

$$\lim_{y\to+\infty}\frac{\theta(iy)}{iy}=\pi$$

one concludes $a=\pi$. Clearly,

$$0\le\Im\theta(x)\le\sup h_k=H,\quad -\infty<x<+\infty$$

$$0\le\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\Im\theta(t)}{(x-t)^2+y^2}dt\le H,\quad -\infty<x<+\infty, y>0$$

$$\pi\Im z\le\Im\theta(z)\le\pi\Im z+H \tag{10.8}$$

In particular,

$$|u(z)|=|\cos\theta(z)|\le\cosh|\Im\theta(z)|\le\cosh(\pi|\Im z|+H)$$

$$\max_{|z|\le R}\log|u(z)|\le\pi R+H \tag{10.9}$$

as claimed. Furthermore,

$$\sup_x|u(x)|=\sup_x|\cos\theta(x)|=\sup_k\cosh h_k=\cosh H \tag{10.10}$$

We turn now to bullet two. One has

$$\theta(\alpha_k^-)=k\pi,\quad \theta(\alpha_{k-1}^+)=(k-1)\pi$$

$$u(\alpha_k^-)-u(\alpha_{k-1}^+)=\cos(k\pi)-\cos((k-1)\pi)=2(-1)^k$$

On the other hand

$$|u(\alpha_k^-)-u(\alpha_{k-1}^+)|\le\left(\max_x|u'|\right)(\alpha_k^- -\alpha_{k-1}^+)$$

Since $u(z)$ is an entire function of exponential type π, and $\sup\{|u(x)|:x\in\mathbb{R}\}\le\cosh H$, the Bernstein inequality say that

$$|u'(x)|\le\pi\cosh H,\quad -\infty<x<+\infty$$

Thus

$$\alpha_k^- -\alpha_{k-1}^+\ge\frac{2}{\pi\cosh H}$$

as claimed. Let $\alpha_k^-\le x\le\alpha_k^+$. One has $|u(\alpha_k^-)|=1$,

$$\Big||u(x)|-1\Big|\le\max_\xi|u'(\xi)|(\alpha_k^+-\alpha_k^-)\le\pi\cosh(H)(\alpha_k^+-\alpha_k^-$$

Now just as in the proof of lemma (10.3) one obtains

$$|\Im\arccos u(x)|\le\pi\cosh(H)(\alpha_k^+-\alpha_k^-)$$

Hence,

$$h_k\le\pi\cosh(H)(\alpha_k^+-\alpha_k^-)$$

as claimed. Now we want to estimate $(\alpha_k^- -\alpha_{k-1}^+)$ from above. Since $\theta(z)$ has an analytic continuation through $[\alpha_{k-1}^+,\alpha_k^-]$, the partial derivatives of θ are well defined for $x\in(\alpha_{k-1}^-\alpha_k^+)$, $y=0$. Note that

$g(z) = \Im\theta(z) - \pi\Im z$ is non-negative in $\Im z > 0, g(z) \geq 0$ and $g(x) = 0$ for $x \in (\alpha_{k-1}^{-}\alpha_k^{+})$. That implies

$$\partial_y g(x+iy)\Big|_{y=0} \geq 0, \quad \text{for } x \in [\alpha_{k-1}^{-}\alpha_k^{+}]$$

Hence,

$$\partial_y \Im\theta(x+iy)\Big|_{y=0} \geq \pi \quad \text{for } x \in [\alpha_{k-1}^{-}\alpha_k^{+}]$$

By Cauchy-Riemann, one obtains

$$\partial_x \Re\theta(x+iy)\Big|_{y=0} \geq \pi, \quad \text{for } x \in [\alpha_{k-1}^{-}\alpha_k^{+}]$$

On the other hand, $\theta(\alpha_k^{-}) - \theta(\alpha_{k-1}^{+}) = \pi$. Thus

$$\pi \geq \int_{\alpha_{k-1}^{-}}^{\alpha_k^{+}} \partial_x \Re\theta(x)dx \geq \pi(\alpha_k^{-} - \alpha_{k-1}^{+})$$

Next we estimate $\alpha_k^{+} - \alpha_k^{-}$ from above. Let $z(\theta)$ be the inverse for $\theta(z)$. Set

$$z_k(\theta) = \frac{1}{\pi}\sqrt{(\theta - k\pi)^2 + h_k^2}$$

The function z_k maps conformly the domain

$$\Theta_k = \{\Im\theta > 0\} \setminus \{\Re\theta = k\pi, 0 \leq \Im\theta \leq h_k\}$$

onto the upper half plane. Clearly for $\theta \in \partial\Theta\{h_j\}$ we have

$$\Im z_k(\theta) - \Im z(\theta) = \Im z_k(\theta) \geq 0$$

Recall also that due to (10.12)

$$\Im z(\theta) \leq \frac{\Im\theta}{\pi}$$

Clearly

$$\Im z_k(\theta) = \frac{\Im\theta}{\pi} + \mathcal{O}\left(\frac{1}{|\theta|}\right), \quad |\theta| \to \infty$$

Thus

$$\lim_{|\theta|\to\infty} (\Im z_k(\theta) - \Im z(\theta)) \geq 0$$

Due to the maximum principle for harmonic functions

$$\Im z_k(\theta) - \Im z(\theta) \geq 0, \quad \text{for } \theta \in \Theta\{h_j\}$$

Let $\vec{n}$ be the normal vector on $\partial\Theta\{h_j\}$ directed inside of the domain $\Theta\{h_j\}$. Let $L_k = \{\Re\theta = \pi k, 0 \leq \Im\theta \leq h_k\}$. Note that $\Im z_k(\theta) - \Im z(\theta) = 0$ on L_k. Note the following. The function $z(\theta)$ maps the straight segments L_k onto a segment $[\alpha_k^{-}, \alpha_k^{+}]$. So, the symmetry principle applies. Namely, for each of the following two domains

The function $z(\theta)$ has an analytic continuation in the domains $D_l \cup L_k \cup D_l^*$, $D_r \cup L_k \cup D_r^*$ respectively. The function $z(\theta)$ itself is discontinuous on L_k, but for symmetrical continuations the partial derivatives are well-defined and Cauchy-Riemann applies. The same applies to $z_k(\theta)$. Denote these continuations as $z^{(l)}, z_k^{(l)}, z^{(r)}, z_k^{(r)}$ respectively.

$$\partial_x \left(\Im z_k^{(l)} - \Im z^{(l)} \right) \Big|_{\Re\theta = k\pi} \leq 0, \quad \partial_x \left(\Im z_k^{(r)} - \Im z^{(r)} \right) \Big|_{\Re\theta = k\pi} \geq 0$$

$$\partial_x \Im z^{(l)} \Big|_{\Re\theta = k\pi} \leq 0, \quad \partial_x \Im z^{(r)} \Big|_{\Re\theta = k\pi} \geq 0$$

That implies

$$|\partial_x \Im z^{(\cdot)}| \leq |\partial_x \Im z_k^{(\cdot)}| \quad \text{on } L_k$$

By Cauchy-Riemann

$$|\partial_y \Re z^{(\cdot)}| \leq |\partial_y z_k^{(\cdot)}| \quad \text{on } L_k$$

Hence,

$$\alpha_k^+ - \alpha_k^- \leq \int_0^{h_k} \left(|\partial_y z_k^{(l)}| + |\partial_y z_k^{(r)}| \right) dy = \frac{2}{\pi} h_k$$

as claimed. One has (with $\alpha_0^\pm = 0$)

$$\alpha_k^+ = \sum_{j=1}^{k} (\alpha_j^+ - \alpha_j^-) + (\alpha_j^- - \alpha_{j-1}^+)$$

$$\alpha_k^+ \leq \frac{2}{\pi} \sum_{j=1}^{k} h_j + k \leq k \left(1 + \frac{2H}{\pi} \right)$$

$$\alpha_k^+ \geq \frac{2k}{\pi \cosh H}$$

as claimed. The estimation for α_k^- is similar. That finishes the second claim. To verify the third, consider

$$f(x) = (-1)^k u(x) - \left[1 + (x - \alpha_k^-)(\alpha_k^+ - x) \frac{\pi^2}{2} \cosh H \right]$$

Recall that $u(\alpha_k^\pm) = (-1)^k$. So, $f(\alpha_k^\pm) = 0$. Furthermore,

$$f''(x) = (-1)^k u''(x) + \pi^2 \cosh H$$

Due to the Bernstein inequalities,

$$|u''(x)| \leq \pi^2 \cosh H$$

Thus $f''(x) \geq 0$. Therefore

$$f(x) \leq 0 \quad \text{for } \alpha_k^- \leq x \leq \alpha_k^+$$

$$0 \leq (-1)^k u(x) \leq 1 + (x - \alpha_k^-)(\alpha_k^+ - x)\frac{\pi^2}{2}\cosh H \leq \cosh(\pi\sqrt{\cosh H(x-\alpha_k^-)(\alpha_k^+ - x)})$$

$\cosh\xi \geq 1 + \xi^2/2$. On the other hand for $x \in [\alpha_k^-, \alpha_k^+]$

$$\theta(x) = k\pi + i\Im\theta(x)$$

$$u(x) = \cos\theta(x) = (-1)^k \cosh(\Im\theta(x))$$

That implies

$$\Im\theta(x) \leq \pi\sqrt{\cosh H(x-\alpha_k^-)(\alpha_k^+ - x)}$$

as claimed. □

Lemma 10.5. (**??**) *Using the notations of the previous lemma, assume*

$$h_{-k} = h_k, \quad \sum (kh_k)^2 < +\infty$$

Then

- $u(z)$ *is an even function,*

$$u(z) = \cos(\pi z) - \frac{d_1}{z}\sin(\pi z) + \frac{g(z)}{z}$$

  where

$$g(z) = \int_0^\pi \tilde{g}(t)\sin(zt)dt, \quad \tilde{g} \in L^2[0,\pi]$$

- $\alpha_k^\pm = \theta^{-1}\{k\pi \pm 0\}$ *obey*

$$\alpha_{-k}^\pm = -\alpha_k^\mp, \quad \alpha_k^\pm = k - \frac{d_1}{\pi k} + l_1^2(k)$$

Proof. Let $\theta_1(z) = -\overline{\theta(-\bar{z})}$. The $\theta_1(z)$ maps conformly $\Im z > 0$ on $\Theta\{h_k\}$ since

$$-\overline{\Theta\{h_k\}} = \Theta\{h_k\}$$

Furthermore, $\lim_{y\to+\infty}(iy)^{-1}\theta(iy) = \pi$ and $\theta_1(0) = 0$. Hence $\theta = \theta_1$. In particular for $x \in (-\infty,\infty)$, $-\theta(-x) = \theta(x)$. That implies

$$-\theta(-z) = \theta(z), \quad u(-z) = \cos(\theta(-z)) = \cos(-\theta(z)) = u(z)$$

Hence $\alpha_{-k}^\pm = -\alpha_k^\mp$. Using the estimates from Lemma (10.4) one has

$$\int_{\alpha_k^-}^{\alpha_k^+} |t|^s \Im\theta(t)dt \leq \left(\max\{|\alpha_k^-|,|\alpha_k^+|\}\right)^s h_k(\alpha_k^+ - \alpha_k^-) \leq \frac{2}{\pi}\left(1 + \frac{2H}{\pi}\right)^s |k|^s h_k^2$$

Since $\sum k^2 h_k^2 < +\infty$,

$$\int_{-\infty}^{\infty} |t|^s \Im\theta(t)dt \leq C < +\infty, \quad s = 0,1,2$$

Due to Poissons Formula for $\Im z > 0$,

$$\theta(z) = \pi z + d + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1+tz}{t-z}\frac{\Im\theta}{1+t^2}dt$$

Note that

$$\frac{1+tz}{t-z}\frac{1}{1+t^2} = -\frac{t}{1+t^2} - \frac{1}{z} + \frac{t}{z(t-z)}$$

Hence

$$\theta(z) = \pi z + d - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{t}{1+t^2}\Im\theta(t)dt - \frac{1}{\pi z}\int_{-\infty}^{\infty}\Im\theta(t)dt + \frac{1}{\pi z}\int_{-\infty}^{\infty}\frac{t}{t-z}\Im\theta(t)dt := \phi(z) + b$$

Since $\theta(-z) = -\theta(z), \phi(-z) = -\phi(z), b = 0$. So

$$\theta(z) = \pi z + \frac{1}{z}(d_1 + \psi(z))$$

$$d_1 = -\frac{1}{\pi}\int_{-\infty}^{\infty}\Im\theta(t)dt < 0$$

$$\psi(z) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{t}{t-z}\Im\theta(t)dt$$

$$\psi(-z) = \psi(z)$$

Note that $\psi(x)$ is well defined for $x \in \mathbb{R}$, moreover,

$$\psi(x) = \lim_{y\to 0}\psi(x+iy)$$

let $x \in (\alpha_{n-1}^+, \alpha_n^-)$. Then

$$\psi(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{t}{t-x}\Im\theta(t)dt = \frac{1}{\pi}\sum_k\int_{\alpha_k^-}^{\alpha_k^+}\frac{t}{t-x}\Im\theta(t)dt$$

The denominator here does not vanish since $x \in (\alpha_{n-1}^+, \alpha_n^-)$. For $k \neq n-1, n, t \in [\alpha_k^-, \alpha_k^+]$ due to lemma (10.4)

$$t - x \geq \frac{2}{\pi}\cosh H$$

$$\int_{\alpha_k^-}^{\alpha_k^+}\frac{|t|}{|t-x|}\Im\theta(t)dt \leq \frac{1}{x}\int_{\alpha_k^-}^{\alpha_k^+}\left(\frac{t^2\pi\cosh H}{2} + |t|\right)\Im\theta(t)dt$$

Hence,

$$\frac{1}{\pi}\left|\sum_{k\neq n-1,n}\int_{\alpha_k^-}^{\alpha_k^+}\frac{t}{t-x}\Im\theta(t)dt\right| \leq \frac{1}{\pi x}\left(\frac{C_2\pi\cosh H}{2} + C_1\right)$$

Furthermore, using the estimates in Lemma (10.4) one has

$$\begin{aligned}\left|\int_{\alpha_{n-1}^-}^{\alpha_{n-1}^+} \frac{t}{t-x}\Im\theta(t)dt\right| \leq& \pi\sqrt{\cosh H}(\max_{\pm}|\alpha_{n-1}^{\pm}|)\int_{\alpha_{n-1}^-}^{\alpha_{n-1}^+}\frac{\sqrt{(t-\alpha_{n-1}^-)(\alpha_{n-1}^+-t)}}{\alpha_{n-1}^+-t}dt\\ \leq& \pi\sqrt{\cosh H}|n-1|\left(1+\frac{2H}{\pi}\right)\sqrt{\alpha_{n-1}^+-\alpha_{n-1}^-}\int_{\alpha_{n-1}^-}^{\alpha_{n-1}^+}\frac{dt}{\sqrt{\alpha_{n-1}^+-t}}\\ \leq& \sqrt{\cosh H}|n-1|\left(1+\frac{2H}{\pi}\right)(4h_{n-1})\end{aligned}$$

The evaluation of the integral over $[\alpha_n^-, \alpha_n^+]$ is completely similar. Thus, for $x \in [\alpha_{n-1}^+, \alpha_n^-]$ one has with some constant B,

$$|\psi(x)| \leq B\left|\frac{1}{|x|} + |n-1|h_{n-1} + |n|h_n\right| \tag{10.11}$$

for $x \in [\alpha_n^-, \alpha_n^+]$, one has

$$\psi(x) = x\left(\theta(x) - \pi x - \frac{d_1}{x}\right) = x\left(n\pi + -\Im\theta(x) - \pi x - \frac{d_1}{x}\right),$$

$$|x||\Im\theta(x)| \leq |x|h_n \leq |n|\left(1+\frac{2H}{\pi}\right)h_n$$

$$\left|\psi(\alpha_n^-) - \alpha_n^-\left(n\pi - \pi\alpha_n^- - \frac{d_1}{\alpha_n^-}\right)\right| \leq n\left(1+\frac{2H}{\pi}\right)h_n,$$

$$\begin{aligned}|\psi(x)-\psi(\alpha_n^-)| \leq& \left|x\left(n\pi - \pi x - \frac{d_1}{x}\right) - \alpha_n^-\left(n\pi - \pi\alpha_n^- - \frac{d_1}{\alpha_n^-}\right)\right| + 2n\left(1+\frac{2H}{\pi}\right)h_n\\ \leq& |n|\pi(x-\alpha_n^-) + \pi(x-\alpha_n^-)(|x|+|\alpha_n^-|) + 2n\left(1+\frac{2H}{\pi}\right)h_n\\ \leq& B|n|h_n\end{aligned}$$

Note that (10.11) applies to $x = \alpha_n^-$. Thus, (10.11) holds for all $x \in (-\infty, \infty)$ (with some adjustment to the constant B). Thus implies $|\psi(x)| \to 0$ with $|x| \to \infty$ and

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx < +\infty$$

Recall that $u(z) = \cos\theta(z)$. Set

$$g(z) = z\left(u(z) - \cos(\pi z) + \frac{d_1\sin(\pi z)}{z}\right)$$

one has

$$\begin{aligned} u(x) = \cos\theta(x) &= \cos\left(\pi x + \frac{d_1}{x} + \frac{\psi(x)}{x}\right) \\ &= \cos(\pi x)\cos\left(\frac{d_1}{x} + \frac{\psi(x)}{x}\right) - \sin(\pi x)\sin\left(\frac{d_1}{x} + \frac{\psi(x)}{x}\right) \\ &= \cos(\pi x)\left(1 + \mathcal{O}\left(\frac{1}{x^2}\right)\right) - \sin(\pi x)\sin\left(\frac{d_1}{x} + \frac{\psi(x)}{x}\right) \\ &= \cos(\pi x) - \sin(\pi x)\sin\left(\frac{d_1}{x} + \frac{\psi(x)}{x}\right) + \mathcal{O}\left(\frac{1}{x^2}\right) \end{aligned}$$

$$\implies g(x) = -\psi(x)\sin(\pi x) + \mathcal{O}\left(\frac{1}{|x|}\right), \quad \& \quad \int_{-\infty}^{\infty} g(x)^2 dx < \infty$$

Furthermore, $g(z)$ is an entire function of exponential type π, i.e

$$|g(z)| \leq \exp(\pi|z|)$$

By the Pely-Wiener Theorem, we have

$$g(z) = \int_0^\pi \tilde{g}_1(t)\exp(-itz)dt$$

where $\tilde{g}_1 \in L^2[0,\pi]$. The function $u(z)$ is even. So $g(z)$ is odd. Therefore,

$$g(z) = \int_0^\pi \sin(tz)\tilde{g}(t)dt, \quad \tilde{g} \in L_2[0,\pi]$$

Finally, one has

$$\theta(\alpha_k^\pm) = k\pi,$$

$$k\pi = \pi\alpha_k^\pm + \frac{1}{\alpha_k^\pm}(d_1 + \psi(\alpha_k^\pm)),$$

$$\alpha_k^\pm = k - \frac{d_1}{\pi\alpha_k^\pm} - \frac{\psi(\alpha_k^\pm)}{\pi\alpha_k^\pm}$$

Since $|\alpha_k^\pm| \gtrsim k$ and $\sum_k |\psi(\alpha_k^\pm)|^2 < +\infty$, one obtains

$$\alpha_k^\pm = k - \frac{d_1}{\pi k} + \frac{\epsilon_k^\pm}{k}, \quad \sum_k |\epsilon_k|^2 < +\infty$$

□

Now we can state the main result, the Marchenko-Ostrovski Theorem (1975)

Theorem 10.1.

$$-\infty < \lambda_0 < \lambda_1^- \leq \lambda_1^+ < \lambda_2^- \leq \lambda_2^+ < \dots$$

In order $\{\lambda_k^+\}_{k=0}^{\infty}$ *and* $\{\lambda_k^-\}_{k=1}^{\infty}$ *be the periodic and anti-perioidc spectra of the Sturm-Liouville operator*

$$-y'' + qy = \lambda y, \quad 0 \le x \le 1$$

with $q \in L^2$*, it is necessary and sufficient that*

$$\lambda_k^{\pm} = \lambda_0 + z^2(k\pi \pm 0)$$

where $z(\theta)$ *is a conformal map from the domain* $\Theta_+\{h_k\}$ *onto the upper half-plane,* $h_0 = 0, h_k = h_{-k}$

$$\sum (kh_k)^2 < +\infty$$

$$z(0) = 0, \quad \lim_{\theta \to +\infty} \frac{1}{i\theta} z(i\theta) = \frac{1}{\pi}$$

Proof. The necessity was already proven. Let $z(\theta)$ be as in the statement of the theorem. We can assume $\lambda_0 = 0$. Let $\theta(z)$ be the inverse function, $u(z) = \cos\theta(z)$. By Lemma (**??**), one has

$$u(z) = \cos(\pi z) - \frac{d_1}{z}\sin(\pi z) + \frac{g(z)}{z}$$

$$g(z) = \int_0^{\pi} \tilde{g}(t)\sin(zt)dt, \quad \tilde{g} \in L^2[0,\pi] \tag{10.12}$$

Moreover, let $\lambda_k^{\pm}$ be the roots of the equation $u(z) = \pm 1$, $\alpha_k^{\pm} = \sqrt{\lambda_k^{\pm}}, k > 0, \alpha_{-k}^{\pm} = -\alpha_k^{\pm}$. Then

$$\alpha_k^{\pm} = k - \frac{d_1\pi}{k} + \frac{\epsilon_k^{\pm}}{k}, \quad \sum |\epsilon_k^{\pm}|^2 < +\infty$$

Let $z(\overline{\theta}) = \overline{z(\theta)}$. Then z maps $\mathcal{C} \setminus \cup_k\{\Re\theta = k\pi : -h_k \le \Im\theta \le h_k\}$ onto $\mathbb{C} \setminus [\cup_{k>0}\{\Im z = 0 : \alpha_k^- \le \Re z \le \alpha_k^+\} \cup \cup_{k<0}\{\Im z = 0 : \alpha_k^+ \le \Re z \le \alpha_k^-\}]$. Pick an arbitrary point,

$$\theta_k = k\pi + ih_k', \quad -h_k \le h_k' \le h_k, \quad k = 1, 2, \ldots$$

on " one side of the slit "

$$\mu_k = z^2(\theta_k')$$

Clearly

$$(\alpha_n^-)^2 = \lambda_n^- \le \mu_n \le \lambda_n^+ = (\alpha_n^+)^2$$

$$\begin{aligned} \alpha_n^{\pm} =& n^2 - \frac{2d_1}{\pi} + 2\epsilon_n^{\pm} + \mathcal{O}\left(\frac{1}{n^2}\right) \\ =& n^2 + C_0 + l^2(n) \\ \mu_n =& n^2 + C_0 + l^2(n) \end{aligned}$$

Set $\sigma_n = \text{sgn} h_n'$,

$$\varkappa_n = \log\left((-1)^n(u(\sqrt{\mu_n}) - \sigma_n\sqrt{u^2(\sqrt{\mu_n}) - 1})\right)$$

We need to estimate $\varkappa_n$. The Bernstein inequality is not good enough for that. Since $u(0) = 0$, $u(\sqrt{z})$ is an entire function of exponential type π. Due to Lemma (4.9)

$$u(\sqrt{z}) - 1 = z\frac{(z-\lambda_n^-)(\lambda_n^+ - z)}{n^4}\prod_{m\neq n, m\geq 1}\frac{(z-\lambda_m^-)(\lambda_m^+ - z)}{m^4}$$

Just as in the proof of Theorem (9.3),

$$\varkappa_n \in l_1^2(n)$$

Therefore exists unique $q(x) \in L^2[0,\pi]$ such that

$$\mu_n(q) = \mu_n, \quad \varkappa_n(q) = \varkappa_n$$

In particular,

$$u(\sqrt{\mu_n}) - \sigma_n\sqrt{u^2(\sqrt{\mu_n}) - 1} = \frac{\Delta(\mu_n, q)}{2} - \sigma_n\sqrt{\left(\frac{\Delta(\mu_n, q)}{2}\right)^2 - 1}$$

Note that

$$\xi \pm \sqrt{\xi^2 - 1} = t \implies \xi = \frac{1}{2}\left(t + \frac{1}{t}\right)$$

Thus

$$u(\sqrt{\mu_n}) = \frac{\Delta(\mu_n, q)}{2}$$

Due to (10.12) the interpolation applies, thus

$$u(z) = y_2(\pi, z^2, q)\sum_{n=1}^{\infty}\frac{\sqrt{\mu_n}\left(\frac{\Delta(\mu_n,q)}{2}\right)}{\partial_\lambda y_2(\pi,\mu_n,q)(\mu_n - z^2)} = \frac{\Delta(z^2)}{2}$$

$$\implies \Delta(z) = 2u(\sqrt{z})$$

The roots of

$$\Delta(z) = \pm 2$$

are $(\alpha_n^{\pm})^2 = \lambda_n, n = 1, 2, \ldots$. □