An optimal lower curvature bound for convex hypersurfaces in Riemannian manifolds

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The purpose of this paper is to provide a reference for the following theorem:

Theorem 1. *Let M be a Riemannian manifold with sectional curvature $\geqslant \kappa$. Then any convex hypersurface $F \subset M$ equipped with the induced intrinsic metric is an Alexandrov's space with curvature $\geqslant \kappa$.*

Here is a slightly weaker statement:

Theorem 2. *[Buyalo] If M is a Riemannian manifold, then any convex hypersurface $F \subset M$ equipped with the induced intrinsic metric is locally an Alexandrov's space.*

In the proof of Theorem 2 in [Buyalo], the (local) lower curvature bound depends on (local) upper as well as lower curvature bounds of M. We show that the proof in [Buyalo] can be modified to give 1.

Definition 3. *A locally Lipschitz function f on an open subset of a Riemannian manifold is called* λ-concave *if for any unit-speed geodesic γ, the function*

$$f \circ \gamma(t) - \lambda t^2/2$$

is concave.

Lemma 4. *Let $f : \Omega \to \mathbb{R}$ be a λ-concave function on an open subset Ω of a Riemannian manifold. Then there is a sequence of nested open domains Ω_i, with $\Omega_i \subset \Omega_j$ for $i < j$ and $\cup_i \Omega_i = \Omega$, and a sequence of λ_i-concave functions $f_i : \Omega_i \to \mathbb{R}$ such that*

(i) on any compact subset $K \subset \Omega$, f_i converges uniformly to f;

(ii) $\lambda_i \to \lambda$ as $i \to \infty$.

This lemma is a slight generalization of [Greene–Wu, Theorem 2] and can be proved exactly the same way.

Proof of Theorem 1. Without loss of generality one can assume that

(a) $\kappa \geqslant -1$,

(b) F bounds a compact convex set C in M,

(c) there is a (-2)-concave function μ defined in a neighborhood of C and $|\mu(x)| < 1/10$ for any $x \in C$,

(d) there is unique minimal geodesic between any two points in C.

(If not, rescale and pass to the boundary of the convex piece cut by F from a small convex ball centered at $x \in F$, taking $\mu = -10 \operatorname{dist}_x^2$.)

Consider the function $f = \operatorname{dist}_F$. From the Rauch comparison, for any unit-speed geodesic γ in the interior of C, $(f \circ \gamma)''$ is bounded in the support sense by the corresponding value in the model case (when $M = \mathbb{H}^2$ and F is a geodesic). In particular,

$$(f \circ \gamma)'' \leqslant f \circ \gamma.$$

Therefore $f + \varepsilon\mu$ is $(-\varepsilon)$-concave in $\Omega_\varepsilon = C \cap f^{-1}((0, \varepsilon))$. Take $K_\varepsilon = f^{-1}([\frac{1}{3}\varepsilon, \frac{2}{3}\varepsilon]) \cap C$. Applying lemma 4, we can find a smooth $(-\frac{\varepsilon}{2})$-concave function f_ε which is arbitrarily close to $f + \varepsilon\mu$ on K_ε and which is defined on a neighborhood of K_ε. Take a regular value $\vartheta_\varepsilon \approx \frac{1}{2}\varepsilon$ of f_ε. (In fact one can take $\vartheta_\varepsilon = \frac{1}{2}\varepsilon$, but it requires a little work.) Since $|\mu|_C| < 1/10$, the level set $F_\varepsilon = f_\varepsilon^{-1}(\vartheta_\varepsilon)$ will lie entirely in K_ε. Therefore F_ε forms a smooth closed convex hypersurface. By the Gauss formula, the sectional curvature of the induced intrinsic metric of F_ε is $\geqslant \kappa$. F_ε bounds a compact convex set C_ε, where $F_\varepsilon \to F$, $C_\varepsilon \to C$ in Hausdorff sense as $\varepsilon \to 0$. By property (d), the restricted metrics from M to C, C_ε are intrinsic, and so C_ε is an Alexandrov space with F_ε as boundary, that converges in Gromov–Hausdorff sense to C. It follows from [Petrunin, Theorem 1.2] (compare [Buyalo, Theorem 1]) that F_ε equipped with its intrinsic metric converges in Gromov–Hausdorff sense to F equipped with its intrinsic metric. Therefore F is an Alexandrov space with curvature $\geqslant \kappa$. □

Remark 5. We are not aware of any proof of theorem 1 which is not based on the Gauss formula. (Although if M is Euclidean space, there is a beautiful purely synthetic proof in [Milka].) Finding such a proof would be interesting on its own, and also could lead to the generalization of theorem 1 to the case when M is an Alexandrov space.

References

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