A note on rational homotopy of biquotients

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Abstract

We construct a natural pure Sullivan model of an arbitrary biquotient which generalizes the Cartan model of a homogeneous space. We also obtain a formula for the rational Poincare polinomial of equal rank biquotients.

1 Introduction

Let G be a compact Lie group and $H \leq G \times G$ be a closed subgroup. Then H acts on G on the left by the formula $(h_1, h_2)g = h_1 g h_2^{-1}$. The orbit space of this action is called a *biquotient* of G by H and denoted by $G//H$. If the action of H on G is free, then $G//H$ is a manifold. This is the only case we consider in this paper. In the special case when H has the form $K_1 \times K_2$ where $K_1 \subset G \times 1 \subset G \times G$ and $K_1 \subset 1 \times G \subset G \times G$ we will sometimes write $K_1 \backslash G / K_2$ instead of $G//(K_1 \times K_2)$.

Biquotients are of interest in Riemannian geometry because they provide one of the main sources of examples of manifolds of nonnegative (and in particular positive) sectional curvature.

From the point of view of rational homotopy theory biquotients provide a large natural class of examples of rationally elliptic spaces which is strictly bigger than the usually considered class of homogeneous spaces. Recall that a simply-connected CW complex is called rationally elliptic if $\dim H^*(X, \mathbb{Q}) < \infty$ and $\dim(\pi_*(X) \otimes \mathbb{Q}) < \infty$. (From here on all cohomology groups are taken with rational coefficients).

The purpose of this note is to point out some well-know facts about rational homotopy of homogeneous spaces which easily generalize to biquotients but have so far remained unnoticed.

First we observe that biquotients admit pure Sullivan models similar to Cartan models of homogeneous spaces. (Recall that a free DGA $(\Lambda V, d)$ is called *pure* if V is finite-dimensional and $d|_{V^{ev}} = 0$ and $d(V^{odd}) \subset V^{ev}$.)

Before we give an explicit formula for computing the Sullivan model of an arbitrary biquotient $G//H$ we need to introduce some notations. It is well-known that G is rationally homotopy equivalent to $S^{2m_1-1} \times \cdots \times S^{2m_k-1}$ where $k = \text{rank}\, G$ and $H^*(B_G, \mathbb{Q}) \cong \mathbb{Q}[x_1, ..., x_k]$. Consequently,

$$H^*(B_G \times B_G) \cong H^*(B_G) \otimes H^*(B_G) \cong \mathbb{Q}[x_1, ..., x_k, y_1, ..., y_k]$$

with $|x_i| = |y_i| = 2m_i$.

Let $f\colon B_H \to B_G \times B_G$ be the map of the classifying spaces induced by the inclusion $H \hookrightarrow G \times G$.

Proposition 1. *In the above notations, a Sullivan model of $G//H$ can be given by the following pure DGA:*

$$(H^*(B_H) \otimes \Lambda(q_1, ...q_k), d)$$

where , $|q_i| = 2m_i - 1$ and d is given by $d|_{H^*(B_H)} = 0$ and $d(q_i) = f^*(x_i - y_i)$.

It is well-known that for a homogeneous space G/H of equal rank groups its Poincare polynomial and its Euler characteristic can be computed by the formulas

$$P_{G//H}(t) = \frac{P_{B_H}(t)}{P_{B_G}(t)} \qquad \chi(G//H) = \frac{|W(G)|}{|W(H)|}$$

It was shown by Singhoff [Sin93], that the formula for the Euler characteristic remains true for equal rank biquotients.

We observe that, in fact, the Poincare polynomial formula (which easily implies the Euler characteristic formula) remains true as well.

Proposition 2. *Let $G//H$ be a biquotient of compact connected Lie groups where rank $G =$ rank H. Then the rational Poincare polynomial of $G//H$ can be computed by the formula*

$$P_{G//H}(t) = \frac{P_{B_H}(t)}{P_{B_G}(t)}$$

2 Proofs of Propositions 2 and 1

We use the textbook [FHT01] as a comprehensive reference (also see [TO97, Chapter1] for a more gentle introduction to rational homotopy theory). We are going to prove the following somewhat stronger version of Propositions 2:

Proposition 3. *Let G, H be connected compact Lie groups such that rank $G =$ rank H and let ρ be a free smooth action of H on G. Then the rational Poincare polynomial of $G/\rho(H)$ can be computed by the formula*

$$P_{G/\rho(H)}(t) = \frac{P_{B_H}(t)}{P_{B_G}(t)}$$

Proof. Let $p\colon G/\rho(H) \to B_H$ be the classifying map of the principle H bundle $H \to G \to G/\rho(H)$. A standard argument (cf. [Esc92a]) shows that $G \to G/\rho(H) \to B_H$ is a Serre fibration (which need not be principal!). Recall that $G \simeq_{\mathbb{Q}} S^{2m_1-1} \times \cdots \times S^{2m_k-1}$. Similarly, $H \simeq_{\mathbb{Q}} S^{2n_1-1} \times \cdots \times S^{2n_l-1}$. Let $(\Lambda(q_1, ..., q_k), d \equiv 0)$ where $|q_i| = 2m_i - 1$ be the minimal model of G and $(Q[x_1, ..., x_l], d \equiv 0)$ where $|x_j| = 2n_j$ be the minimal model of B_H. Note that $k = l$ since rank $G =$ rank H. Then according to [FHT01, Proposition 15.5], $G/\rho(H)$ admits a Sullivan model of the form

$$(\Lambda(q_1, ..., q_k) \otimes Q[x_1, ..., x_l], \bar{d}) \tag{2.1}$$

for some appropriately defined $\bar{d}$. Since this model is elliptic and $k = l$, by [FHT01, Corollary to the Proposition 32.10], the Poincare polynomial and the Euler characteristic of $G/\rho(H)$ can be computed as follows

$$P_{G//H}(t) = \frac{\Pi_{i=1}^{k}(1-t^{2m_j})}{\Pi_{i=1}^{k}(1-t^{2n_j})} = \frac{\frac{1}{\Pi_{i=1}^{k}(1-t^{2n_j})}}{\frac{1}{\Pi_{i=1}^{k}(1-t^{2m_j})}} = \frac{P_{B_H}(t)}{P_{B_G}(t)} \tag{2.2}$$

$$\chi(G//H) = \frac{\Pi_{i=1}^{k} m_i}{\Pi_{i=1}^{k} n_i} \tag{2.3}$$

It is well-known (see for example [Oni63, Proposition11.4]) that $\Pi_{i=1}^{k} m_i = |W(G)|$ and $\Pi_{i=1}^{k} n_i = |W(H)|$ and therefore (2.3) implies

$$\chi(G//H) = \frac{|W(G)|}{|W(H)|} \tag{2.4}$$

□

Example 4. Consider the Eschenburg manifold $M^6 = S^1\backslash U(3)/T^2$. Here the left S^1 is given by diag $(\lambda_1, \lambda_1, \bar{\lambda}_1)$ and the right T^2 is given by diag $(\lambda_1, \lambda_2, 1)$ where $|\lambda_i| = 1$. (According to [Esc92b] M^6 admits a metric of positive sectional curvature). $U(3)$ is rationally equivalent to $S^1 \times S^3 \times S^5$. Therefore by Proposition 2,

$$P_{M^6}(t) = \frac{P_{BT^3}(t)}{P_{BU(3)}(t)} = \frac{(1-t^2)(1-t^4)(1-t^6)}{(1-t^2)^3} = (1+t^2)(1+t^2+t^4) = 1 + 2t^2 + 2t^4 + t^6$$

Observe that one of the consequences of Proposition 2 is that the Poincare polynomial of $G//H$ depends only on G and H but not on the inclusion $H \hookrightarrow G$ and thus M^6 has the same rational betti numbers as the homogeneous complex flag $U(3)/T^3$.

The proof of Proposition 3 does not require the explicit knowledge of the differential $\bar{d}$ in the model of $G/\rho(H)$ given by (2.1). Proposition 1 which will be proved next provides an explicit formula for $\bar{d}$ in case of a biquotient and shows that the model in this case is pure.

Proof of Proposition 1. As was observed by Eschenburg [Esc92a], any biquotient $G//H$ is diffeomorphic to a biquotient of $G \times G$ by $G \times H$ written as $\Delta G\backslash G \times G/H$, where ΔG stands for the diagonal embedding of G into $G \times G$. Let $p\colon G//H \to B_H$ be the classifying map of the principle H bundle $H \to G \to G//H$. Recall that $G \to G//H \to B_H$ is a Serre fibration. Moreover, this fibration fits into the following fibered square (see [Esc92a] and [Sin93])

$$\begin{array}{ccc} G//H & \longrightarrow & B_G \\ \downarrow & & \downarrow \\ B_H & \longrightarrow & B_{G\times G} \end{array} \tag{2.5}$$

where both vertical arrows are fibrations with fiber G and both horizontal arrows are fibrations with fiber $(G \times G)/H$. In particular, the fibration $G//H \to B_H$ is the pullback of the fibration $G \to B_G \xrightarrow{\phi} B_{G\times G}$ which following Eschenburg, we will refer to as the *reference fibration.*

We begin by constructing the canonical model of the reference fibration. Since this fibration is induced by the diagonal map $\Delta\colon G \to G\times G$, it follows that ϕ is the diagonal embedding $\Delta_{B_G}\colon B_G \to B_G \times B_G$. Consider the map $\phi^*\colon H^*(B_G \times B_G) \to H^*(B_G)$. Recall that $G \simeq_{\mathbb{Q}} S^{2m_1-1} \times \cdots \times S^{2m_k-1}$ and the minimal model of B_G is isomorphic to $H^*(B_G, \mathbb{Q}) \cong \mathbb{Q}[x_1, ..., x_k]$ with zero differentials and with $|x_i| = 2m_i$. Similarly, the minimal model of $B_G \times B_G$ is isomorphic to its cohomology ring $B = \mathbb{Q}[x_1, ..., x_k, y_1, ..., y_k]$ with $|x_i| = |y_i| = 2m_i$. Thus ϕ^* can be viewed as a DGA-homomorphism of minimal models of B_G and $B_{G\times G}$.

Let us construct a Sullivan model of ϕ^*. Since $\phi = \Delta_{B_G}$ we compute $\phi^*(x_i) = \phi^*(y_i) = x_i$ for all $i = 1, \ldots, k$. Consider the relative Sullivan algebra $(B \otimes \Lambda(q_1, ...q_k), d)$ where $dx_i = dy_i = 0$ and $dq_i = x_i - y_i$. Then it is immediate to check that this algebra is a Sullivan model (in fact, a minimal one!) of ϕ^* with the quasi-isomorphism $B\otimes\Lambda(q_1, ...q_k) \to H^*(B_G)$ given by $x_i \to x_i$, $y_i \to x_i$, $q_i \to 0$.

By the naturality of models of maps [FHT01, page 204, Proposition 15.8], from the fibered square (2.5), we obtain that a Sullivan model of the map $G//H \to B_H$ can be given by the pushout of $(B \otimes \Lambda(q_1, ...q_k), d)$ via the homomorphism $f^*\colon B \to H^*(B_H)$; i.e. it can be written as

$$(H^*(B_H), 0) \otimes_{(B,d)} (B \otimes \Lambda(q_1, ...q_k), d) = (H^*(B_H) \otimes \Lambda(q_1, ...q_k), \bar{d})$$

where $\bar{d}$ is given by $\bar{d}|_{H^*(B_H)} = 0$ and $\bar{d}(q_i) = f^*(x_i - y_i)$. In particular, $M(G//H) = (H^*(B_H) \otimes \Lambda(q_1, ...q_k), \bar{d})$ is a model for $G//H$. Notice that $H^*(B_H)$ is a free polynomial algebra on a finite number of even-dimensional generators, and thus the model $M(G//H)$ is pure. $\square$

Remark 5. It is easy to see that the minimal model of a pure Sullivan model is again pure. Therefore, Proposition 1 implies that minimal models of biquotients are pure.

The pure model $M(G//H)$ constructed in the proof of Proposition 1 provides an effective way of computing rational cohomology of biquotients. We refer to $M(G//H)$ as the *Cartan model* of $G//H$. This method of computing $H^*(G//H)$ is essentially equivalent to the method developed by Eschenburg [Esc92b] (when applied to $\mathbb{Q}$ coefficients) who computed the Serre spectral sequence of the fibration $G \to G//H \to B_H$. In fact, it is easy to recover this spectral sequence from the Cartan model by looking at the standard filtration of $M(G//H)$ by the wordlength in the number of odd degree generators. Also note that in case when $G//H$ is an ordinary homogeneous space (i.e. when $H \subset G \times G$ has the form $H \times 1 \subset G \times G$) this model is easily seen to reduce to the standard Cartan model of G/H.

Proposition 1 also implies that all general facts about pure spaces (such as the description of formal pure spaces) are applicable to biquotients (also see [BK] for some other possible applications).

Let $H^*_i(G//H)$ be the lower filtration of $H^*(G//H)$ coming from the above mentioned filtration of $M(G//H)$. Note that by construction,
$H^*_0(G//H) \cong H^*(B_H)/I \cong H^*(B_H) \otimes_{H^*(B_G \times B_G)} H^*(B_G)$ where I is the ideal in $H^*(B_H)$

generated by $\{f^*(x_i - y_i), i = 1, ..., n\}$. By [GHV76, Chapter 2], $H^*_i(G//H) = 0$ for $i > k = -\chi_\pi(G//H) = \#$ of odd degree generators of $M(G//H) - \#$ of even degree generators of $M(G//H) = \text{rank }(G) - \text{rank }(H)$ (cf. [Sin93, Proposition 6.4].

When rank (G) = rank (H) we see that $H^*(G//H) = H^*_0(G//H) \cong H^*(B_H)/I$ (cf. [Sin93, Theorem 6.5]).

Example 6. Cartan model of $S^1\backslash Sp(3)/SU(3)$. Here the left S^1 is given by diag $((e^{2\pi it}, 1, 1)$. Let $T^3 = \text{diag }(e^{2\pi it_1}, e^{2\pi it_2}, e^{2\pi it_3})$ be the standard maximal torus of $Sp(2)$. Then $H^*(B(Sp(3)) \subseteq H^*(BT^2) = \mathbb{Q}[t_1, t_2, t_3]$ (with $|t_i| = 2$) consists of all symmetric polynomials in t_i^2, i.e, $H^*(B(Sp(2)) = \mathbb{Q}[x_1, x_2, x_3]$ where $x_1 = \Sigma t_i^2, x_2 = \Sigma_{i\neq j} t_i^2 t_j^2, x_3 = t_1^2 t_2^2 t_3^2$. We will view the Lie algebra $\mathfrak{t}^2$ of the maximal torus of $SU(3)$ as the hyperplane $t_1 + t_2 + t_3 = 0$ in $Lie(T^3)$. Then $H^*(BSU(3))$ is the free polynomial algebra $\mathbb{Q}[\sigma_2|_{\mathfrak{t}^2}, \sigma_3|_{\mathfrak{t}^2}] = \mathbb{Q}[\tilde{\sigma_2}, \tilde{\sigma_2}]$ where $\sigma_1, \sigma_2, \sigma_3$ are the canonical symmetric polynomials in t_1, t_2, t_3 and $\tilde{\sigma_i} = \sigma_i|_{\mathfrak{t}^2}$.

Observe that $f\colon BS^1 \times BSU(3) \to BSp(3) \times BSp(3)$ can be written as $f = f_1 \times f_2$ where $f_1\colon BS^1 \to BSp(3)$ and $f_2\colon BSU(3) \to BSp(3)$. We will denote by x_1, x_2, x_3 the generators (described above) of the cohomology of the first $BSp(3)$ factor and by y_1, y_2, y_3 the corresponding generators of the cohomology of the second $BSp(3)$ factor. It is easy to see that $f_1^*(x_1) = t^2, f_1^*(x_1) = 0, f_1^*(x_1) = 0$ (here $H^*(BS^1) = \mathbb{Q}[t]$ with $|t| = 2$). To compute f_2^* observe that $y_1 = \sigma_1^2 - 2\sigma_2, y_2 = \sigma_2^2 - 2\sigma_1\sigma_3, y_3 = \sigma_3^2$. Therefore

$$f_2^*(y_1) = y_1|_{\mathfrak{t}^2} = -2\tilde{\sigma_2}, f_2^*(y_2) = y_2|_{\mathfrak{t}^2} = \tilde{\sigma_2}^2, f_2^*(y_3) = y_3|_{\mathfrak{t}^2} = \tilde{\sigma_3}^2$$

By Proposition 1, the Cartan model of $S^1\backslash Sp(3)/SU(3)$ is given by $(\mathbb{Q}[t, \tilde{\sigma_2}, \tilde{\sigma_3}] \otimes \Lambda(q_3, q_7, q_{11}), d)$ where $|t| = 2, |\tilde{\sigma_i}| = 2i, |q_i| = i$ and $dt = d\tilde{\sigma_i} = 0, dq_3 = t^2 + 2\tilde{\sigma_2}, dq_7 = -\tilde{\sigma_2}^2, dq_{11} = -\tilde{\sigma_3}^2$. This model is not minimal. Application of Sullivan's algorithm reduces it to the following minimal model of $S^1\backslash Sp(3)/SU(3)$:

$$(\mathbb{Q}[t, \tilde{\sigma_3}] \otimes \Lambda(q_7, q_{11}), d), \text{ where } dt = d\tilde{\sigma_i} = 0, dq_7 = -t^4/4, dq_{11} = -\tilde{\sigma_3}^2$$

and the degrees of the generators are the same as before. Note that this model is easily identified as the minimal model of $\mathbb{CP}^3 \times S^6$ and thus $S^1\backslash Sp(3)/SU(3)$ is rationally equivalent to $\mathbb{CP}^3 \times S^6$. A similar computation shows that the minimal model of $Sp(3)/U(3)$ is given by

$$(\mathbb{Q}[t, \tilde{\sigma_3}] \otimes \Lambda(q_7, q_{11}), d), \text{ where } dt = d\tilde{\sigma_i} = 0, dq_7 = t^4 - 2t\tilde{\sigma_3}, dq_{11} = \tilde{\sigma_3}^2$$

It is easy to see that these two minimal models are not isomorphic and thus $Sp(3)/U(3)$ and $S^1\backslash Sp(3)/SU(3)$ have different rational homotopy types. Moreover, it also implies that $H^*(Sp(3)/U(3))$ and $H^*(S^1\backslash Sp(3)/SU(3))$ are not isomorphic as algebras since both spaces are formal (it is well-known that any elliptic space of positive Euler characteristic is formal [Hal77]) even though they are isomorphic as graded vector spaces by Proposition 2.

A similar computation (cf. [FT94] and [GHV76, Chapter 11] for details) shows that the Cartan model of $S^1\backslash Sp(n)/SU(n)$ is given by

$$(\mathbb{Q}[t, \sigma_2, ..\sigma_n] \otimes \Lambda(q_3, ..., q_{4n-1}), d) \text{ where } |t| = 2, |\sigma_i| = 2i, |q_i| = i \text{ and} \tag{2.6}$$
$$dt = d\sigma_i = 0, dq_3 = 2\sigma_2 - t^2, dq_{4i-1} = 2\sigma_{2i} + \sum_{r+s=2i} (-1)^s \sigma_r \sigma_s \text{ for } i > 1$$

3 Open questions

Proposition 1 implies that many results about rational homotopy of homogeneous spaces hold true for biquotients.

It is worth mentioning some interesting problems that have been solved for homogeneous spaces but remain open for biquotients.

Let $\mathcal{H}$ be the class of simply-connected spaces whose rational cohomology algebras are finite dimensional and have no nonzero derivations of negative degree.

According to Meier [Mei83], $\mathcal{H}$ contains any compact simply-connected manifold whose cohomology ring satisfies the hard Lefschetz duality (in particular any simply connected Kähler manifold).

Halperin conjectured that any elliptic space C of positive Euler characteristic belongs to $\mathcal{H}$. It is well-known (cf. [Mei82]) that a formal space belongs to $\mathcal{H}$ iff every orientable rational fibration with fiber X is totally non-cohomologous to zero or equivalently iff its Serre spectral sequence collapses on the E_2 term.

The Halperin conjecture, which is considered one of the central problems in rational homotopy theory, has been confirmed in several important cases [FHT01, page 516].

In particular, it was shown by Shiga and Tezuka [ST87] that it holds for homogeneous spaces of equal rank groups.

Problem 7. *Show that the Halperin conjecture holds for any biquotient $G//H$ where* $\mathrm{rank}(H) = \mathrm{rank}(G)$.

The verification of the Halperin conjecture appears to be nontrivial even for some of the most basic examples such as $S^1\backslash Sp(n)/SU(n)$ despite the relative simplicity of its Cartan model (2.6).

In view of the result of Meier mentioned above, one possible way to prove it in this case would be by showing that $H^*(S^1\backslash Sp(n)/SU(n))$ satisfies the hard Lefschetz duality. Note, that the homogeneous space $Sp(n)/Un)$ is Kähler and hence its cohomology does satisfy hard Lefschetz. Note also that $Sp(n)/Un)$ and $S^1\backslash Sp(n)/SU(n)$ have the same betti numbers by Proposition 2. However, as Example 6 indicates their cohomology algebras are distinct. The above discussion naturally leads us to the following question which appears to be interesting from other points of view as well.

Question 8. *When is a biquotient $G//H$ Kähler?*

More specifically, suppose that a homogeneous space G/H_1 is Kähler. Let $G//H_2$ be a biquotient such that $H_1 \cong H_2$. Is $G//H_2$ Kähler?

Remark 9. The answer to Question 8 is unknown even for biquotients that do satisfy the hard Lefschetz such as the Eschenburg biquotient $M^6 = S^1\backslash U(3)/T^2$.

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