Assignment 5

APM462 – Nonlinear Optimization – Summer 2016

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SOLUTIONS

Question 1 Let $u_*(\cdot)$ be an extremal(e.g. minimizer) of the functional $F[u(\cdot)] = \int_a^b L(x, u(x), u'(x))dx$. Find the Euler-Lagrange equation satisfied by $u_*(\cdot)$ where $L(x, z, p)$ is given by the following functions:

(a) $L(x, z, p) = xz + p^2$

(b) $L(x, z, p) = \frac{p}{\sqrt{1+p^2}}$

(c) $L(x, z, p) = z + xp^2$

(d) $L(x, z, p) = xp - p^2$

Solution Recall the Euler-Lagrange equation is

$$\frac{\partial L}{\partial u} - \frac{d}{dx}\frac{\partial L}{\partial u'} = 0$$

If you work out the derivatives you should obtain:

(a) E-L $\implies x - 2u'' = 0$

(b) E-L $\implies u'' = 0$

(c) E-L $\implies 1 - 2u' - 2xu'' = 0$

(d) E-L $\implies 1 - 2u'' = 0$

Question 2 Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly convex function, and F be the functional defined by

$$F[u(\cdot)] = \frac{1}{2}\int_0^1 f(u'(x))dx,$$

acting on functions in the set

$$\mathcal{A} = \{C^1 \text{ functions } u : [0, 1] \to \mathbb{R} \text{ such that } u(0) = 2, u(1) = 1\}$$

Find the minimizer of F in $\mathcal{A}$.

Solution Euler-Lagrange dictates (when $u \in C^2$)

$$-f''(u'(x))u''(x) = 0$$

and since f is strictly convex, we have $f'' > 0$, thus

$$u'' = 0 \implies u(x) = c_1 x + c_2, \quad c_1, c_2 \in \mathbb{R}$$

and the boundary conditions on u give

$$u(x) = 2 - x$$

Otherwise, we have from Euler-Lagrange (when $u \in C^1$)

$$f'(u') = const$$

Thus either

$$f' = const \quad \text{or} \quad u' = c_1, \quad c_1 \in \mathbb{R}$$

but if $f' = const$, then f isn't strictly convex, which is a contradiction. Thus

$$u' = c_1 \implies u(x) = c_1 x + c_2, \quad c_1, c_2 \in \mathbb{R}$$

and the boundary values imply $u(x) = 2 - x$.

□

Question 3 Find the first order necessary conditions satisfied by a solution $u_* \in C^2$ to a minimization problem where one of the endpoints is fixed and the other is free:

$$\text{minimize } F[u(\cdot)] = \int_a^b L(x, u(x), u'(x))dx$$

$$\text{subject to: } u \in \mathcal{A} = \{u : [a, b] \to \mathbb{R} : u \in C^1[a, b], u(a) = A\}$$

Solution Take $v \in \{u : [a, b] \to \mathbb{R} : u \in C^1[a, b], v(a) = 0\}$, then

$$\begin{aligned}
\frac{\delta F}{\delta u} = \frac{d}{ds} F[u(x) + sv(x)]\Big|_{s=0} &= \int_a^b \left[\frac{\partial L}{\partial u} v(x) + \frac{\partial L}{\partial u'} v'(x)\right] dx \\
&= \int_a^b \left[\frac{\partial L}{\partial u} - \frac{d}{dx}\frac{\partial L}{\partial u'}\right] v(x)dx + \frac{\partial L}{\partial u'} v(x)\Big|_a^b, \qquad \text{integration by parts} \\
&= \int_a^b \left[\frac{\partial L}{\partial u} - \frac{d}{dx}\frac{\partial L}{\partial u'}\right] v(x)dx + \frac{\partial L}{\partial u'}(b)v(b), \qquad \text{since } v(a) = 0
\end{aligned}$$

We now see the extra term from the integration by parts when $x = b$ is free, so we require

$$\frac{\partial L}{\partial u'}(b) = 0$$

as an additional necessary condition to Euler-Lagrange. □

Question 4 For each of the following functionals,

(a) compute $\delta F/\delta u$ "from first principles", that is by considering $\frac{d}{ds}F[u(\cdot)+sv(\cdot)]$ for function $v(\cdot) \in C^1$ which are zero t the endpoints.

(b) Also compute $\delta F/\delta u$ by using the general formula we have for $\delta F/\delta u$, when F is a functional of the form $F[u(\cdot)] = \int_a^b L(x, u(x), u'(x))dx$.

(c) Solve the problem of minimizing F in the set

$$\mathcal{A} = \{C^1 \text{ functions } u : [0,1] \to \mathbb{R} \text{ such that } u(0) = 1, u(1) = 3\}$$

(d) Solve the problem of minimizing F in the set

$$\mathcal{A} = \{C^1 \text{ functions } u : [0,1] \to \mathbb{R} \text{ such that } u(0) = 1\}$$

1.
$$F_1[u(\cdot)] = \int_0^1 \left[\frac{1}{2}u'(x)^2 + 2u(x)\right] dx$$

2.
$$F_2[u(\cdot)] = \int_0^1 \left[\frac{1}{2}u'(x)^2 + u'(x) + 2u(x)\right] dx$$

3.
$$F_3[u(\cdot)] = \int_0^1 \frac{(u'(x)-1)^2}{x^2+1} dx$$

Solution

1. The first variation is (assuming $v(0) = v(1) = 0$)
$$\frac{\delta F_1}{\delta u} = \lim_{s\to 0} \frac{F_1[u+sv] - F_1[u]}{s} = \int_0^1 (u'(x)v'(x) + 2v(x))dx = \int_0^1 (2 - u''(x))v(x)dx$$
We see the minimum is found via
$$2 - u''(x) = 0 \implies u(x) = x^2 + c_1 x + c_2$$
For c), we see $u = x^2 + x + 1$. For d), we find $u = x^2 - 2x + 1$.

2. a),b),c) are the same as the previous question since
$$F_2[u(\cdot)] = F_1[u(\cdot)] + u(1) - u(0)$$
For d), u takes the general form as in 1), and $c_2 = 1$ still... but
$$\frac{\partial L}{\partial u'} = u' + 1 = 0 \implies c_1 = -3$$
Thus $u(x) = x^2 - 3x + 1$.

3. The first variation is (assuming $v(0) = v(1) = 0$)
$$\frac{\delta F_3}{\delta u} = \lim_{s\to 0} \frac{F_3[u+sv] - F_3[u]}{s} = 2\int_0^1 \frac{(u'(x)-1)}{x^2+1}v'(x)dx = -2\int_0^1 v(x)\frac{d}{dx}\left(\frac{u'(x)-1}{x^2+1}\right)dx$$
We see the minimum is found via
$$\frac{d}{dx}\left(\frac{u'(x)-1}{x^2+1}\right) = 0 \implies u(x) = \frac{c_1}{3}x^3 + (c_1+1)x + c_2$$
For c), we see $u(x) = \frac{1}{4}x^3 + \frac{7}{4}x + 1$. For d), we find $u(x) = x + 1$. □