# NILPOTENCY, ALMOST NONNEGATIVE CURVATURE AND THE GRADIENT PUSH

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ABSTRACT. We show that almost nonnegatively curved $m$-manifolds are, up to finite cover, nilpotent spaces in the sense of homotopy theory and have $C(m)$-nilpotent fundamental groups. We also show that up to a finite cover almost nonnegatively curved manifolds are fiber bundles with simply connected fibers over nilmanifolds.

## 1. INTRODUCTION

Almost nonnegatively curved manifolds were introduced by Gromov in the late 70s [Gro80], with the most significant contributions to their study made by Yamaguchi in [Yam91] and Fukaya and Yamaguchi in [FY92]. Building on their ideas, in the present article we establish several new properties of these manifolds which yield, in particular, new topological obstructions to almost nonnegative curvature.

A closed smooth manifold is said to be almost nonnegatively curved if it can Gromov-Hausdorff converge to a single point under a lower curvature bound. By rescaling, this definition is equivalent to the following one, which we will employ throughout this article.

**Definition 1.1.** *A closed smooth manifold $M$ is called almost nonnegatively curved if it admits a sequence of Riemannian metrics $\{g_n\}_{n\in\mathbb{N}}$ whose sectional curvatures and diameters satisfy* $\sec(M, g_n) \geqslant -1/n$ *and* $\operatorname{diam}(M, g_n) \leqslant 1/n$.

Almost nonnegatively curved manifolds generalize almost flat as well as nonnegatively curved manifolds. One main source of examples comes from a theorem of Fukaya and Yamaguchi. It states that if $F \to E \to B$ is a fiber bundle over an almost nonnegatively curved manifold $B$ whose fiber $F$ is compact and admits a nonnegatively curved metric which is invariant under the structure group, then the total space $E$ is almost nonnegatively curved [FY92]. Further examples are given by closed manifolds which admit a cohomogeneity one action of a compact Lie group (compare [ST04]).

In this work we combine collapsing techniques with a non-smooth analogue of the gradient flow of concave functions which we call the "gradient push". This notion is based on the construction of gradient curves of $\lambda$-concave functions used in [PP96] and bears many similarities to the Sharafutdinov retraction [Sha78]. The gradient push plays a key role in the proofs of two of the three main results in this paper, and we believe that it should also prove useful for dealing with other problems related to collapsing under a lower curvature bound.

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1.1. To put the main theorems of the present work into perspective, let us first briefly recall some previously known results:

Let $M = M^m$ be an almost nonnegatively curved $m$-manifold.

* Gromov proved in [Gro78] that the minimal number of generators of the fundamental group $\pi_1(M)$ of $M$ can be estimated by a constant $C_1(m)$ depending only on $m$, and in [Gro81] that the sum of Betti numbers of $M$ with respect to any field of coefficients does not exceed some uniform constant $C_2 = C_2(m)$.
* Yamaguchi showed that, up to a finite cover, $M$ fibers over a flat $b_1(M;\mathbb{R})$-dimensionsal torus and that $M^m$ is diffeomorphic to a torus if $b_1(M;\mathbb{R}) = m$ [Yam91].
* Fukaya and Yamaguchi proved that $\pi_1(M)$ is almost nilpotent, i.e., contains a nilpotent subgroup of finite index, and also that $\pi_1(M)$ is $C_3(m)$-solvable, i.e, contains a solvable subgroup of index at most $C_3(m)$ [FY92].
* If a closed manifold has negative Yamabe constant, then it cannot volume collapse with scalar curvature bounded from below (see [Sch89, LeB01]). In particular, no such manifold can be almost nonnegatively curved.
* The $\hat{A}$-genus of a closed spin manifold $X$ of almost nonnegative Ricci curvature satisfies the inequality $\hat{A}(X) \leqslant 2^{\dim(X)/2}$ ([Gal83], [Gro96, page 41]).

Let us now state the main results of this article.

1.2. Our first result concerns the hitherto unexplored relation between curvature bounds and the actions of the fundamental group on the higher homotopy groups.

Recall that an action by automorphisms of a group $G$ on an abelian group $V$ is called nilpotent if $V$ admits a finite sequence of $G$-invariant subgroups

$$V = V_0 \supset V_1 \supset \ldots \supset V_k = 0$$

such that the induced action of $G$ on $V_i/V_{i+1}$ is trivial for any $i$. A connected CW-complex $X$ is called *nilpotent* if $\pi_1(X)$ is a nilpotent group that operates nilpotently on $\pi_k(X)$ for every $k \geqslant 2$.

Nilpotent spaces play an important role in topology since they enjoy some of the best homotopy-theoretic properties of simply connected spaces, like a Whitehead theorem or reasonable Postnikov towers. Furthermore, unlike the category of simply connected spaces, the category of nilpotent ones is closed under many constructions such as the based loop space functor or the formation of function spaces, and group-theoretic functors, like localization and completion, have topological extensions in this category.

**Theorem A** (Nilpotency Theorem). *Let $M$ be a closed almost nonnegatively curved manifold. Then a finite cover of $M$ is a nilpotent space.*

It would be interesting to know whether the order of this covering can be estimated solely in terms of the dimension of $M$.

**Example 1.2.** Let $h\colon S^3 \times S^3 \to S^3 \times S^3$ be defined by $h : (x, y) \mapsto (xy, yxy)$. This map is a diffeomorphism with the inverse given by $h^{-1}\colon (u, v) \mapsto (u^2v^{-1}, vu^{-1})$. The induced map $h_*$ on $\pi_3(S^3 \times S^3)$ is given by the matrix $A_h = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Notice that the eigenvalues of $A_h$ are different from 1 in absolute value. Let $M$ be the mapping cylinder of $h$. Clearly,

$M$ has the structure of a fiber bundle $S^3 \times S^3 \to M \to S^1$, and the action of $\pi_1(M) \cong \mathbb{Z}$ on $\pi_3(M) \cong \mathbb{Z}^2$ is generated by $A_h$. In particular, $M$ is not a nilpotent space and hence, by Theorem A, it does not admit almost nonnegative curvature. This fact doesn't follow from any previously known results.

1.3. Our next main result provides an affirmative answer to a conjecture of Fukaya and Yamaguchi [FY92, Conjecture 0.15].

**Theorem B** ($C$-Nilpotency Theorem for $\pi_1$). *Let $M$ be an almost nonnegatively curved $m$-manifold. Then $\pi_1(M)$ is $C(m)$-nilpotent, i.e., $\pi_1(M)$ contains a nilpotent subgroup of index at most $C(m)$.*

Notice that Theorem B is new even for manifolds of nonnegative curvature.

**Example 1.3.** For any $C > 0$ there exist prime numbers $p > q > C$ and a finite group $G_{pq}$ of order $pq$ which is solvable but not nilpotent. In particular, $G_{pq}$ does not contain any nilpotent subgroup of index less than or equal to $C$.

Whereas none of the results mentioned so far excludes $G_{pq}$ from being the fundamental group of some almost nonnegatively curved $m$-manifold, Theorem B shows that for $C > C(m)$ none of the groups $G_{pq}$ can be realized as the fundamental group of such a manifold.

1.4. In [FY92] Fukaya and Yamaguchi also conjectured that a finite cover of an almost nonnegatively Ricci curved manifold $M$ fibers over a nilmanifold with a fiber which has nonnegative Ricci curvature and whose fundamental group is finite. This conjecture was later refuted by Anderson [And92].

It is, on the other hand, very natural to consider this conjecture in the context of almost nonnegative *sectional* curvature. In fact, here Yamaguchi's fibration theorem ([Yam91]) and the results of [FY92] easily imply that a finite cover of an almost nonnegatively curved manifold is the total space of a Serre fibration over a nilmanifold with simply connected fibers.

From mere topology, it is, however, not clear whether this fibration can actually always be made into a genuine fiber bundle. Our next result shows that this is indeed true, and that for manifolds of almost nonnegative sectional curvature Fukaya's and Yamaguchi's original conjecture essentially does hold.

**Theorem C** (Fibration Theorem). *Let $M$ be an almost nonnegatively curved manifold. Then a finite cover $\tilde{M}$ of $M$ is the total space of a fiber bundle*

$$F \to \tilde{M} \to N$$

*over a nilmanifold $N$ with a simply-connected fiber $F$. Moreover, the fiber $F$ is almost nonnegatively curved in the sense of the following definition.*

**Definition 1.4.** *A closed smooth manifold $M$ is called almost nonnegatively curved in the generalized sense if for some nonnegative integer $k$ there exists a sequence of complete Riemannian metrics $g_n$ on $M \times \mathbb{R}^k$ and points $p_n \in M \times \mathbb{R}^k$ such that*

*(1) the sectional curvatures of the metric balls of radius $n$ around $p_n$ satisfy*

$$\sec(B_n(p_n)) \geqslant -1/n;$$

(2) *for* $n \to \infty$ *the pointed Riemannian manifolds* $((M \times \mathbb{R}^k, g_n), p_n)$ *converge in the pointed Gromov-Hausdorff distance to* $(\mathbb{R}^k, 0)$*;*
(3) *the regular fibres over* $0$ *are diffeomorphic to* $M$ *for all large* $n$*.*

Due to Yamaguchi's fibration theorem [Yam91], manifolds which are almost nonnegatively curved in the generalized sense play the same central role in collapsing under a lower curvature bound as almost flat manifolds do in the Cheeger-Fukaya-Gromov theory of collapsing with bounded curvature (see [CFG92]).

It is not known whether all manifolds which are almost nonnegatively curved in the generalized sense are almost nonnegatively curved. Clearly, if $k = 0$, this definition reduces to the standard one. Moreover, it is easy to see that all results of the present article, as well as all results about almost nonnegatively curved manifolds mentioned earlier (except possibly for the ones concerning the $\hat{A}$-genus and Yamabe constant), hold for manifolds which are almost nonnegatively curved in the sense of Definition 1.4.

1.5. Let us now describe the structure of the remaining sections of this article.

In section 2, after providing some necessary background from Alexandrov geometry, we introduce the gradient push, which is, roughly speaking, the gradient flow of the square of a distance function. It serves as one of the main technical tools in the proofs of theorem A and theorem B.

In section 3 we prove Theorem A by a direct application of the gradient push technique.

In section 4 we prove Theorem B. The proof is also based on the gradient push, but is more involved and employs further technical tools such as "limit fundamental groups" of Alexandrov spaces.

In section 5 we prove Theorem C. This section is completely independent from the rest of the article.

In section 6 we discuss some further open questions related to our results.

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## 2. Alexandrov geometry and the gradient push

This section provides necessary background in Alexandrov geometry and introduces the gradient push. The results of sections 2.1-2.3 are mostly repeated from [PP96] and [Pet95]. The reader may consult [BGP92] for a general reference on Alexandrov spaces.

### 2.1. $\lambda$-concave functions.

**Definition 2.1.** *(for a space without boundary) Let* $A$ *be an Alexandrov space without boundary. A Lipschitz function* $f\colon A \to \mathbb{R}$ *is called* $\lambda$*-concave if for any unit speed minimizing geodesic* $\gamma$ *in* $A$*, the function*

$$f \circ \gamma(t) - \lambda t^2/2$$

*is concave.*

If $A$ is an Alexandrov space with boundary, then its double $\tilde{A}$ is also an Alexandrov space (see [Per91, 5.2]). Let $\mathrm{p} : \tilde{A} \to A$ be the canonical map. Given a function $f$ on $A$, set $\tilde{f} = f \circ \mathrm{p}$.

**Definition 2.2.** *(for a space with boundary) Let $A$ be an Alexandrov space with boundary. A Lipschitz function $f \colon A \to \mathbb{R}$ is called $\lambda$-concave if for any unit speed minimizing geodesic $\gamma$ in $\tilde{A}$, the function*

$$\tilde{f} \circ \gamma(t) - \lambda t^2/2$$

*is concave.*

**Remark 2.3.** Notice that the restriction of a linear function on $\mathbb{R}^n$ to a ball is not 0-concave in this sense.

**Remark 2.4.** In the above definitions, the Lipschitz condition is only technical. With some extra work, all results of this section can be extended to continuous functions.

2.2. **Tangent cone and differential.** Given a point $p$ in an Alexandrov space $A$, we denote by $T_p = T_p(A)$ the tangent cone at $p$.

If $d$ denotes the metric of an Alexandrov space $A$, let us denote by $\lambda A$ the space $(A, \lambda d)$. Let $i_\lambda : \lambda A \to A$ be the canonical map. The limit of $(\lambda A, p)$ for $\lambda \to \infty$ is the tangent cone $T_p$ at $p$ (see [BGP92, 7.8.1]).

**Definition 2.5.** *For any function $f \colon A \to \mathbb{R}$ the function $d_p f \colon T_p \to \mathbb{R}$ such that*

$$d_p f = \lim_{\lambda \to \infty} \lambda (f \circ i_\lambda - f(p))$$

*is called the differential of $f$ at $p$.*

It is easy to see that for a $\lambda$-concave function $f$ the differential $d_p f$ is defined everywhere, and that $d_p f$ is a 0-concave function on the tangent cone $T_p$.

**Definition 2.6.** *Given a $\lambda$-concave function $f \colon A \to \mathbb{R}$, a point $p \in A$ is called critical point of $f$ if $d_p f \leqslant 0$.*

2.3. **Gradient curves.** With a slight abuse of notation we will call elements of the tangent cone $T_p$ the "tangent vectors" at $p$. The origin of $T_p$ plays the role of the zero vector and is denoted by $o = o_p$. For a tangent vector $v$ at $p$ we define its absolute value $|v|$ as the distance $|ov|$ in $T_p$. For two tangent vectors $u$ and $v$ at $p$ we can define their "scalar product"

$$\langle u, v \rangle = (|u|^2 + |v|^2 - |uv|^2)/2 = |u| \cdot |v| \cos \alpha,$$

where $\alpha = \angle uov$ in $T_p$.

For two points $p, q \in A$ we define $\log_p q$ to be a tangent vector $v$ at $p$ such that $|v| = |pq|$ and such that the direction of $v$ coincides with a direction from $p$ to $q$ (if such a direction is not unique, we choose any one of them). Given a curve $\gamma(t)$ in $A$, we denote by $\gamma^+(t)$ the right and by $\gamma^-(t)$ the left tangent vectors to $\gamma(t)$, where, respectively,

$$\gamma^\pm(t) \in T_{\gamma(t)}, \quad \gamma^\pm(t) = \lim_{\varepsilon \to +0} \frac{\log_{\gamma(t)} \gamma(t \pm \varepsilon)}{\varepsilon}.$$

For a real function $f(t)$, $t \in \mathbb{R}$, we denote by $f^+(t)$ its right derivative and by $-f^-(t)$ its left derivative. Note that our sign convention (which is chosen to agree with the notion of right and left derivatives of curves) is not quite standard. For example,

$$\text{if } f(t) = t \text{ then } f^+(t) \equiv 1 \text{ and } f^-(t) \equiv -1.$$

**Definition 2.7.** *Given a $\lambda$-concave function $f$ on $A$, a vector $g \in T_p(A)$ is called a gradient of $f$ at $p \in A$ (in short: $g = \nabla_p f$) if*
*(i) $d_p f(x) \leqslant \langle g, x\rangle$ for any $x \in T_p$, and*
*(ii) $d_p f(g) = \langle g, g\rangle$.*

It is easy to see that any $\lambda$-concave function has a uniquely defined gradient vector field. Moreover, if $d_p f(x) \leqslant 0$ for all $x \in T_p$, then $\nabla_p f = o$ (here $o$ denotes the origin of the tangent cone $T_p$); otherwise,

$$\nabla_p f = d_p f(\xi)\xi$$

where $\xi$ is the (necessarily unique) unit vector for which the function $d_p f$ attains its maximum.

Moreover, for any minimizing geodesic $\gamma : [a, b] \to U$ parameterized by arclength, the following inequality holds:

$$\langle \gamma^+(a), \nabla_{\gamma(a)} f\rangle + \langle \gamma^-(b), \nabla_{\gamma(b)} f\rangle \geqslant -\lambda(b-a). \tag{2.1}$$

Indeed,

$$\begin{aligned}\langle \gamma^+(a), \nabla_{\gamma(a)} f\rangle + \langle \gamma^-(b), \nabla_{\gamma(b)} f\rangle &\geqslant d_{\gamma(a)} f(\gamma^+(a)) + d_{\gamma(b)} f(\gamma^-(b)) = \\ &= (f\circ\gamma)^+|_a + (f\circ\gamma)^-|_b \geqslant -\lambda(b-a).\end{aligned}$$

**Definition 2.8.** *A curve $\alpha\colon [a, b] \to A$ is called an $f$-gradient curve if for any $t \in [a, b]$*

$$\alpha^+(t) = \nabla_{\alpha(t)} f.$$

**Proposition 2.9.** *Given a $\lambda$-concave function $f\colon A \to \mathbb{R}$ and a point $p \in A$ there is a unique gradient curve $\alpha\colon [0, \infty) \to A$ such that $\alpha(0) = p$.*

*Moreover, if $\alpha$ and $\beta$ are two $f$-gradient curves, then*

$$|\alpha(t_1)\beta(t_1)| \leqslant |\alpha(t_0)\beta(t_0)| \exp(\lambda(t_1 - t_0)) \quad \textit{for all} \quad t_1 \geqslant t_0.$$

The gradient curve can be constructed as a limit of broken geodesics, made up of short segments with directions close to the gradient. The convergence, uniqueness, as well as the last inequality in Proposition 2.9 follow from inequality (2.1) above, while Corollary 2.11 below guarantees that the limit is indeed a gradient curve, having a unique right tangent vector at each point.

**Lemma 2.10.** *Let $A_n \overset{GH}{\to} A$ be a sequence of Alexandrov spaces with curvature $\geqslant k$ which Gromov-Hausdorff converges to an Alexandrov space $A$.*

*Let $f_n \to f$, where $f_n : A_n \to \mathbb{R}$ is a sequence of $\lambda$-concave functions converging to $f : A \to \mathbb{R}$.*

*Let $p_n \to p$, where $p_n \in A_n$ and $p \in A$.*

*Then*

$$|\nabla_p f| \leqslant \liminf_{n\to\infty} |\nabla_{p_n} f_n|.$$

**Corollary 2.11.** *Given a $\lambda$-concave function $f$ on $A$ and a sequence of points $p_n \in A$, $p_n \to p$, we have*

$$|\nabla_p f| \leqslant \liminf_{n\to\infty} |\nabla_{p_n} f|.$$

*Proof of Lemma 2.10.* Fix an $\varepsilon > 0$ and choose $q$ near $p$ such that

$$\frac{f(q) - f(p)}{|pq|} > |\nabla_p f| - \varepsilon.$$

Now choose $q_n \in A_n$ such that $q_n \to q$. If $|pq|$ is sufficiently small and $n$ is sufficiently large, the $\lambda$-concavity of $f_n$ then implies that

$$\liminf_{n\to\infty} \frac{d_{p_n} f_n(v_n)}{|v_n|} \geqslant |\nabla_p f| - 2\varepsilon \quad \text{for} \quad v_n = \log_{p_n}(q_n) \in T_{p_n}(A_n).$$

Therefore,

$$\liminf_{n\to\infty} |\nabla_{p_n} f_n| \geqslant |\nabla_p f| - 2\varepsilon \quad \text{for any} \quad \varepsilon > 0,$$

i.e.,

$$\liminf_{n\to\infty} |\nabla_{p_n} f_n| \geqslant |\nabla_p f|.$$

□

**Lemma 2.12.** *Let $f$ be a $\lambda$-concave function, $\lambda \geqslant 0$ and $\alpha(t)$ be an $f$-gradient curve, and let $\bar\alpha(s)$ be its reparameterization by arclength. Then $f \circ \bar\alpha$ is $\lambda$-concave.*

*Proof.*

$$(f \circ \bar\alpha)^+(s_0) = |\nabla_{\bar\alpha(s_0)} f| \geqslant \frac{d_{\bar\alpha(s_0)} f(\log_{\bar\alpha(s_0)}(\bar\alpha(s_1))}{|\bar\alpha(s_1)\,\bar\alpha(s_0)|} \geqslant$$

$$\geqslant \frac{f(\bar\alpha(s_1)) - f(\bar\alpha(s_0)) - \lambda|\bar\alpha(s_1)\,\bar\alpha(s_0)|^2/2}{|\bar\alpha(s_1)\,\bar\alpha(s_0)|} \geqslant \frac{f(\bar\alpha(s_1)) - f(\bar\alpha(s_0))}{s_1 - s_0} - \lambda|\bar\alpha(s_1)\,\bar\alpha(s_0)|/2.$$

Since $\frac{|\bar\alpha(s_1)\,\bar\alpha(s_0)|}{(s_1 - s_0)} \to 1$ as $s_1 \to s_0+$, it follows that $f \circ \bar\alpha$ is $\lambda$-concave.

□

2.4. **Gradient push.** Let $f$ be a $\lambda$-concave function on an Alexandrov space $A$. Consider the map $\Phi_f^T : A \to A$ defined as follows: $\Phi_f^T(x) = \alpha_x(T)$, where $\alpha_x : [0, \infty) \to A$ is the $f$-gradient curve with $\alpha_x(0) = x$. The map $\Phi_f^T$ is called *$f$-gradient push at time $T$*. From Proposition 2.9 it is clear that $\Phi_f^T$ is an $\exp(\lambda T)$-Lipschitz map. Next we want to prove that this map behaves nicely under Gromov-Hausdorff-convergence.

**Theorem 2.13.** *Let $A_n \overset{GH}{\to} A$ be a sequence of Alexandrov spaces with curvature $\geqslant k$ which converges to an Alexandrov space $A$.*

*Let $f_n \to f$, where $f_n : A_n \to \mathbb{R}$ is a sequence of $\lambda$-concave functions and $f : A \to \mathbb{R}$.*

*Then $\Phi_{f_n}^T \to \Phi_f^T$.*

Theorem 2.13 immediately follows from the following Lemma:

**Lemma 2.14.** *Let $A_n \stackrel{GH}{\to} A$ be a sequence of Alexandrov spaces with curvature $\geqslant k$ which converges to an Alexandrov space $A$.*

*Let $f_n \to f$, where $f_n : A_n \to \mathbb{R}$ is a sequence of $\lambda$-concave functions and $f : A \to \mathbb{R}$.*

*Let $\alpha_n : [0, \infty) \to A_n$ be the sequence of $f_n$-gradient curves with $\alpha_n(0) = p_n$ and let $\alpha : [0, \infty) \to A$ be the $f$-gradient curve with $\alpha(0) = p$.*

*Then $\alpha_n \to \alpha$.*

*Proof.* We may assume without loss of generality that $f$ has no critical points. (Otherwise consider instead the sequence $A'_n = A_n \times \mathbb{R}$ with $f'_n(a \times x) = f_n(a) + x$.)

Let $\bar\alpha_n(s)$ denote the reparameterization of $\alpha_n(t)$ by arc length. Since all $\bar\alpha_n$ are 1-Lipschitz, we can choose a converging subsequence from any subsequence of $\bar\alpha_n$. Let $\bar\beta : [0, \infty) \to A$ be its limit.

Clearly, $\bar\beta$ is also 1-Lipschitz and hence $|\bar\beta^+| \leqslant 1$. Therefore, by Lemma 2.10,

$$\lim_{n\to\infty} f_n \circ \bar\alpha_n|_a^b = \lim_{n\to\infty} \int_a^b |\nabla_{\bar\alpha_n(s)} f_n| ds \geqslant$$

$$\geqslant \int_a^b |\nabla_{\bar\beta(s)} f| \geqslant \int_a^b d_{\beta(t)} f(\beta^+(t)) = f \circ \beta|_a^b$$

On the other hand, since $\bar\alpha_n \to \bar\beta$ and $f_n \to f$ we have $f_n \circ \bar\alpha_n|_a^b \to f \circ \bar\beta|_a^b$. Therefore, in both of these inequalities in fact equality holds.

Hence, $|\nabla_{\bar\beta(s)} f| = \lim_{n\to\infty} |\nabla_{\bar\alpha_n(s)} f_n|$, $|\bar\beta^+(s)| = 1$ and the directions of $\bar\beta^+(s)$ and $\nabla_{\bar\beta(s)} f$ coincide almost everywhere. This implies that $\bar\beta(s)$ is a gradient curve reparameterized by arc length. In other words, if $\bar\alpha(s)$ denotes the reparameterization of $\alpha(t)$ by arc length, then $\bar\beta(s) = \bar\alpha(s)$ for all $s$. It only remains to show that the original parameter $t_n(s)$ of $\alpha_n$ converges to the original parameter $t(s)$ of $\alpha$.

Notice that $|\nabla_{\bar\alpha_n(s)} f_n| dt_n = ds$ or $dt_n/ds = ds/d(f_n \circ \bar\alpha_n)$. Likewise, $dt/ds = ds/d(f \circ \bar\alpha)$. Then the convergence $t_n \to t$ follows from the $\lambda$-concavity of $f_n \circ \bar\alpha_n$ (see Lemma 2.12) and the convergence $f_n \circ \bar\alpha_n \to f \circ \bar\alpha$. $\square$

**Remark 2.15.** (A few words about the name "gradient push")

At first sight, it might look strange why we call "gradient push" something which is normally referred to as the gradient flow. The reason for this name stems from the relation of this notion to moving furniture inside a room by pushing it around. If a piece of furniture is located in the middle of a room, one can push it to any other place in the room. But as soon as it is pushed to a wall one cannnot push it back to the center; and once it is pushed into a corner one cannot push it anywhere. The same is true for the gradient push in an Alexandrov space, where the role of walls and corners is played by extremal subsets.

### 2.5. **Gradient balls.**

Let $A$ be an Alexandrov space and let $S \subset A$ be a subset of $A$. A function $f : A \to \mathbb{R}$ which can be represented as

$$f = \sum_i \theta_i \frac{\mathrm{dist}^2_{a_i}}{2} \quad \text{with} \quad \theta_i \geqslant 0, \ \sum_i \theta_i = 1 \ \text{ and } \ a_i \in S$$

will be called *cocos-function with respect to* $S$ (where "cocos" stands for **co**nvex **co**mbination of **s**quares of distance functions). A broken gradient curve for a collection of such functions will be called cocos-curve with respect to $S$.

For $p \in A$ and $T, r \in \mathbb{R}_+$, let us define "the gradient ball with center $p$ and radius $T$ with respect to $B_r(p)$", $\beta_T^r(p)$, as the set of all end points of cocos-curves with respect to $B_r(p)$ that start at $p$ with total time $\leqslant T$.

**Lemma 2.16.**

*(I) There exists $T = T(m) \in \mathbb{R}_+$ such that for any $m$-dimensional Alexandrov space $A$ with curvature $\geqslant -1$ and any $q \in A$ there is a point $p \in A$ such that*
 *(i) $|pq| \leqslant 1$, and*
 *(ii) $B_1(p) \subset \beta_T^1(p)$.*

*(II) There exists $T' = T'(m) \in \mathbb{R}_+$ such that the following holds. Let $A$ be an Alexandrov space which is a quotient $A = \tilde{A}/\Gamma$ of an $m$-dimensional Alexandrov space $\tilde{A}$ with curvature $\geqslant -1$ by a discrete action of a group of isometries $\Gamma$. Let $q \in A$ and $p = p(q) \in A$ be as in part I above.*
 *Then for any lift $\tilde{p} \in \tilde{A}$ of $p$ one has that $B_1(\tilde{p}) \subset \beta 1_{T'}(\tilde{p})$.*

*Proof.* The proof is similar to the construction of a strained point in an Alexandrov space (see [BGP92]).

Set $\delta = 10^{-m}$. Take $a_1 = q$ and take $b_1$ to be a farthest point from $a_1$ in the closed ball $\bar{B}_1(a_1)$. Take $a_2$ to be a midpoint of $a_1b_1$ and let $b_2$ be a farthest point from $a_2$ such that $|a_1b_2| = |a_1a_2|$ and $|a_2b_2| \leqslant \delta|a_1b_1|$, etc. On the $k$-th step we have to take $a_k$ to be a midpoint of $a_{k-1}b_{k-1}$ and $b_k$ to be a farthest point from $a_k$ such that $|a_ib_k| = |a_ia_k|$ for all $i < k$ and $|a_kb_k| \leqslant \delta|a_{k-1}b_{k-1}|$.

After $m$ steps, take $p$ to be a midpoint of $a_mb_m$. We only have to check that we can find $T = T(m)$ such that $\beta 1_T(p) \supset B_1(p)$.

Let $t_i$ be the minimal time such that $B_{|a_ib_i|/\delta^m}(p) \subset \beta 1_{t_i}(p)$. Then one can take $T = t_1$. Therefore it is enough to give estimates for $t_m$ and $t_{k-1}/t_k$ only in terms of $\delta$ and $m$. Looking at the ends of broken gradient curves starting at $p$ for the functions $\operatorname{dist}_p^2/2$, $\operatorname{dist}_{a_i}^2/2$ and $\operatorname{dist}_{b_i}^2/2$, we easily see that $t_n \leqslant 1/\delta^m$. Now, looking at the ends of broken gradient curves starting at $B_{|a_{k-1}b_{k-1}|/\delta^m}(p)$ for the functions $\operatorname{dist}_p^2/2$, $\operatorname{dist}_{a_i}^2/2$ and $\operatorname{dist}_{b_i}^2/2$, we have that $t_{k-1}/t_k \leqslant 1/\delta^m$. Therefore $t_1 \leqslant 1/\delta^{m2} = 10^{-m^3}$. This finishes the proof of part (I).

For part (II), notice that

* for any $r, t > 0$ we have $\beta_t^r(p) \subset B_{re^t}(p)$;
* if $\beta_t^r(p) \supset B_\rho(p)$, then $\beta_{t+\tau}^r(p) \supset B_{\rho e^\tau}(p)$;
* if $\rho = |px|$ and $x \in \beta_t^{r+\rho}(p)$, then $\beta_\tau^r(x) \subset \beta_{t+\tau}^{r+\rho}(p)$.

Take $\varepsilon = e^{-T}/4$ and apply part (I) of the lemma to $\frac{1}{\varepsilon}\tilde{A}$ to find a point $p' \in \tilde{A}$ such that $|\tilde{p}p'| \leqslant \varepsilon$ and $B_\varepsilon(p') \subset \beta_T^\varepsilon(p') \subset \tilde{A}$. Then for some deck transformation $\gamma$ we have $\gamma p' \in \beta_T^\varepsilon(p) \subset B_{\varepsilon e^T}(p)$. Therefore it holds that $\gamma p' \in B_{1/2}(\tilde{p})$. Hence, taking

$$T' = 2T + 1/\varepsilon = 2T + 4e^T,$$

we obtain

$$\beta 1_{T'}(p) \supset \beta^{\varepsilon}_{T+1/\varepsilon}(\gamma p') \supset B_1(\tilde{p}).$$

□

## 3. Nilpotency of almost nonnegatively curved manifolds

In this section we prove Theorem A.

3.1. **Short basis.** We will use the following construction due to Gromov.

Given an Alexandrov space $A$ with a marked point $p \in A$, and a group $\Gamma$ acting discretely on $A$ one can define a short basis of the action of $\Gamma$ at $p$ as follows:

For $\gamma \in \Gamma$ define $|\gamma| = d(p, \gamma(p))$. Choose $\gamma_1 \in \Gamma$ with the minimal norm in $\Gamma$. Next choose $\gamma_2$ to have minimal norm in $\Gamma \backslash < \gamma_1 >$. On the $n$-th step choose $\gamma_n$ to have minimal norm in $\Gamma \backslash < \gamma_1, \gamma_2, ..., \gamma_{n-1} >$. The sequence $\{\gamma_1, \gamma_2, ...\}$ is called a *short basis* of $\Gamma$ at $p$. In general, the number of elements of a short basis can be finite or infinite. In the special case of the action of the fundamental group $\pi_1(A,p)$ on the universal cover of $A$ one speaks of the short basis of $\pi_1(A,p)$.

It is easy to see that for a short basis $\{\gamma_1, \gamma_2, ...\}$ of the fundamental group of an Alexandrov space $A$ the following is true:

(1) If $A$ has diameter $d$ then $|\gamma_i| \leqslant 2d$.
(2) If $A$ is compact then $\{\gamma_i\}$ is finite.
(3) For any $i > j$ we have $|\gamma_i| \leqslant |\gamma_j^{-1}\gamma_i|$.

The third property implies that if $\tilde{p} \in \tilde{A}$ is in the preimage of $p$ in the universal cover $\tilde{A}$ of $A$ and $\tilde{p}_i = \gamma_i(\tilde{p})$, then

$$|\tilde{p}_i\tilde{p}_j| \geqslant \max\{|\tilde{p}\tilde{p}_i|, |\tilde{p}\tilde{p}_j|\}.$$

As was observed by Gromov, if $A$ is an Alexandrov space with curvature $\geqslant \kappa$ and diameter $\leqslant d$, the last inequality implies that $\angle \tilde{p}_i\tilde{p}\tilde{p}_j > \delta = \delta(\kappa, d) > 0$. This yields an upper bound on the number of elements of a short basis in terms of $\kappa, d$ and the dimension of $A$.

3.2. In the proof, we will use the following simple observation. Let $G$ be a group acting on an abelian group $V$ via a representation $\rho\colon G \to \mathrm{Aut}(V)$. It is obvious from the definition that the action of $G$ on $V$ is nilpotent iff the actions of $G$ on $Tor(V)$ and $V/Tor(V)$ are nilpotent. It easily follows from Engel's theorem that if $V$ is finitely generated and if for any $g \in G$ all eigenvalues of the induced action of $g$ on $V \otimes \mathbb{R}$ are equal to 1, then $G$ contains a finite index subgroup whose action on $V$ is nilpotent.

Let $M$ be an almost nonnegatively curved manifold. Let us denote by $M_n = (M, g_n)$, $n \in \mathbb{N}$, a sequence of Riemannian metrics on $M$ such that $K_{M_n} \geqslant -1/n$ and $\mathrm{diam}(M_n) \leqslant 1/n$. Let us denote by $\tilde{M}$ the universal covering of $M$, and by $\tilde{M}_n \to M_n$ the universal Riemannian covering of $M_n$ (i.e., $\tilde{M}_n$ is $\tilde{M}$ equipped with the pullback of the Riemannian metric $g_n$).

**Key Lemma 3.1.** *Given $\varepsilon > 0$ and $r_2 > r_1 > 0$, let $\tilde{M}_n \supset B_{r_2}(p_n) \supset B_{r_1}(p_n)$. Then, for $n$ sufficiently large, there is a $(1+\varepsilon)$-Lipschitz map $\Phi_n : B_{r_2}(p_n) \to B_{r_1}(p_n)$ which is homotopic to the identity on $B_{r_2}(p_n)$.*

*Proof.* Fix $R \gg r_2$ (here $R > 1000(1+1/\varepsilon)r_2$ will suffice). Notice that as $n \to \infty$, we have that $B_R(p_n) \to B_R \subset \mathbb{R}^q$. Choose a finite $R/1000$-net $\{a_i\}$ of $\partial B_R \subset \mathbb{R}^q$. Let $a_{i,n} \in M_n$ be sequences such that $a_{i,n} \to a_n$. Consider the sequence of functions $f_n : M_n \to \mathbb{R}$ with $f_n = \min_i \operatorname{dist}^2_{a_{i,n}}$.

For large $n$, the functions $f_n$ are 2-concave in $B_R(p_n)$, so that, in particular, the gradient pushs $\Phi^T_{f_n}|_{B_{r_2}(p_n)}$ are $e^{2T}$-Lipschitz. Moreover, if $\xi_x$ denotes the starting vector of a unit speed shortest geodesic from $x$ to $p_n$, then for any $x \in B_{r_2}(p_n)\backslash B_{r_1}(p_n)$ we have $\langle \xi_x, \nabla f\rangle \geqslant R/2$. Therefore, if $T = 2r_2/R$, then $\Phi^T_{f_n}(B_{r_2}(p_n)) \subset B_{r_1}(p_n)$. Thus $\Phi_n = \Phi^{2r_2/R}_{f_n}$ provides a $4r_2/R$-Lipschitz map $B_{r_2}(p_n) \to B_{r_1}(p_n)$, and it is $(1+\varepsilon)$-Lipschitz if one chooses $R$ sufficiently large. □

**Corollary 3.2.** *Let $M$ be almost nonnegatively curved manifold. Let $h : \pi_1(M) \to Aut(H^*(\tilde M, Z)/tor)$ be the natural action of $\pi_1(M)$ on $(H^*(\tilde M)$. Then there is a sequence of norms $\|*\|_n$ on $H^*(\tilde M, \mathbb{Z})/tor$ such that the following holds. Given any $\varepsilon > 0$, there is $n \in \mathbb{Z}_+$ such that for any $\gamma \in \pi_1(M)$ with $|\gamma|_n \leqslant 2diam(M_n)$ we have $\|h(\gamma)\|_n \leqslant 1+\varepsilon$.*

*Proof.* [FY92, theorem 0.1] and Yamaguchi's fibration theorem [Yam91] imply that if $n$ is sufficiently large, for any fixed $r \in \mathbb{R}_+$ we have that for any $p_n \in \tilde M_n$ the inclusion map $B_r(p_n) \to \tilde M_n$ is a homotopy equivalence.

Let $\|*\|_{n,r}$ denote the $L_\infty$-norm on differential forms on $B_r(p_n) \subset \tilde M_n$.

Fix $r_2 > r_1 > 0$. If $\omega$ is a differential form on $B_{r_1}(p_n) \subset M_n$ and $n$ is sufficiently large, Lemma 3.1 implies that

$$\|\Phi_n^*(\omega)\|_{n,r_2} \leqslant (1+\varepsilon)\|\omega\|_{n,r_1} \quad \text{and} \quad 2\operatorname{diam}(M_n) \leqslant r_2 - r_1.$$

If now $\omega$ is a form on $B_{r_2}(p_n) \in \tilde M_n$ and $\gamma \in \pi_1(M)$ such that

$$|\gamma|_n = |p_n\, \gamma(p_n)| \leqslant 2\operatorname{diam}(M_n) \leqslant r_2 - r_1,$$

then $B_{r_1}(p_n) \subset B_{r_2}(\gamma(p_n)) \subset \tilde M_n$, whence

$$\|\Phi_n^*(\gamma^*(\omega))\|_{n,r_2} \leqslant (1+\varepsilon)\|\gamma^*(\omega)\|_{n,r_1} \leqslant (1+\varepsilon)\|\omega\|_{n,r_2}.$$

Thus, for the induced norms on the de Rham cohomology of $\tilde M$ (and on its integral subspace $H^*(\tilde M, \mathbb{Z})/tor)$) we have

$$\|[\gamma^*(\omega)]\|_{n,r_2} \leqslant (1+\varepsilon)\|[\omega]\|_{n,r_2}.$$

Therefore the sequence of norms $\|*\|_n = \|*\|_{n,r_2}$ satisfies the conditions of the Corollary. □

**Lemma 3.3.** *There exists a constant $N = N(n,k) \in \mathbb{Z}_+$ such that the following holds. If $G$ is a subgroup of $GL(n,\mathbb{Z})$ and $S$ is a set of generators of $G$ with $\#(S) \leqslant k$ such that the eigenvalues of each element of $S^N$ are all equal to 1 in absolute value, then the same is true for the eigenvalues of all elements of $G$.*

*Proof.* Let $B$ be the set of all matrices in $GL(n,\mathbb{Z})$ for which all of their eigenvalues are equal to 1 in absolute value. Since the characteristic polynomials of such matrices are uniformly bounded and have integer coefficients, there are only finitely many of them. Let $\bar B$ be the Zariski closure of $B$ in the set of all real $n\times n$ matrices. By the above, all elements of $\bar B$ satisfy that the absolute values of all of their eigenvalues are equal to 1.

Consider now the space $V=\mathbb{R}^{kn2}$ of $k$-tuples of real $n\times n$ matrices.

Consider a collection of matrices $(M_1,M_2,...,M_k)\in V$, where $M_i\in GL(n,\mathbb{R})$. Let $F_k$ be a free group on $k$ generators, generated by $S=\{\gamma_1,\gamma_2,...,\gamma_k\}$, and let $h: F_k\to GL(n,\mathbb{R})$ be the homomorphism defined by $h(\gamma_i)=M_i$. The property that for any $\gamma\in F_k$ $h(\gamma)$ be an element of $\bar{B}$ then describes an algebraic subset $A_\gamma\subset V$.

The intersection $A=\cap_{\gamma\in F_k}A_\gamma$ is also algebraic, and therefore there is a finite number $N=N(n,k)$ such that for $S^N\subset F_k$, $A=\cap_{\gamma\in S^N}A_\gamma$. □

**Lemma 3.4.** *Let $\Gamma$ be a subgroup of $GL(n,\mathbb{Z})$ such that the eigenvalues of each element of $G$ are equal to $1$ in absolute value. Then $\Gamma$ contains a subgroup $\Gamma'$ of finite index all of whose elements have eigenvalues equal to $1$.*

*Proof.* Let $G$ denote the Zariski closure of $\Gamma$ in $GL(n,\mathbb{R})$ . Then $G$, being an algebraic group, is a Lie group with finitely many components. Let $G_\circ$ be the identity component of $G$. By the same argument as in the proof of the previous lemma, the set of all characteristic polynomials of the elements of $G$ is finite. Therefore the characteristic polynomial of any element of $G_\circ$ is identically equal to $(x-1)^n$.

Therefore, the subgroup $\Gamma'=\Gamma\cap G_\circ$ satisfies all conditions of the Lemma. □

*Proof of Theorem A.* Let $M$ be an almost nonnegatively curved manifold. Denote, as usual, by $M_n=(M,g_n)$, $n\in\mathbb{N}$, a sequence of Riemannian metrics on $M$ such that $K_{M_n}\geqslant -1/n$ and $\operatorname{diam}(M_n)\leqslant 1/n$, by $\tilde{M}$ the universal covering of $M$, and by $\tilde{M}_n\to M_n$ the universal Riemannian covering of $M_n$.

After passing to a finite cover of $M$, by [FY92] we may assume that $\pi_1(M)$ is nilpotent.

Fix $p\in M$ and let $\{\gamma_{i,n}\}$ be a short basis of $\pi_1(M_n,p)$. Then, if $n$ is sufficiently large, the short basis $\{\gamma_{i,n}\}$ has at most $k=k(\dim M)$ elements and its elements satisfy $|\gamma_{i,n}|_n\leqslant 2/n$ for every $i$. Moreover, Corollary 3.2 implies that given $\varepsilon>0$, for all large $n$ and every $i$ we have $\|h(\gamma_{i,n})\|_n<1+\varepsilon$ and $\|h(\gamma_{i,n}^{-1})\|_n<1+\varepsilon$.

Take $N=N(k,m)$ as in Lemma 3.3. One can choose $\varepsilon>0$ so small that if $p$ is a polynomial with integer coefficients for which all of its roots have absolute values lying between $1/(1+\varepsilon)^N$ and $(1+\varepsilon)^N$, then all roots of $p$ have absolute values equal to $1$. This follows from the fact that the total number of integer polynomials all of whose roots are contained in a fixed bounded region is finite.

Set $S_n:=\{\gamma_{i,n}\}$. Then for any $\gamma\in S_n^N$ we have $\|h(\gamma)\|_n<(1+\varepsilon)^N$ and $\|h(\gamma^{-1})\|_n<(1+\varepsilon)^N$. Therefore the absolute values of all eigenvalues lie between $1/(1+\varepsilon)^N$ and $(1+\varepsilon)^N$. Since the characteristic polynomial of $h(\gamma)$ has integer coefficients, the absolute values of all the eigenvalues of $h(\gamma)$ are in fact equal to $1$.

Apply now Lemma 3.3. It follows that for any $\gamma\in\pi_1(M)$ the absolute values of all eigenvalues of $h(\gamma)$ are equal to $1$.

Then Lemma 3.4 implies that after passing to a finite cover $M'$ of $M$, for any $\gamma\in\pi_1(M')$ all eigenvalues of $h(\gamma)$ are equal to $1$. By Engel's theorem, one can choose an integral basis of $H^*(\tilde{M},\mathbb{R})$ such that the action of $\pi_1(M)$ on $H^*(\tilde{M},\mathbb{Z})/tor$ is given by upper triangular matrices.

Therefore, by passing to a finite cover $M''$ of $M'$, we can assume that the action of $\pi_1(M'')$ on $H^*(\tilde{M},\mathbb{Z})$ (and on $H_*(\tilde{M},\mathbb{Z})$) is nilpotent.

Recall (see e.g. [HMR75, 2.19]) that a connected CW complex with nilpotent fundamental group is nilpotent if and only if the action of its fundamental group on the homology of its universal cover is nilpotent. Thus $M''$ is a nilpotent space, whence the proof of Theorem $A$ is complete. □

## 4. C-nilpotency of the fundamental group

In this section we will prove Theorem B. It will follow from the following somewhat stronger result.

**Theorem 4.1.** *For any integer $m$ there exist constants $\epsilon(m) > 0$ and $C(m) > 0$ such that the following holds. If $M^m$ is a closed smooth $m$-manifold which admits a Riemannian metric with $\sec(M^m) > -\epsilon(m)$ and $\operatorname{diam}(M^m) < 1$, then the fundamental group of $M^m$ is $C(m)$-nilpotent, i.e., $\pi_1(M^m)$ contains a nilpotent subgroup of index $\leqslant C(m)$.*

**Remark 4.2.** The proofs of Theorems A and C show that corresponding versions of those results also do hold when these theorems are reformulated in a fashion similar to Theorem 4.1.

By an argument by contradiction, Theorem 4.1 follows from the following statement:

Given a sequence of Riemannian $m$-manifolds $(M_n, g_n)$ with diameters $\operatorname{diam}(M_n, g_n) \leqslant 1/n$ and sectional curvatures $K_{g_n} \geqslant -1/n$, one can find $C \in \mathbb{R}$ such that $\pi_1(M_n)$ is $C$-nilpotent for all sufficiently large $n$.

To prove this statement, we will make use of the following two algebraic lemmas.

Recall that the group of outer automorphisms $\operatorname{Out}(G)$ of a group $G$ is defined as the quotient of its automorphism group $\operatorname{Aut}(G)$ by the subgroup of inner automorphisms $\operatorname{Inn}(G)$.

**Lemma 4.3** (*A characterization of $C$-nilpotent groups*)**.** *Let*

$$\{1\} = G_l \subseteq \ldots \subseteq G_1 \subseteq G_0 = G$$

*be a sequence of groups satisfying the following properties:*

*For any $i$*

*(i) $G_i \trianglelefteq G$ is normal in $G$;*

*(ii) the image of the conjugation homomorphism $h_i : G \to \operatorname{Out}(G_i/G_{i+1})$ is finite of order at most $C_i$;*

*(iii) $\operatorname{Inn}(G_i/G_{i+1})$ has order $\leqslant c_i$.*

*Then $G$ contains a nilpotent subgroup $N$ of index at most $C = \prod_i c_i C_i$, where $N$ is of nilpotency class $\leqslant l$.*

**Proof.** First of all, notice that property (i) assures that the objects described in parts (ii) and (iii) of the lemma are well-defined.

Properties (ii) and (iii) imply that the image of the conjugation homomorphism $f_i : G \to \operatorname{Aut}(G_i/G_{i+1})$ is finite, and that this image has order at most $C_i c_i$.

Let $N = \cap_i \ker f_i$. Then $N$ satisfies the conclusion of the Lemma. Indeed, it is clear that $[G : N] \leqslant C = \prod_i C_i c_i$. Furthermore , let $N_i = N \cap G_i$. Then we obviously have that $N_i \trianglelefteq N$ for any $i$. By construction, we also have that $N_i/N_{i+1}$ is contained in the center of $N/N_{i+1}$, which means that $N$ is nilpotent of nilpotency class $\leqslant l$.

**Trivial Lemma 4.4** (*A characterization of finite actions*)**.** *If $S$ is a finite set of generators of a group $G$ with $S^{-1} = S$, and $h : G \to H$ is a homomorphism with $|h(S^n)| < n$ for some $n > 0$, then $h(S^n) = h(G)$ and, in particular, $|h(G)| < n$.*

Let now $\Gamma$ be a group which acts discretely by isometries on an Alexandrov space $A$ with curvature $\geqslant -1$. Choose a marked point $p \in A$. Assume that $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$ is a finite short basis of $\Gamma$ at $p$, and that $\theta \leqslant |\gamma_i| \leqslant 1$, where $|\gamma| \stackrel{\text{def}}{=} |p\gamma(p)|$. Let $\#(R)$ denote the number of delements $\gamma \in \Gamma$ with $|\gamma| \leqslant R$. The Bishop-Gromov inequality implies that

$$\#(R) \leqslant v^m_{-1}(R)/v^m_{-1}(\theta),$$

where $m = \dim A$ and $v^m_{-1}(r)$ is the volume of the ball of radius $r$ in the $m$-dimensional simply connected space form of curvature $-1$. Therefore, if $\#\#(L)$ denotes the number of homomorphisms $h : \Gamma \to \Gamma$ with norm $\leqslant L$ (i.e., the number of homomorphisms for which it holds that for any $\gamma \in \Gamma$ one has that $|h(\gamma)| \leqslant L|\gamma|$), then

$$\#\#(L) \leqslant \#(L)^n \leqslant \Big[\frac{v^m_{-1}(L)}{v^m_{-1}(\theta)}\Big]^n. \tag{4.1}$$

### 4.1. The construction.

For $n \to \infty$, the manifolds $M_n$ clearly converge to a point $=: A_0$.

Set $M_{n,1} := M_n$ and $\vartheta_{n,1} := \mathrm{diam} M_{n,1}$.

Rescale now $M_{n,1}$ by $\frac{1}{\vartheta_{n,1}}$ so that $\mathrm{diam}(M_{n,1}/\vartheta_{n,1}) = 1$. Passing to a subsequence if necessary, one has that the manifolds $\frac{1}{\vartheta_{n,1}} M_{n,1}$ converge to $A_1$, where $A_1$ is a compact nonnegatively curved Alexandrov space with diameter $1$.

Now choose a regular point $\bar p_1 \in A_1$, and consider distance coordinates around $\bar p_1 \in U_1 \to \mathbb{R}^{k_1}$, where $k_1$ is the dimension of $A_1$. The distance functions can be lifted to $U_{n,1} \subset \frac{1}{\vartheta_{n,1}} M_{n,1}$.

Let $M_{n,2}$ be the level set of $U_{n,1} \to \mathbb{R}^{k_1}$ that corresponds to $\bar p_1$. Clearly, $M_{n,2}$ is a submanifold of codimension $k_1$. Set $\vartheta_{n,2} := \mathrm{diam} M_{n,2}$.

Passing again to a subsequence if necessary, one has that the sequence $\frac{1}{\vartheta_{n,2}} M_{n,2}$ converges to some Alexandrov space $A_2$. As before, $A_2$ is a compact nonnegatively curved Alexandrov space with diameter 1. Set $k_2 := k_1 + \dim A_2$. If one now chooses a marked point in $M_{n,2}$, then, as $n \to \infty$, $M_n/\vartheta_{n,2}$ converges to $A_2 \times \mathbb{R}^{k_1}$, which is of some dimension $k_2 > k_1$.

We repeat this procedure until, at some step, $k_l = m$.

As a result one obtains a sequence $\{A_i\}$ of compact nonnegatively curved Alexandrov spaces with diameter $1$ that satisfies

$$\mathrm{dim} A_i = k_i - k_{i-1}, \quad \text{so that} \quad \sum_{i=1}^{l} \mathrm{dim} A_i = m.$$

We also obtain a sequence of rescaling factors $\vartheta_{n,i} = \mathrm{diam} M_{n,i}$, and a nested sequence of submanifolds

$$\{p_n\} = M_{n,l} \subset \dots \subset M_{n,2} \subset M_{n,1} = M_n,$$

which in turn induces a sequence of homomorphisms

$$\{1\} = \pi_1(M_{n,l}) \xrightarrow{\imath} \cdots \xrightarrow{\imath} \pi_1(M_{n,2}) \xrightarrow{\imath} \pi_1(M_{n,1}) = \pi_1(M_n).$$

Let $G_i := G_i(n) := \imath^i \pi_1(M_{n,i})$.

For $n$ sufficiently large, the subgroups $G_i(n)$ are clearly those which are generated by elements of norm $\leqslant 3\vartheta_{n,i}$. Equivalently, if one takes a short basis $\{\gamma_i\}$ of $G(n)$, then $G_i$ is the subgroup generated by all elements $\gamma_i$ such that $|\gamma_i| \leqslant 3\vartheta_{n,i}$.

4.2. **Limit fundamental groups of Alexandrov spaces.**

We will now define the "limit" or "L-fundamental groups" of the Alexandrov spaces $A_i$ constructed above. This notion is similar to the notion of the fundamental group of an orbifold. However, we note in advance that the construction of the L-fundamental group does not only depend on the spaces $A_i$, but also on the chosen rescaled subsequence of $M_n$. In fact, the following construction shows that the limit fundamental group of $A_i$, $\pi_1^L(A_i)$, is isomorphic to $\pi_1(M_{n,i}, M_{n,i+1})$ for all sufficiently large $n$. But, unlike $\pi_1(M_{n,i})$, the groups $\pi_1^L(A_i)$ will not depend on $n$.

*The limit fundamental groups of* $A_i$. Consider the converging sequence $(M_n/\vartheta_{n,i}, p_n) \to (A_i \times \mathbb{R}^{k_{i-1}}, \bar{p}_i \times 0)$ (here the interesting case is collapsing). Recall that $\bar{p}_i \in A_i$ is a regular point. Fix $\varepsilon > 0$ such that $\mathrm{dist}_{\bar{p}_i}$ on $A_i$ does not have critical values in $(0, 2\varepsilon)$. Take a sequence $R_n$ which converges very slowly to infinity (here we will need $R_n\vartheta_{n,i}/\vartheta_{n,i-1} \to 0$ and $R_n \to \infty$).

Consider then a sequence of Riemannian coverings $\Pi\colon (\tilde{B}_n, \tilde{p}_n) \to (B_{R_n}(p_n), p_n)$ of $B_{R_n}(p_n) \subset M_n/\vartheta_{n,i}$ with $\pi_1(\tilde{B}_n, \tilde{p}_n) = \pi_1(B_\varepsilon(p_n), p_n)$, where $B_\varepsilon(p_n) \subset M_n/\vartheta_{n,i}$.

After passing to a subsequence if necessary, the sequence $(\tilde{B}_n, \tilde{p}_n)$ converges to a nonnegatively curved Alexandrov space $\tilde{A}_i \times \mathbb{R}^{k_{i-1}}$, where the space $\tilde{A}_i$ has the same dimension as $A_i$. Indeed, by construction it contains an isometric copy of $B_\varepsilon(p_{n,i})$, and therefore

$$\dim \tilde{A}_i + k_{i-1} = \dim \lim_{i\to\infty} B_\varepsilon(p_{n,i}) = \dim A_i + k_{i-1}.$$

Let us show that for all sufficiently large $n$,

$$\imath(\pi_1(M_{n,i+1})) \trianglelefteq \pi_1(M_{n,i}).$$

Assume that $\Pi(\tilde{q}_n) = \tilde{p}_n$ and that $\tilde{q}_n \to \bar{q}_n \in \tilde{A}_i$. Connect $\bar{p}_n$ and $\bar{q}_n$ by a geodesic which, by [Pet98], only passes through regular points. Note that in a small neighborhood of this geodesic in $M_n$ we have two copies of $M_{n,i+1}$, near $\tilde{p}_n$ and $\tilde{q}_n$. Therefore, applying Yamaguchi's Fibration Theorem in a small neighborhood of this geodesic, we can construct a diffeomorphism from $M_{n,i+1}$ to itself. This implies that for any loop $\gamma$ which after lifting connects $\tilde{p}_n\tilde{q}_n$, we have $\gamma^{-1}\imath\pi_1(M_{n,i+1})\gamma \subset \imath\pi_1(M_{n,i+1})$, i.e., $\imath\pi_1(M_{n,i+1}) \triangleleft \pi_1(M_{n,i})$ (for an alternative argumen see also [FY92]).

This easily yields that $A_i = \tilde{A}_i/\Gamma_i$, where $\Gamma_i$ is a group of isometries which acts discretely on $\tilde{A}_i$. The group $\Gamma_i$ is denoted by $\pi_1^L(A_i)$ (the *limit* or *L-fundamental group* of $A_i$). This group is clearly isomorphic to

$$\pi_1(M_{n,i}, M_{n,i+1}) = \pi_1(M_{n,i})/\imath(\pi_1(M_{n,i+1}))$$

for all sufficiently large $n$, and the space $\tilde{A}_i$ will be called the *universal covering* of $A_i$. Since $\tilde{A}_i$ is nonnegatively curved and $A_i = \tilde{A}_i/\pi_1^L(A_i)$ is compact, by Toponogov's splitting theorem $\tilde{A}_i$ isometrically splits as $\tilde{A}_i = K_i \times \mathbb{R}^{s_i}$, where $K_i$ is a compact Alexandrov space with curv $\geqslant 0$. Since $\pi_1^L(A_i)$ is a group of isometries that acts discretely on $\tilde{A}_i$, it follows that $\pi_1^L(A_i)$ is a virtually abelian group.

Consider now the corresponding series

$$\{1\} = G_l(n) \subset \ldots \subset G_1(n) \subset G_0(n) = \pi_1(M_n).$$

The theorem then follows from the following

**Lemma 4.5.** *For all sufficiently large $n$, the series*

$$\{1\} = G_l(n) \subset \ldots \subset G_1(n) \subset G_0(n)$$

*constructed above satisfies the assumptions of Lemma 4.3 for numbers $C_i$ and $c_i$ which do not depend on $n$.*

We first prove the following

**Sublemma 4.6.** *Each subgroup $G_i$ is normal in $G$.*

*Proof.* We will show by reverse induction on $k$ that $G_i \trianglelefteq G_k$ for any $k \leqslant i$. Let us assume that we already know that $G_i \trianglelefteq G_{k+1}$. Since

$$\imath\pi_1(M_{n,k+1}) \trianglelefteq \pi_1(M_{n,k}),$$

we know that $G_{k+1} \trianglelefteq G_k$. Consider the covering $\Pi_{k+1}\colon (\tilde{M}_{n,k+1}, \tilde{p}_{n,k+1}) \to (M_n, p_n)$ with covering group $\Gamma_{k+1}$.

Clearly $(\tilde{M}_{n,k+1}, \tilde{p}_{n,k+1}) \to \mathbb{R}^{s_i}$ for some integer $s_i$. Applying Lemma 2.16, it follows that for any $a \in G$ with $|a| < 1$ there is a cocos-curve $\gamma$ in $\tilde{M}_{n,k+1}$ with total time $T$ connecting $\tilde{p}_n$ and $a(\tilde{p}_n)$ in $\tilde{M}_{n,k+1}$. Then clearly $\gamma \sim ga$ for some $g \in G_{k+1}$. Let us denote by $\Phi^T : \tilde{M}_{n,i} \to \tilde{M}_{n,i}$ the gradient push corresponding to $\gamma$.

Let $\gamma_j$ be a loop from the short basis of $G_i$. As was mentioned in 4.1, if $n$ is large, then $\operatorname{length}\gamma_j \leqslant 3\vartheta_{n,i}$. Let us denote by $\tilde{\gamma}_j$ a lift of $\gamma_j$ to $\tilde{M}_{n,i}$. Let $\tilde{p}_{n,j} \in \tilde{M}_{n,i}$ be its starting point. Since $[\gamma_j] \in G_i$, we have that $\tilde{\gamma}_j$ is a loop in $\tilde{M}_{n,i}$. Consider then the loop $\gamma'_j = \Pi \circ \Phi^T \circ \tilde{\gamma}_j$. Clearly,

$$[\gamma_j] = a^{-1}g^{-1}[\gamma'_j]ga, \quad \text{or} \quad [\gamma'_j] = ga[\gamma_j]a^{-1}g^{-1}.$$

Now Proposition 2.9 implies that

$$\operatorname{length}(\gamma'_j) \leqslant \exp(2T)\operatorname{length}(\gamma_j).$$

Thus, for sufficiently large $n$,

$$ga[\gamma_j]a^{-1}g^{-1} \in G_i,$$

and since $g \in G_i \trianglelefteq G_{k+1}$ it follows that

$$a[\gamma_j]a^{-1} \in G_i,$$

i.e., $G_i \trianglelefteq G_k$.

$\square$

*Proof of Lemma 4.5.* The group

$$\pi_1^L(A_i) = \pi_1(M_{n,i}, M_{n,i+1}) = \pi_1(M_{n,i})/\imath(\pi_1(M_{n,i}))$$

is virtually abelian. Let $d_i$ be the minimal index of an abelian subgroup of $\pi_1^L(A_i)$. The epimorphism $\imath^i : \pi_1(M_{n,i}) \to G_i$ induces an epimorphism $\pi_1^L(A_i) \to G_i(n)/G_{i+1}(n)$. Therefore, $G_i(n)/G_{i+1}(n)$ is $d_i$-abelian for all large $n$. Thus, setting $c_i = d_i!$, we have $|\operatorname{Inn}(G_i(n)/G_{i+1}(n))| \leqslant c_i$.

Consider the covering $\Pi_i : \tilde{M}_{n,i} \to M_n$ with covering group $G_i$, and let $\tilde{p}_{n,i}$ be a preimage of $p_n$. Clearly $(\tilde{M}_{n,i}, \tilde{p}_{n,i}) \to \mathbb{R}^{s_i}$ for some integer $s_i$. Applying Lemma 2.16, it follows that for any $a \in G$ with $|a| < 1$ there is a cocos-curve $\gamma$ in $\tilde{M}_{n,i}$ which connects $p$ and $a(p)$. Then clearly $\gamma \sim ga$ for some $g \in G_i$. Let us denote by $\Phi^T : \tilde{M}_{n,i} \to \tilde{M}_{n,i}$ the gradient push corresponding to $\gamma$.

Let $b \in G_i$ and $\beta$ be a loop representing $b$. Let us denote by $\tilde{\beta}$ a lift of $\beta$ to $\tilde{M}_{n,i}$. Let $\tilde{p}_{n,i} \in \tilde{M}_{n,i}$ be its starting point. Since $[\beta] \in G_i$, we have that $\tilde{\beta}$ is a loop in $\tilde{M}_{n,i}$.

Consider now the loop $\beta' = \Pi \circ \Phi^T \circ \tilde{\beta}$. Clearly,

$$b = [\beta] = a^{-1}g^{-1}[\beta']ga, \quad \text{or} \quad [\beta'] = gaba^{-1}g^{-1}.$$

Proposition 2.9 then implies that

$$\operatorname{length}(\beta') \leqslant \exp(2T)\operatorname{length}(\beta)$$

Therefore, if $h_a : G_i/G_{i+1} \to G_i/G_{i+1}$ is induced by the conjugation $b \to aba^{-1}$, then for any $a \in G$ there is $g \in G_i$ such that $|h_{ga}| \leqslant \exp(2T)$.

Let now $\delta_i$ be the minimal norm of the elements of $\pi_1^L(A_i)$, where $\pi_1^L(A_i)$ acts on $\tilde{A}_i$. Then (4.1) implies that the image of the action of $G$ by conjugation in $Out(G_i/G_{i+1})$ is $C_i$-finite, where $C_i$ depends only on $c_i$, $T$, and $\delta_i$. □

*End of the proof of Theorem 4.1.* Apply now Lemma 4.3 to obtain that $G$ is $C$-nilpotent for $C = \prod_i C_i c_i$, whence the proof of Theorem 4.1 is complete. □

**Remark 4.7.** Theorem 4.1 can be reformulated as follows: There exists a constant $\epsilon(m) > 0$ such that if $N^m$ is a Riemannian manifold which admits a discrete free isometric action by a group $\Gamma$ such that $\sec(N) > -\epsilon(m)$ and $\operatorname{diam}(N/\Gamma) < 1$, then $\Gamma$ is $C(m)$-nilpotent.

As was pointed out to us by B. Wilking, in the above reformulation of Theorem 4.1 one can actually easily remove the assumption that the $\Gamma$ action be free.

**Corollary 4.8.** *There exists a constant $\epsilon(m) > 0$ such that if $N^m$ is a Riemannian manifold which admits a discrete isometric action by a group $\Gamma$ such that $\sec(N) > -\epsilon(m)$ and $\operatorname{diam}(N/\Gamma) < 1$, then $\Gamma$ is $C(m)$-nilpotent.*

*Proof.* Let $\epsilon = \epsilon(m)$ be as provided by Theorem 4.1 and suppose $N$ satisfies the assumptions of the corollary for this $\epsilon$. Let $F$ be the frame bundle of $N$. Then the action of $\Gamma$ on $N$ lifts to a free isometric action on $F$. As was observed in [FY92], using Cheeger's rescaling trick $F$ can be equipped with a $\Gamma$ invariant metric satisfying $\sec(F) > -\epsilon(m)$ and $\operatorname{diam}(F/\Gamma) < 1$. Since the induced action of $\Gamma$ on $F$ is free, the claim of the corollary now follows from Theorem 4.1. □

## 5. Proof of the Fibration Theorem

Let $M$ be an almost nonnegatively curved manifold. Let us denote by $M_n = (M, g_n)$ a sequence of Riemannian metrics on $M$ such that $K_{M_n} \geqslant -1/n$ and $\operatorname{diam}(M_n) \leqslant 1/n$.

Let us denote by $\tilde{M}$ the universal cover of $M$ and by $\tilde{M}_n \to M_n$ the universal Riemannian covering of $M_n$ (i.e., $\tilde{M}$ equipped with the pull back of the metric $g_n$ on $M$).

By [FY92], passing to a finite cover we may assume that $\Gamma = \pi_1(M)$ is a nilpotent group without torsion. Hence, to prove the topological part of Theorem $C$, it is enough to show the following:

**Theorem 5.1.** *Let $M$ be a closed almost nonnegatively curved $m$-manifold such that $\Gamma = \pi_1(M)$ is a nilpotent group without torsion. Then $M$ is the total space of a fiber bundle*

$$F \to M \to N$$

*where the base $N$ is a nilmanifold and the fiber $F$ is simply connected.*

The assumption on $\Gamma$ implies that we can fix a series $\Gamma = \Gamma_0 \rhd \Gamma_1 \rhd \Gamma_2 \rhd \ldots \rhd \Gamma_l = \{1\}$ such that $\Gamma_i$ is normal in $\Gamma$ and $\Gamma_i/\Gamma_{i+1} \cong \mathbb{Z}$.

Let us first us give an informal proof.

5.1. **An informal proof of Theorem 5.1.** We use induction to construct the bundles $F_i \to M \xrightarrow{f_i} N_i$, where each $N_i$ is a nilmanifold with $\pi_1(N_i) = \Gamma/\Gamma_i$ and $\pi_1(F_i) \cong \Gamma_i$. Since the base of induction is trivial, we are only interested in the induction step.

Fix $p \in N_i$, and let $F_i(p)$ be the fiber over $p$. For any sufficiently large $n$ choose a subgroup $G_i = G_i(n)$ such that $\Gamma_i \lhd G_i \lhd \Gamma_{i+1}$ and $[\Gamma_i : G_i]$ is finite, but large enough so that the cover $\bar{F}_i(p)$ of $F_i(p)$ corresponding to $G_i$ is Hausdorff close to a unit circle $S^1$.

Construct now a bundle map $\varphi_p\colon \bar{F}_i(p) \to S^1$ by lifting distance functions from $S^1$ (This can be done by a slight generalization of a construction in [FY92] and [BGP92]). Let $\omega_p = d\varphi_p$.

Then $\omega_p$ is a closed integral non-degenerate one-form on $F_i(p)$. Since deck transformations are isometries, after averaging by $\mathbb{Z}_a$, where $a = [\Gamma_i : G_i]$, we can assume that $\omega_p$ is $\mathbb{Z}_a$-invariant. Thus $\omega_p$ descends to a form on $F_i(p)$ which when integrated gives a bundle map $F_i(p)$ onto a small $S^1$.

Note that although this bundle is defined only up to rotations of $S^1$, its fibers are well-defined.

Since $\Gamma_{i+1}$ is normal in $\Gamma$, the choice of the covering $\bar{F}_i(p)$ of $F_i(p)$ is unambiguous for all $p \in N_i$. By using a partition of unity on $N_i$ we can glue the forms $\omega_p$ into a global 1-form on $M$ which satisfies the following properties:

a) $\omega|_{F(p)}$ is closed and integral for any $p$;
b) $\omega|_{F(p)}$ is non-degenerate.

Integrating $\omega$ over the various $F(p)$'s we construct a continuous family of bundles $F_p \to S^1$. The level sets partition each $F(p)$ and hence the whole $M$ into fibers of a fiber bundle, whose quotient space is then a circle bundle $N_{i+1}$ over $N_i$.

5.2. This gives a good idea of the proof. However, to make it precise some extra work has to be done. In particular, one has to be careful with the construction of $\omega$. To make this construction possible we have to keep track of how $F(p)$ was obtained. Namely, we have to use that the fiber $F(p)$ was obtained by a construction as in Yamaguchi's fibration theorem (see [Yam91] or [BGP92]). This makes the induction proof quite technical.

We now proceed with the rigorous proof of Theorem 5.1.

5.3. **Proof of Theorem 5.1.** Let us denote by $\tilde{M}_{n,i}$ the Riemannian covering of $M_n$ with respect to $\Gamma_i$.

For any choice of marked points $p_n$ we have that $(\tilde{M}_{n,i}, p_n, \pi_1(M)) \to (\mathbb{R}^i, 0, \mathbb{R}^i)$ in equivariant Gromov-Hausdorff convergence, where $\mathbb{R}^i$ acts on itself by translations. Indeed, the limit space must be a nonnegatively curved simply connected Alexandrov space, and since $\operatorname{diam} M_n \to 0$ we have that it possesses a transitive group action by a nilpotent group. Then Euclidean space, acting as a group of translations, is here the only choice, and it is easy to see that the dimension of the limit must be equal to $i$.

Therefore $(\tilde{M}_n, p_n, \pi_1(M)) \to (\mathbb{R}^l, 0, \mathbb{R}^l)$, and we may also assume that for each $i$ we have that $(\tilde{M}_n, p_n, \Gamma_i) \to (\mathbb{R}^l, 0, \mathbb{R}^{l-i})$, where $\mathbb{R}^{l-i}$ is the coordinate subspace of $\mathbb{R}^l$ which corresponds to the first $l-i$ elements of the standard basis.

Now, let us give a technical definition:

If $R$ is a Riemannian manifold, let us denote by $\widetilde{\operatorname{dist}}_p$ the average of a distance function over a small ball around $p$. This enables us to work with the $C1$ function $\widetilde{\operatorname{dist}}_p$ instead of the Lipschitz function $\operatorname{dist}_p$.

**Definition 5.2.** *Let $R_n \to R$ be a sequence of Riemannian $m$-manifolds with curvature $\geqslant \kappa$ which Gromov-Hausdorff converges to a Riemannian $m'$-manifold $R$, where $m' \leqslant m$. A sequence of forms $\omega_n$ on $R_n$ is said to $\varepsilon$-approximate a form $\omega$ on $R$, if*

*(i) for any point $p \in R$ there is a neighborhood $U \ni p$ which admits a distance chart $f : U \to \mathbb{R}^{m'}$,*

$$f(x) = (\operatorname{dist}_{a_1}(x), \operatorname{dist}_{a_2}(x), ..., \operatorname{dist}_{a_{m'}}(x))$$

*which is a smooth regular map, and*

*(ii) smooth lifts of $f$ to $U_n \subset R_n$ give, for $n$ large enough, regular maps*

$$f_n(x) = (\widetilde{\operatorname{dist}}_{a_{1,n}}(x), \widetilde{\operatorname{dist}}_{a_{2,n}}(x), ..., \widetilde{\operatorname{dist}}_{a_{m',n}}(x))$$

*with $a_{i,n} \in M_n$, $a_{i,n} \to a_n$ such that*

$$|(f_n \circ f^{-1})^*(\omega) - \omega_n|_{C0} < \varepsilon$$

*for all sufficiently large $n$.*

Theorem 5.1 now easily follows from the following lemma:

**Lemma 5.3.** *Given $\varepsilon > 0$ there is a sequence of one-forms $\{\omega_{1,n}, \omega_{2,n}, \cdots, \omega_{k,n}\}$ on $\tilde{M}_n$ with the following properties:*

*(i) For each $i$, $\omega_{i,n}$ is a $\pi_1(M)$-invariant form on $\tilde{M}_n$.*

*(ii) The forms $\omega_{i,n}$ $\varepsilon$-approximate the coordinate forms $dx_i$ on $\mathbb{R}^k$. In particular, the forms $\{\omega_{i,n}\}$ are nowhere zero and almost orthonormal at each point.*

(iii) *If for any* $j < i$ *it holds that* $\omega_{j,n}(X) = \omega_{j,n}(Y) = 0$*, then* $d\omega_{i,n}(X,Y) = 0$*. In particular, for each* $i$ *and all sufficiently large* $n$*, the distribution corresponding to the system of equations*

$$\omega_{j,n}(X) = 0 \quad \textit{for all} \quad j \leqslant i$$

*defines on* $\tilde{M}_n$ *a foliation* $\mathcal{F}_{i,n}$*.*

(iv) *If* $\tilde{F}_{i,n}(x) \subset \tilde{M}_n$ *denotes the fiber of the foliation* $\mathcal{F}_{i,n}$ *through the point* $x \in \tilde{M}_n$*, then each* $\tilde{F}_{i,n}(x)$ *is* $\Gamma_i$*-invariant, i.e., for any* $\gamma \in \Gamma_i$ *one has that* $\tilde{F}_{i,n}(x) = \tilde{F}_{i,n}(\gamma x)$*. Moreover, the action of* $\Gamma_i$ *on* $\tilde{F}_{i,n}(x)$ *is cocompact for each* $i$*. In particular,* $\mathcal{F}_{i,n}$ *induces on* $M_n$ *the structure of a fiber bundle.*

*Proof.* We will construct these forms by induction. Assume that we have already constructed one-forms $\omega_1, \omega_2, \dots, \omega_{i-1}$ which meet all the required properties. They give a $\pi_1(M)$-invariant fibration of $\tilde{M}_n$ by submanifolds $\tilde{F}_{i-1,n}(x)$ through each point $x \in \tilde{M}_n$, with tangent spaces defined by the equations $\omega_j(X) = 0$ for $j = 1, \dots, i-1$.

Denote by $\theta : \mathbb{R} \to [0,1]$ a smooth monotone function which is equal to $1$ before $0$ and $0$ after $1$. Choose numbers $\delta_n > 0$ slowly converging to $0$, and let $\Theta_{i,n} : \tilde{M}_n \to \mathbb{R}_+$ be the function defined by

$$\Theta_{i,n}(x) = \min_{y \in F_{i-1,n}(x)} \theta(|p_n y|/\delta_n).$$

Clearly $\Theta_{i,n}$ is a continuous $\Gamma_{i-1}$-invariant function which is constant on each $F_{i-1,n}(x)$. Moreover, for large $n$, $\Theta_{i,n}$ has support in some $C_i\delta_n$-neighborhood of $F_{i-1,n}(p_n)$, and is equal to $1$ in some $c_i\delta_n$-neighborhood of $F_{i-1,n}(p_n)$.

Now let $\varphi : \mathbb{R} \to [0,1]$ be a smooth nondecreasing function which is $0$ before $1/2$ and $1$ after $3/2$. Consider the form

$$\omega'_{i,n} = \Theta_{i,n} \cdot d(\varphi \circ \widetilde{\mathrm{dist}}_{\Gamma_i a_{i,n}}),$$

where $a_{i,n} \in \tilde{M}_n$ is a sequence of points converging to $-e_i \in \mathbb{R}^l$, and $\widetilde{\mathrm{dist}}_{\Gamma_i a_{i,n}}$ is the average of $\mathrm{dist}_{\Gamma_i x}$ for $x$ in a small ball around $a_{i,n}$. The support of $\omega'_i$ has two components, one which contains $p_n$ (notice here that $p_n \to 0 \in \mathbb{R}^l$), and another which does not. (It follows from the construction that the limit of $F_{i-1,n}(p_n)$ is a coordinate plane in $\mathbb{R}^l$).

Set $\omega''_{i,n} := \omega'_{i,n}$ on the component of $p_n$, and let this form be $0$ otherwise. Clearly, $\omega''_{i,n}$ is then a continuous $\Gamma_i$-invariant form whose restriction to $\tilde{F}_{i-1,n}(x)$ is exact. Moreover, each level set of its integral over $\tilde{F}_{i-1,n}(x)$ is $\Gamma_i$-invariant.

By construction, the form $\omega''/|\omega''|$ is now (in the sense of definition 5.2) close to $dx_i$ at the points where $|\omega''| \neq 0$. Take

$$\omega_{i,n} = c \sum_{\gamma \in \Gamma/\Gamma_i} \gamma\omega',$$

where the coefficient $c$ is chosen in such a way that $|\omega_{i,n}(p_n)| = 1$. As $\delta_n$ is a sequence slowly converging to zero, we may assume that $\mathrm{diam}(M_n)/\delta_n \to 0$. Therefore, $\omega_{i,n}$ is the form we need. □

Notice that the proof of Theorem 5.1 actually also shows that the fibers in Theorem 5.1 are almost nonnegatively curved manifolds in the generalized sense with $k = l$. Therefore, the proof of Theorem $C$ is complete. □

## 6. Open questions

We would like to conclude this work by posing a number of related open questions.

**Question 6.1.** *Is it true that manifolds which are almost nonnegatively curved in the generalized sense are almost nonnegatively curved?*

In view of Theorems A and B it is reasonable to ask the following questions:

**Question 6.2.** *Is it true that almost nonnegatively curved $m$-manifolds $M^m$ are $C(m)$-nilpotent spaces?*

It is clear from the proof of Theorems A and B that this is true if the universal cover of $M^m$ has torsion free integral cohomology.

In view of Theorem B it is natural to ask the following question:

**Question 6.3.** *Can one give an explicit bound on $C(m)$ in Theorem B ?*

Theorem A also gives rise to the following question.

**Question 6.4.** *Do almost nonnegatively curved manifolds admit Riemannian metrics with zero topological entropy ?*

We take interest in this question since it has been shown in [PP04] that for the pointed loop space $\Omega M$ of a closed nilpotent manifold $M$ which admits a Riemannian metric with zero topological entropy, $\pi_*(\Omega M) \otimes \mathbb{Q}$ is finite dimensional. Notice that for simply connected manifolds this last condition is equivalent to saying that $M$ is a rationally elliptic space. Moreover, to include the case of infinite fundamental groups Totaro has proposed a general definition of an elliptic space as follows: A topological space $X$ is *elliptic* if it is homotopy equivalent to a finite CW complex, it has a finite covering which is a nilpotent space and the loop space homology of the universal covering of $X$ grows polynomially with any field of coefficients. If the above question has a positive answer, Theorem A and the above result from [PP04] will show that almost nonnegatively curved manifolds are rationally elliptic in this broader sense.

As was pointed out in the discussion in the Introduction before Theorem C, it already follows from Yamaguchi's fibration theorem and [FY92] that a finite cover of an almost nonnegatively curved manifold admits a Serre fibration onto a nilmanifold with simply connected fibers. While this is formally weaker than the statement of Theorem C, it would be interesting to have an answer to the following, purely topological, question:

**Question 6.5.** *Let $F \to M \xrightarrow{f} N$ be a Serre fibration of closed manifolds where $N$ is a nilmanifold and $F$ is simply connected. Is it true that after passing to a finite cover, the map $f$ becomes homotopic to a fiber bundle projection ?*

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