Ramsey-theoretic analysis of the conditional structure of weakly-null sequences

Stevo Todorčević*

Abstract. Understanding the possible conditional structure in a given weakly-null sequence (x_i) in some normed space X lies in the heart of several classical problems of this area of mathematics. We will expose the set-theoretic and Ramsey-theoretic methods relevant to both the lack and the existence of this conditional structure. We will concentrate on more recent results and will point out problems for further study.

2010 Mathematics Subject Classification. 03E05,05D10,46B20.

Keywords. Ramsey spaces, Unconditional Sequences.

1. The unconditional basic sequence problem

Recall that a sequence (x_i) in some normed space X is unconditional if there is a constant $C \geq 1$ such that

$$\Big\| \sum_{i \in I} a_i x_i \Big\| \leq C \Big\| \sum_{j \in J} a_j x_j \Big\|$$

for any pair $I \subseteq J$ of (finite) subsets of the index-set of (x_i) and for every sequence $(a_j : j \in J)$ of scalars. The unconditional basic sequence problem asking whether an arbitrary infinite-dimensional[1] normed space contains an *infinite* unconditional basic[2] sequence has played a prominent role both before and after its eventual solution by Gowers and Maurey [18].

Theorem 1.1 ([18]). *There is a separable reflexive infinite-dimensional space X with no infinite unconditional basic sequence.*

In [1], Argyros, Lopez-Abad and Todorcevic were able to extend this to the level of non-separable spaces as well.

*The author is grateful to the Fields Institute for hospitality during the writing of this paper.

[1] Unless otherwise stated, from now on, all normed spaces are implicitly assumed to be infinite-dimensional although we shall keep stressing this from time to time.

[2] The 'basic' here refers to the notion of Schauder basic sequence defined at the beginning of the next Section.

Theorem 1.2 ([1]). *There is also a non-separable reflexive space X with no infinite unconditional basic sequence.*

The Ramsey-theoretic nature of the unconditional basic sequence problem was apparent quite early but the following result of Gowers [16] that immediately followed [18] required a new infinite-dimensional Ramsey theorem commonly known today as *Gowers Dichotomy* (see also [17]).

Theorem 1.3 ([16]). *An infinite-dimensional Banach space contains either an infinite unconditional basic sequence or a hereditarily indecomposable Banach space.*[3]

Concerning this result we note that the space of Theorem 1.1 is actually hereditarily indecomposable while the space of Theorem 1.2 being reflexive and non-separable must have many decompositions as sum of two closed infinite-dimensional subspaces. In this article, we shall discuss the following two general versions of the problem.

Problem 1.4. (1) When does an infinite-dimensional normed space contains an infinite unconditional basic sequence?

(2) When does an infinite normalized weakly null sequence in some normed space contains an infinite unconditional subsequence?

In view of the solution of the unconditional basic sequence problem these problems may appear a bit wage but here is one example that shows that even a partial result in this direction sheds some light to another classical problem in this area, the separable quotient problem.

Theorem 1.5 ([20], [24]). *If the dual X^* of some Banach space X contains an infinite unconditional basic sequence then X admits a quotient with an unconditional basis.*

2. Finite and partial unconditionality

Recall that a sequence $(x_i)_{i=0}^{\infty}$ in some normed space is a Schauder basic sequence if it is normalized and if there is a constant $C \geq 1$ such that $\| \sum_{i<m} a_i x_i \| \leq C \| \sum_{j<n} a_j x_j \|$ for all $m < n$ and all choices $(a_j : j < n)$ of scalars. The classical procedure of Mazur for selecting a Schauder basic sequence inside an arbitrary Banach space when applied to subsequences of a given weakly null sequence gives us the following result of Bessaga and Pelczynski [5].

[3]Recall that an infinite-dimensional Banach space X is *indecomposable* if it cannot be written as sum $Y \bigoplus Z$ of two closed infinite-dimensional subspaces Y and Z. We say that X is *hereditarily indecomposable* if all closed infinite-dimensional subspaces of X are indecomposable.

Theorem 2.1 ([5]). *For every* $\varepsilon > 0$, *every normalized weakly null sequence* (x_n) *contains an infinite* $(1+\varepsilon)$*-Schauder basic subsequence* (x_{n_i}).

A further application of Ramsey theorem will give us the following multidimensional version of a result of Odell [38].

Theorem 2.2 ([8], [38]). *Let* k *a positive integer and* $\varepsilon > 0$. *Suppose that for every* $i < k$ *we are given a normalized weakly null sequence* $(x_n^i)_{n=0}^\infty$ *in some Banach space* X. *Then, there exists an infinite set* M *of integers such that for every* $\{n_0 < \cdots < n_{k-1}\} \subseteq M$ *the* k*-sequence* $(x_{n_i}^i)_{i<k}$ *is* $(1+\varepsilon)$*-unconditional.*

As indicated before, in general, it's not possible to improve this result and get an *infinite* unconditional basic sequence starting from infinitely many weakly null sequences. The following early result of Maurey and Rosenthal [31] identifies the minimal counterexample.

Theorem 2.3 ([31]).

(1) *For every* $\varepsilon > 0$ *and every* $\alpha < \omega^\omega$, *every normalized weakly null sequence in* $\mathcal{C}(\alpha+1)$ *has a* $(2+\varepsilon)$*-unconditional subsequence.*

(2) *For every* $\varepsilon > 0$ *every normalized weakly null sequence in* $\mathcal{C}(\omega^\omega+1)$ *has a* $(4+\varepsilon)$*-unconditional subsequence.*

(3) *There is a normalized weakly null sequence in* $\mathcal{C}(\omega^{\omega^2}+1)$ *with no unconditional subsequence.*

There are several results in the literature that give sufficient conditions on a given weakly null sequence in order to contain an infinite unconditional subsequence. Of these we mention the following result that uses the Nash-Williams theory of fronts and barriers.

Theorem 2.4 ([3],[14], [45]). *Suppose that* (x_n) *is a normalized weakly-null sequence in* $\ell_\infty(\Gamma)$ *with the property that*

$$\inf\{|x_n(\gamma)| : n \in \mathbb{N},\, \gamma \in \Gamma\} > 0.$$

Then (x_n) *contains an infinite unconditional basic subsequence.*

There is indeed a very natural relation between weakly null sequences $(x_n)_{n=0}^\infty$ and compact and precompact families of finite subsets of $\mathbb{N}$ that are subject to the Nash-Williams theory. To see this, assume, without loss of generality that (x_n) is a weakly null sequence in some space of the form $\ell_\infty(\Gamma)$. For $\gamma \in \Gamma$, set

$$F_\gamma = \{n \in \mathbb{N} : x_n(\gamma) \neq 0\}.$$

Let $\mathcal{F} = \{F_\gamma : \gamma \in \Gamma\}$. Then $\mathcal{F}$ is a *precompact family* of finite subsets of $\mathbb{N}$, i.e., all the pointwise limits of this family are finite sets, or to put it combinatorially, every infinite subset M of $\mathbb{N}$ contains a finite initial segment s such that s is *not* a proper initial segment of any element of the family $\mathcal{F}$. Let $\mathcal{B}$ be the collection of all finite subsets s of $\mathbb{N}$ that have no proper end-extensions in $\mathcal{F}$ and are minimal with respect to this property, i.e., every proper initial segment of s has an end-extension in $\mathcal{F}$. First of all note that $\mathcal{B}$ is a *thin family*, i.e., forms an antichain relative to the ordering $\sqsubseteq$ of end-extension. However, note that $\mathcal{B}$ is a *front*, i.e., every infinite subset M of $\mathbb{N}$ has an initial segment in $\mathcal{B}$. These are the notions introduced originally by Nash-Williams [36], where he proved that thin families have the Ramsey property in the following sense.

Theorem 2.5 ([36]). *Suppose $\mathcal{H} = \mathcal{H}_0 \cup \cdots \cup \mathcal{H}_l$ is a finite partition of a thin family $\mathcal{H}$ of finite subsets of $\mathbb{N}$. Then there is an infinite set $M \subseteq \mathbb{N}$ and $i < l$ such that $\mathcal{H} \upharpoonright M \subseteq \mathcal{H}_i$.*[4]

Note that for a fixed positive integer k the family $[\mathbb{N}]^k$ of all k-element subsets of $\mathbb{N}$ is a thin family (and, in fact it is a front) and that in this case Nash-Williams' theorem reduces to Ramsey's theorem. However, Nash-Williams' theorem is in fact a far-reaching extension of Ramsey's theorem that initiated the study of Ramsey theory of infinite dimension, the Ramsey theory most relevant to the questions we discuss here. So, going back to our family $\mathcal{F}$ associated to the weakly null sequence (x_n) and the front $\mathcal{B}$ and applying Nash-Williams' theorem we find an infinite set such that

$$\mathcal{F}[M] = \overline{\mathcal{B} \upharpoonright M}.$$[5]

From this we conclude that any study of further subsequences of $(x_n)_{n\in M}$ must involve the front $\mathcal{B}$ on M. A closer examination reveals that one has to study mappings with domains $\mathcal{B} \upharpoonright M$. The following important result of Pudlak and Rödl [40] reveals the true complexity of any such study.

Theorem 2.6 ([40]). *For every front $\mathcal{B}$ on $\mathbb{N}$ and every mapping $f : \mathcal{B} \to \mathbb{N}$ there exist an infinite subset M of $\mathbb{N}$ and a mapping $\varphi : \mathcal{B} \upharpoonright M \to \overline{\mathcal{B} \upharpoonright M}$ such that:*

(1) *φ is an* internal mapping*, i.e., $\varphi(s) \subseteq s$ for all $\in \mathcal{B} \upharpoonright M$,*

(2) *$\varphi(s) \not\sqsubseteq \varphi(t)$ for all $s, t \in \mathcal{B} \upharpoonright M$ such that $\varphi(s) \neq \varphi(t)$ and*

(3) *for $s, t \in \mathcal{B} \upharpoonright M$, $f(s) = f(t)$ iff $\varphi(s) = \varphi(t)$.*

This shows that the complexity weakly null subsequence $(x_n)_{n\in M}$ is captured by the complexity of internal mappings on fronts like $\mathcal{B} \upharpoonright M$. It should also be

[4]Here, $\mathcal{H} \upharpoonright M = \{s \in \mathcal{H} : s \subseteq M\}$.

[5]Here, $\mathcal{F}[M] = \{F \cap M : F \in \mathcal{F}\}$ and $\overline{\mathcal{B} \upharpoonright M}$ is the topological closure of the restriction $\mathcal{B} \upharpoonright M$.

mentioned that the mapping φ satisfying the conclusion of Theorem 2.6 must be a *unique* such a mapping with domain $\mathcal{B} \restriction M$. More precisely, suppose that

$$\varphi_0 : \mathcal{B} \restriction M_0 \to \overline{\mathcal{B} \restriction M_0} \text{ and } \varphi_1 : \mathcal{B} \restriction M_1 \to \overline{\mathcal{B} \restriction M_1}$$

are two mappings satisfying the conclusion of Theorem 2.6. If the intersection $M_0 \cap M_1$ is infinite, then there is an infinite set $N \subseteq M_0 \cap M_1$ such that

$$\varphi_0 \restriction (\mathcal{B} \restriction N) = \varphi_1 \restriction (\mathcal{B} \restriction N).$$

Another useful consequence of Theorem 2.5 is that for every front $\mathcal{B}$ there is an infinite set M such that $\mathcal{B} \restriction M$ is, in fact, a *barrier* on M, i.e., that every infinite subset of M has an initial segment in $\mathcal{B}$ and that, moreover, $\mathcal{B} \restriction M$ is *Sperner*, i.e., that $s \not\subseteq t$ for all $s \neq t$ in $\mathcal{B} \restriction M$. Thus, without loss of generality we may work with barriers instead with fronts. One useful property of barriers $\mathcal{B}$ is that for every infinite set M the topological closure $\overline{\mathcal{B} \restriction M}$ is simply equal to the $\subseteq$-downwards closure of $\mathcal{B} \restriction M$. Note that the finite rank fronts $[\mathbb{N}]^k$ are also barriers, but there are barriers $\mathcal{B}$ whose topological closures have arbitrary countable Cantor-Bendixon ranks. One important example of a barrier of rank ω is the *Schreier barrier*

$$\mathcal{S} = \{s \subseteq \mathbb{N} : |s| = \min(s) + 1\}$$

that forms the initial stage of a well studied transfinite hierarchy $\mathcal{S}_\xi$ $(1 \leq \xi < \omega_1)$ of Schreier barriers of higher ranks. Part of their importance in this area is based on the fact that their topological closures are *spreading*, i.e., the property that if some s belongs to $\overline{\mathcal{S}_\xi}$ then so does every finite set t of the same cardinality as s such that for every $i < |t|$, the ith element of t is bigger or equal than the ith element of s. We refer the reader to [28] which attempts towards a systematic study of combinatorial and topological properties of barriers as well as systematic study of internal mappings on barriers that are relevant to problems about weakly null sequences.

We finish this section by mentioning the well-known result of Elton [10] about the unconditional structure found inside arbitrary weakly null sequences.

Theorem 2.7 ([10]). *For every $0 < \varepsilon \leq 1$ there is a constant $C(\varepsilon) \geq 1$ such that every normalized weakly null sequence (x_n) has an infinite subsequence (x_{n_i}) such that $\| \sum_{i \in I} a_i x_{n_i} \| \leq C(\varepsilon) \| \sum_{j \in J} a_j x_{n_j} \|$ for every pair $I \subseteq J$ of subsets of $\mathbb{N}$ and every choice $(a_j : j \in J)$ of scalars such that $\varepsilon \leq |a_j| \leq 1$ for all $j \in J$.*

The following problem is in the literature known as the *Elton unconditional constant problem* (see, for example, [6]).

Problem 2.8. Is $\sup_{0 < \varepsilon \leq 1} C(\varepsilon) < \infty$?

3. w^*-null sequences and the quotient problem

Recall that a sequence $(f_\gamma)_{\gamma\in\Gamma}$ of bounded linear functionals on some normed space X is *w^*-null* if for every $x \in X$ and $\varepsilon > 0$, the set $\{\gamma \in \Gamma : |f_\gamma(x)| \geq \varepsilon\}$ is finite. That nontrivial such sequences always exist is a theorem due to Josefson [25] and Nissenzweig [37].

Theorem 3.1 ([25], [37]). *For every infinite-dimensional normed space X there is a normalized w^*-null sequence $(f_n)_{n=0}^\infty$ in X^*.*

Having such normalized sequence $(f_n)_{n=0}^\infty$ in X^* one is tempted to apply the Bessaga-Pelczynski technique to try to select a Schauder basic subsequence. This is exactly what Johnson and Rosenthal [24] did when they realized that one should also be looking for such Schauder basic subsequence (f_{n_i}) that has some sequence (x_i) in X as the corresponding sequence of biorthogonal functionals on the closed norm span of (f_{n_i}). This is how they proved the following well-known result.

Theorem 3.2 ([24]). *Every separable infinite-dimensional space has an infinite-dimensional quotient with a Schauder basis.*

It is quite natural to ask if this result can be extended to arbitrary spaces and this is what became known as the *separable quotient problem.* If one tries using the Ramsey-theoretic or set-theoretic analysis of this problem one will observe that the arguments in [24] are enough for getting separable quotients for spaces of density $< \mathfrak{b}$. Recall that $\mathfrak{b}$ is the minimal cardinality of a subset of $\mathbb{N}^\mathbb{N}$ unbounded in the ordering of eventual dominance. Recall also the similar number $\mathfrak{p}$, the minimal cardinality of a family $\mathcal{F}$ of infinite subsets of $\mathbb{N}$ such that $\bigcap \mathcal{F}_0$ is infinite for all finite $\mathcal{F}_0 \subseteq \mathcal{F}$ but there is no infinite $M \subseteq \mathbb{N}$ such that $M \setminus N$ is finite for all $N \in \mathcal{F}$. Recall also that $\mathfrak{m}$ is the minimal cardinality of a family of nowhere dense subsets that cover some nonempty compact T_2-space K which has no isolated points and which satisfies the countable chain condition. It is easily seen that $\omega_1 \leq \mathfrak{m} \leq \mathfrak{p} \leq \mathfrak{b}$. Then we have the following extension of the result in [24].

Theorem 3.3 ([46]). *Suppose that a Banach space X has density $< \mathfrak{m}$ and that its dual X^* has an uncountable normalized w^*-null sequence. Then X has a quotient with a Schauder basis of length ω_1.*

Remark 3.4. Given an w^*-null sequence $\{f_\gamma : \gamma < \omega_1\} \subseteq X^*$, the proof finds an uncountable subsequence $\{f_\gamma : \gamma \in \Gamma\}$ that forms a Schauder basis of its norm-closed linear span $\overline{\text{span}}\{f_\gamma : \gamma \in \Gamma\}$ and a quotient map

$$T : X \to (\overline{\text{span}}\{f_\gamma : \gamma \in \Gamma\})^*$$

onto the dual of this space which itself has a Schauder basis $\{f^*_\gamma : \gamma \in \Gamma\}$ formed by the biorthogonal functionals of the Schauder basis $\{f_\gamma : \gamma \in \Gamma\}$. This feature of the proof is of independent interest and has already been used in appplications some of which will be mentioned below.

To satisfy the hypothesis of Theorem 3.3 one needs to invoke a set-theoretic dichotomy, PID. To introduce this dichotomy, we need to recall some standard definitions.

Definition 3.5. Recall than an *ideal* on an index set S is simply a family $\mathcal{I}$ of subsets of S closed under taking subsets and finite unions of its elements. We shall consider only ideals of *countable* subsets of S and assume that all our ideals include the ideal of all finite subsets of S.

Definition 3.6. We say that such an ideal $\mathcal{I}$ is a *P-ideal* if for every sequence (a_n) in $\mathcal{I}$ there is $b \in \mathcal{I}$ such that $a_n \setminus b$ is finite for all n.

Example. (a) The ideal $[S]^{<\aleph_0}$ of finite subsets of S is the trivial P-ideal.

(b) The ideal $[S]^{\leq\aleph_0}$ of all countable subsets of S is a P-ideal and, in fact, a maximal ideal in the category of ideals of countable sets.

(c) Ignoring the ideal of finite subsets of S there is another class of small P-ideals $\mathcal{I} = \mathcal{F}^\perp$, determined by a family $\mathcal{F}$ of cardinality $< \mathfrak{b}$ of subsets of S in the following manner: $\mathcal{F}^\perp = \{x \in [S]^{\leq\aleph_0} : (\forall Y \in \mathcal{F})|x \cap Y| < \aleph_0\}$.

The *P-ideal dichotomy* states that every ideal $\mathcal{I}$ on an arbitrary index-set S either contains the ideal of all countable subsets of some uncountable subset T of S or there is a countable decomposition (S_n) of S such that $I \subseteq \{S_n : n < \omega\}^\perp$. More precisely we have the following definition.

Definition 3.7. The *P-ideal dichotomy*, PID, is the statement that for every P-ideal $\mathcal{I}$ of countable subsets of some index set S, either

(1) there is uncountable $T \subseteq S$ such that $[T]^{\aleph_0} \subseteq \mathcal{I}$, or

(2) there is a decomposition $S = \bigcup_{n<\omega} S_n$ such that $\mathcal{I} \subseteq \{S_n : n < \omega\}^\perp$.

We refer the reader to [49] for more about this interesting dichotomy which, on one hand, follows from the strong Baire category principles such as $\mathfrak{mm} > \omega_1$ but, on the other hand, is consistent with the Continuum Hypothesis. But let us only mention one of its applications to problems of our interest here.

Theorem 3.8 ([46]). *Assume PID. Then every Banach space X of density $< \mathfrak{m}$ has a quotient with a Schauder basis which can be assumed to be of length ω_1 if X is not separable.*

Corollary 3.9 ([46]). *Assume PID and* $\mathfrak{m} > \omega_1$. *Then every non-separable Banach space has un uncountable biorthogonal system.*

Recall that an *Asplund space* is a Banach space X with the property that separable subspaces of X have separable duals. They were originally introduces as spaces X with the property that every convex continuous function defined on a convex open subset U of X is Fréchet differentiable on a dense G_δ-subset of U. Recall also that a Banach space X has the *Mazur intersection property* if every closed convex subset of X is the intersection of closed balls of X. Mazur [32] proved that every Banach space with a Fréchet differentiable norm has the Mazur intersection property, so it was quite natural to ask if Asplund spaces have this property as well. The following two fact connects Theorem 3.8 to this problem.

Theorem 3.10 ([23]). *Suppose that a Banach space* X *has a biorthogonal system* $\{(x_i, f_i) : i \in I\} \subseteq X \times X^*$ *such that* $X^* = \overline{\mathrm{span}}\{f_i : i \in I\}$. *then* X *admits an equivalent norm with the Mazur intersection property.*

Theorem 3.11 ([4]). *Suppose* X *is an Asplund space of density* $\aleph_1$ *with an uncountable biorthogonal system. Then there is a normalized sequence* $\{x_\xi : \xi < \omega_1\}$ *of elements of* X *such that the operator* $f \mapsto (f(x_\xi) : \xi < \omega_1)$ *maps* X^* *into a nonseparable subset of* $c_0(\omega_1)$.

Remark 3.12. Note that this result is giving us a particular instance of the hypothesis of Theorem 3.3. So applying (the proof of) Theorem 3.3 (see Remark 3.4), we get an uncountable subsequence $\{x_\gamma : \gamma \in \Gamma\}$ forming a Shauder basis of its norm-closed linear span $\overline{\mathrm{span}}\{x_\gamma : \gamma \in \Gamma\}$ and a quotient map

$$T : X^* \to (\overline{\mathrm{span}}\{x_\gamma : \gamma \in \Gamma\})^*$$

onto the dual of this space which itself is spanned by the basis $\{x^*_\gamma : \gamma \in \Gamma\}$ formed by the biorthogonal functionals of the basis $\{x_\gamma : \gamma \in \Gamma\}$. So if in addition X (and therefore X^*) has density $\aleph_1$, we satisfy the hypothesis of Theorem 3.10.

Combining this with Theorems 3.3 (Remarks 3.4 and 3.12), 3.10 and 3.8, we get the following.

Corollary 3.13 ([4]). *Assume PID. Then every Asplund space of density* $< \mathfrak{m}$ *admits an equivalent norm with the Mazur intersection property.*

Assumptions like $\mathfrak{m} > \omega_1$ are necessary in Corollary 3.9 in view of the following fact[6].

[6] Recall that PID is consistent with the equality $\mathfrak{b} = \omega_1$.

Theorem 3.14 ([43]). *If $\mathfrak{b} = \omega_1$ then there is a non separable Asplund space of the form $X = C(K)$ with no uncountable biorthogonal systems.* [7]

We finish this section with another application of Theorem 3.8.

Theorem 3.15 ([46]). *Assume PID and $\mathfrak{m} > \omega_1$. Then every non-separable Banach space contains a closed convex subset supported*[8] *by all of its points.*

Remark 3.16. In [41], Rolewicz showed that a separable Banach space does not contain such a convex subset. There are examples that show that some assumption is needed in Theorem 3.15 (see [27] and [29]). However, given the assumption PID, we feel that it would be of independent interest to determine the exact extra assumptions that are needed for each of the three problems from the geometry of Banach spaces. For example, at this stage it is unclear if PID itself is sufficient for solving the problem of Rolewicz about support sets. On the other hand, it seems plausible that, assuming PID, the set-theoretic assumption $\mathfrak{b} > \omega_1$ is equivalent to the the existence of an uncountable biorthogonal system in every nonseparable Asplund space and also equivalent to the statement that every Asplund space of density not bigger than $\aleph_1$ admits an equivalent norm with the Mazur intersection property.

4. Weakly null sequences on Polish spaces

When the weakly null sequence lives in $\ell_\infty(\Gamma)$ and Γ is a Polish space unconditionality results can be obtained using the Ramsey theory of trees based on the Halpern-Läuchli theorem [21] (see [48]). We spend this section to give some explanation of this.

Definition 4.1. Fix a rooted finitely branching tree U with no terminal nodes. A subtree T of U will be called a **strong subtree** if the levels of T are subsets of the levels of U and if for every $t \in T$ every immediate successor of t in U is extended by a unique immediate successor of t in T.

Theorem 4.2 ([21]). *For every sequence $U_0, ..., U_{d-1}$ of rooted finitelly branching trees with no terminal nodes and for every finite colouring of the* level product $U_0 \otimes \cdots \otimes U_{d-1}$, *we can find for each $i < d$ a strong subtree T_i of U_i such that the T_i's share the same level set and such that the level product $T_0 \otimes \cdots \otimes T_{d-1}$ is monochromatic.*

[7] In fact, $X = C(K)$ is hereditarily Lindelöf relative to its weak topology so it admits no equivalent norm with the Mazur intersection property.

[8] Recall that x in C *supports* C if there is $f \in X^*$ such that $f(x) = \min\{f(y) : y \in C\} < \sup\{f(y) : y \in C\}$.

This theorem serves as a pigeonhole principle behind the topological Ramsey space $\mathcal{S}_\infty(U)$ of strong subtrees of U (see, [48]). The following result of Milliken [34] is the analogue of the well-known result of Galvin and Prikry [13] about the space of all infinite subsets of $\mathbb{N}$, the space that was relevant in the previous section of this paper.

Theorem 4.3 ([34]). *For every finite Borel colouring of the space $\mathcal{S}_\infty(U)$ of all strong subtrees of U there is a strong subtree T of U such that the set $\mathcal{S}_\infty(T)$ of strong subtrees of T is monochromatic.*

In applications one usually colours some specific subsets F of U. This theorem is relevant because the "shape" of F uniquely determines its *strong subtree envelope*, so the colouring can be induced to $\mathcal{S}_\infty(U)$. For more information about this, the reader is referred to the relevant Chapter of [48].

When the trees $U_0, U_1, ..., U_{d-1}$ are uniformly branching then the corresponding version of the Halpern-Läuchli theorem is closely related to another well-known pigeonhole principle, the Hales-Jewett theorem [20], and consequently also to the Hindman theorem ([22]) and the Gowers theorem ([15]) which also have Ramsey spaces associated to them (see [48]). Here we mention one of these because of its relevance to the problems we treat here.

Let FIN be the collection of all *nonempty* finite subsets of $\mathbb{N}$. A *block-sequence* in FIN is a sequence $X = (x_n) \subseteq \mathrm{FIN}$ such that $x_m < x_n$ whenever $m < n$. We say that $X = (x_m)$ is a *block-subsequence* of $Y = (y_n)$ and write $X \leq Y$ whenever every x_m can be written as a union of some of the y_m's. Let $\mathrm{FIN}^{[\infty]}$ be the space of all infinite block-sequences in FIN. The Hindman theorem is the pigeonhole principle behind the important fact that $\mathrm{FIN}^{[\infty]}$ forms a topological Ramsey space. We just mention here a consequence of this fact.

Theorem 4.4 ([33]). *For every finite Borel colouring of* $\mathrm{FIN}^{[\infty]}$ *there is* $Y = (y_n) \in \mathrm{FIN}^{[\infty]}$ *such that the collection of all infinite block subsequences of Y is monochromatic.*

Here is a typical application of this result that shows its relationship to both the space $[\mathbb{N}]^\infty$ of all infinite subsets of $\mathbb{N}$ and the space of all perfect subtrees[9] of the complete binary tree $2^{<\mathbb{N}}$.

Theorem 4.5. *For every Borel colouring of the product $2^{\mathbb{N}} \times [\mathbb{N}]^\infty$ with either finitely many colours, or countably many colours that are invariant on finite changes on the second coordinate, there is a perfect set $P \subseteq 2^{\mathbb{N}}$ and an infinite set $M \subseteq \mathbb{N}$ such that the product $P \times [M]^\infty$ is monochromatic.*

[9]Recall that a subtree T of $2^{<\mathbb{N}}$ is *perfect* if every node of t has at least two incomparable successors. This notion naturally corresponds to the notion of a *perfect subset* of $2^{\mathbb{N}}$, a nonempty compact subset without isolated points.

Back to the analysis of weakly null sequences in $\ell_\infty(\Gamma)$ when Γ is a Polish space. The following classical result of Mycielski [35] is quite useful in reducing the complexity of the problem to its Ramsey-theoretic core.

Theorem 4.6 ([35]). *Suppose X is a Polish space and that $M_n (n < \omega)$ is a sequence of subsets of some finite powers X^{k_n} of X such that M_n is meager in X^{k_n} for all $n < \omega$. Then there is a perfect set $P \subseteq X$ such that $[P]^{k_n} \cap M_n = \emptyset$ for all n.*[10]

The following result of Argyros, Dodos and Kanellopoulos [2] is an inspiring application of Mycielski's theorem to Problem 1.4 (2) above.

Theorem 4.7 ([2]). *Suppose X is a Polish space and that $(f_a)_{a\in 2^{\mathbb{N}}}$ is a bounded sequence in $\ell_\infty(X)$ such that $(x,a) \mapsto f_a(x)$ is a Borel function from $X \times 2^{\mathbb{N}}$ into $\mathbb{R}$ and that $|\{a \in 2^{\mathbb{N}} : f_a(x) \neq 0\}| \leq \aleph_0$ for all $x \in X$. Then there is a perfect set $P \subseteq 2^{\mathbb{N}}$ such that the sequence $(f_a)_{a\in P}$ is 1-unconditional.*

The Ramsey theory of trees based on the Halpern-Läuchli theorem was brought to this area of mathematics by the following result of the author.

Theorem 4.8 ([44]). *Suppose K is a separable compact set of Baire class-1 functions defined on some Polish space X. Let D be a countable dense subset of K, and let f be a point of K that is not G_δ in K. Then there is a homeomorphic embedding*

$$\Phi : \mathbb{P} \to K$$

such that $\Phi(\infty) = f$ and $\Phi[2^{<\mathbb{N}}] \subseteq D$.

Example. We recall that $\mathbb{P}$ here is Pol's compactum [39], the space

$$\mathbb{P} = 2^{<\mathbb{N}} \cup 2^{\mathbb{N}} \cup \{\infty\},$$

where the points of the Cantor tree $2^{<\mathbb{N}}$ are isolated, the nodes of a branch of this tree converge to the corresponding member of $2^{\mathbb{N}}$ and ∞ is the point that compactifies the rest of the space. This is of course a standard space but what Pol [39] shows is that $\mathbb{P}$ is homeomorphic to a compact subset of the collection of the Baire-class-1 functions on the Cantor set $2^{\mathbb{N}}$.

Combining Theorems 4.7 and 4.8, we get the following result.

Theorem 4.9 ([2]). *Suppose X is a separable Banach space that contains no ℓ_1 but whose dual X^* is not separable. Then its double dual X^{**} contains a normalized 1-unconditional sequence of the form $(f_a)_{a\in 2^{\mathbb{N}}}$.*

[10] Here $[P]^{k_n}$ denote the collection of all k_n-tuples $(x_i : i < k_n)$ of elements of P such that $x_i \neq x_j$ whenever $i \neq j$.

Using this and Theorem 1.5, we get the following application to the separable quotient problem.

Corollary 4.10 ([2]). *Every infinite-dimensional* dual *Banach space has a separable infinite-dimensional quotient.*

5. Product-Ramsey property

We introduce two closely related and well-studied Ramsey-theoretic properties of an index set Γ, or the corresponding cardinal $|\Gamma|$.

Definition 5.1. We say that Γ has the *free-set property*, and write $\mathrm{FSP}(\Gamma)$, if every algebra $\mathcal{A}$ on Γ with no more than countably many operations has an infinite *free set*, an infinite subset X of Γ such that no $x \in X$ is in the sub algebra of $\mathcal{A}$ generated by $X \setminus \{x\}$.

Our interest here in this property is based on the following result from [8].

Theorem 5.2. *If a normalized weakly null sequence* $(x_i)_{i\in I}$ *is indexed by a set* I *that has the free-set property then it contains an infinite unconditional basic subsequence.*

Definition 5.3. We say that an index-set Γ is *polarized*, or it has the *product-Ramsey property*, and write $\mathrm{PRP}(\Gamma)$, if for every colouring χ of the set $\Gamma^{<\omega}$ of all finite sequences of the index-set Γ into 2 colours there exists an infinite sequence (X_i) of 2-element subsets of Γ or, equivalently, an infinite sequence (X_i) of *infinite* subsets of Γ, such that χ is constant on $\prod_{i<n} X_i$ for all n.

It is easily seen (see [7]) that if the index set Γ has the product-Ramsey property then it has the free-set property. The converse is not true since the index-sets Γ satisfying $\mathrm{PRP}(\Gamma)$ must be of cardinality bigger than the continuum while it is possible that $\Gamma = \mathbb{R}$ has the free-set property (see, for example, [30]). On the other hand, the $\mathrm{PRP}(\Gamma)$ is an interesting and useful Ramsey-theoretic property of an index-set which also has its Borel analogues (see [48]) as well as its density analogue (see [50]). However, as indicated above, the index set Γ satisfying the PRP has to be relatively large. This is also true about index-sets satisfying the free-set property.

Theorem 5.4 ([12]). *The free-set property fails for index-sets that have cardinalities* $< \aleph_\omega$.

It turns out that this lower bound on cardinalities of index-sets Γ satisfying FSP and PRP is the best possible as the following result shows.

Theorem 5.5 ([26], [7]). *It is consistent*[11] *with the axioms of set theory that every index set* Γ *of cardinality at least* $\aleph_\omega$ *satisfies the product Ramsey property and, therefore, also the free-set property.*

Corollary 5.6 ([8]). *It is consistent with the axioms of set theory that every normalized weakly null sequence of length at least* $\aleph_\omega$ *has an infinite unconditional subsequence.*

We shall also need the 2-dimensional version of $\mathrm{PRP}(\Gamma)$.

Definition 5.7. We say that an index-set Γ is *2-polarized* (or, it has the *2-dimensional product-Ramsey property*), and write in short as $\mathrm{PRP}_2(\Gamma)$, if for every colouring χ of the set $([\Gamma]^2)^{<\omega}$ of all finite sequences of 2-element subsets of the index-set Γ into 2 colours there exist an infinite sequence (X_i) of *infinite* subsets of Γ) such that χ is constant on $\prod_{i<n}[X_i]^2$ for all n.

We have the following analogue of Theorem 5.5 which, however, uses a stronger consistency assumption.

Theorem 5.8 ([42], [9]). *It is consistent with the axioms of set theory that an (every) index set* Γ *of cardinality* $\aleph_\omega$ *satisfies the two-dimensional product-Ramsey property.*

Our interest in this property is based on the following result of [8].

Theorem 5.9 ([8]). *Suppose that* $(x_\gamma)_{\gamma\in\Gamma}$ *is a normalized and separated sequence in some Banach space* X *containing no* ℓ_1. *If the index-set* Γ *satisfies* $\mathrm{PRP}_2(\Gamma)$ *then there is an infinite sequence* (β_n,γ_n) *of pairs of elements of* Γ *such that the semi-normalized sequence* $(x_{\beta_n}-x_{\gamma_n})$ *is unconditional.*

Corollary 5.10 ([8]). *If* $\mathrm{PRP}_2(\kappa)$ *holds then for every* $\varepsilon>0$ *every Banach space of density at least* κ *contains an infinite* $(1+\varepsilon)$*-unconditional basic sequence.*

Corollary 5.11 ([8]). *It is consistent with the axioms of set theory that every Banach space of density at least* $\aleph_\omega$ *has an infinite unconditional basic sequence as well as an infinite-dimensional quotient with an unconditional basis.*

Problem 5.12. Is the weaker assumption $\mathrm{PRP}(\Gamma)$ sufficient for the conclusion of Theorem 5.9?

Problem 5.13. Can in some of these results the bound $\aleph_\omega$ be replaced by $\aleph_n$ for $n<\omega$?

We shall see below that this cannot be done in case of Corollary 5.6.

[11] In fact, equiconsistent with the existence of an index set I which supports a σ-additive probability measure $\mu:\mathcal{P}(I)\to[0,1]$.

6. Positional graphs and conditional weakly null sequences

Fix an ordinal Γ. For two finite subsets I and J of $Gamma$, let $I < J$ denote the fact that every ordinal in I is smaller than any ordinal in J. Let $I \sqsubseteq J$ denote the fact that I is an initial segment of J.

Definition 6.1. For an integer n, we say that two subsets F and G of Γ are in $\Delta(n)$-*position* if there is a decomposition $F \cap G = I \cup J$ such that

(1) $I \sqsubseteq F$ and $I \sqsubseteq G$,

(2) $I < J$,

(3) $|J| \leq n$.

For a family $\mathbf{V} \subseteq [\Gamma]^{<\omega}$, we associate the corresponding *positional graph*

$$\mathcal{G}_n(\mathbf{V}) = (\mathbf{V},\ \Delta(n)^c),$$

where we put an edge between two finite $F, G \in \mathbf{V}$ if they are *not* in the $\Delta(n)$-position. Let $\mathcal{G}_n(\Gamma) = ([\Gamma]^{<\omega},\ \Delta(n)^c)$.

We shall be particularly interested in answers to the following general question.

Problem 6.2. For which Γ and $\mathbf{V} \subseteq [\Gamma]^{<\omega}$, the positional graph $\mathcal{G}_n(\mathbf{V})$ is *countably chromatic*?[12]

Here is one partial answer to this question.

Theorem 6.3 ([47]). *$\mathcal{G}_0(\omega_1)$ is countably chromatic.*

Unfortunately, this result cannot be extended to other ω_k's as it is easily seen that $\mathcal{G}_n(\omega_2)$ is not countably chromatic for any integer n. In fact, it is easily seen that for every inter n the graph $\mathcal{G}_n(\omega_2)$ contains an uncountable complete subgraph. This gives us the reason for the following definition.

Definition 6.4. We say that a family $\mathbf{V}$ of finite subsets of Γ is *dense* if for every infinite $A \subseteq \Gamma$ there is infinite $B \subseteq A$ such that $[B]^{<\omega} \subseteq \mathbf{V}$.

This leads us to the following more specific version of Problem 6.2.

Problem 6.5. For which Γ there exist dense $\mathbf{V} \subseteq [\Gamma]^{<\omega}$ and an integer n such that the corresponding positional graph $\mathcal{G}_n(\mathbf{V})$ is countably chromatic?

Finally, we mention the reason for our interest in positional graphs and these questions about them.

[12] Recall that a graph $\mathcal{G} = (V, E)$ is *countably chromatic* if there is a $\chi : V \to \omega$ such that $\chi(x) \neq \chi(y)$ for every pair of vertices from V that form an edge in E.

Theorem 6.6 ([30]). *If for some integer n there is a dense family $\mathbf{V} \subseteq [\Gamma]^{<\omega}$ such that $\mathcal{G}_n(\mathbf{V})$ is countably chromatic, then there is a normalized weakly null sequence indexed by Γ without infinite unconditional basic subsequence.*

The idea of constructing conditional norms using special functionals is already present in the proof of Theorem 2.3(3) above. It has been used and modified in many constructions that followed afterwards. Here, we seem to have the optimal combinatorial hypothesis that makes this idea work. To see this fix an infinite subset M of $\mathbb{N}$ such that $\min(M) \geq n$ and such that $\sum_{k<l \text{ in } M} \sqrt{\frac{k}{l}} \leq 1$. Since $\mathcal{G}_n(\mathbf{V})$ is countably chromatic, we can fix $c : \mathbf{V} \to M$ such that:

(1) $c(F) = c(G)$ implies that F and G are in $\Delta(n)$-position and

(2) $c(F) = c(G)$ implies that $|F| = |G|$.

Definition 6.7. We say that a finite block sequence $(s_i)_{i<k}$ of subsets of Γ is *c-special* whenever

(a) $\bigcup_{i<j} s_i \in \mathbf{V}$ for every $j < k$.

(b) $|s_j| = c(\bigcup_{i<j} s_i)$ for every $j < k$.

Note that the assumption that $\mathbf{V}$ is a dense family of finite subsets of Γ ensures that every infinite subset A of Γ contains an arbitrarily long special block sequence of finite sets. The following set collects all the *special functionals* that we need for defining our conditional norm.

$$\mathcal{F} = \{\sum_{i<k} |s_i|^{-1/2} \mathbf{1}_{s_i} : (s_i)_{i<k} \text{ is a finite } c\text{-special block-sequence}\}.$$

Using this we can define the following norm on the vector space $c_{00}(\Gamma)$ of all finitely supported real functions on Γ.

$$\|x\|_{\mathcal{F}} := \max\{\|x\|_\infty, \sup_{f\in\mathcal{F}} \langle x, f\rangle\}.$$

Then for every subset t of Γ whose cardinality $|t|$ belongs to M, we have that $\|\mathbf{1}_t\|_{\mathcal{F}} \leq |t|^{1/2}$, so $(e_\gamma)_{\gamma\in\Gamma}$ is a normalized weakly null sequence in $(c_{00}(\Gamma), \|\cdot\|_{\mathcal{F}})$.[13] The definition of a positional graph is made in order to facilitate the proof of the following crucial fact.

Lemma 6.8 ([30]). *The weakly null sequence $(e_\gamma)_{\gamma\in\Gamma}$ contains no infinite unconditional basic subsequence.*

[13] Here, e_γ is the vector with support $\{\gamma\}$ such that $e_\gamma(\gamma) = 1$.

We note that the constructed normed space $(c_{00}(\Gamma), \|\cdot\|_{\mathcal{F}})$ as well as its completion are both c_0-saturated spaces. It is possible to modify the construction to get a reflexive example. The following problem suggests itself.

Problem 6.9. Is there a similar combinatorial condition on uncountable Γ that ensures the existence of a reflexive space of density Γ with no infinite unconditional basic sequences?

Recall that Theorem 1.2 above gives such an example for $\Gamma = \omega_1$.

7. Constructing countably chromatic positional graphs

The construction crucially depends also on the following concept from [47].

Definition 7.1. A function $\varrho : [\kappa^+]^2 \to \kappa$ is called an (injective version of) ϱ-function whenever:

(a) ϱ is *subbaditive,* i.e. for every $\alpha < \beta < \gamma < \kappa^+$

(a.1) $\varrho(\alpha, \beta) \leq \max\{\varrho(\alpha, \gamma), \varrho(\beta, \gamma)\}$,

(a.2) $\varrho(\alpha, \gamma) \leq \max\{\varrho(\alpha, \beta), \varrho(\beta, \gamma)\}$.

(b) $\varrho(\alpha, \beta) \neq \varrho(\bar{\alpha}, \beta)$ for every $\alpha \neq \bar{\alpha} < \beta$.

(c) $\varrho(\alpha, \beta) \neq \varrho(\beta, \gamma)$ for every $\alpha < \beta < \gamma$.

It is proved in [47] that such a map exists for every regular cardinal κ and so, in particular, for $\kappa = \omega_n$ for every non-negative integer n. So, from now on, for each positive integer n we fix an injective ϱ function $\varrho^{(n)} : [\omega_n]^2 \to \omega_{n-1}$.

Definition 7.2. For integers $i \leq n$ we define $f_i^{(n)} : [\omega_n]^{i+1} \to \omega_{n-i}$ recursively as follows. Let $f_0^{(n)} := \mathrm{Id}_{\ \omega_n}$ and let

$$f_i(\alpha_0, \alpha_1, \ldots, \alpha_i) = \varrho^{(n-(i-1))}(f_{i-1}(\alpha_0, \ldots, \alpha_{i-1}), f_{i-1}(\alpha_1, \ldots, \alpha_i))$$

for $\alpha_0 < \cdots < \alpha_i$ in ω_n and $0 < i \leq n$. Let $f_n = f_n^{(n)} : [\omega_n]^{n+1} \to \omega$.

These functions are used in selecting a dense family $\mathbf{V}_n \subseteq [\omega_n]^{<\omega}$ that spans a countably chromatic positional graph. The following two notions play an important role in this. For two sets of ordinals A and W and a positive integer k a function of the form $f : [A]^k \to W$ is *shift-increasing* whenever

$$f(\alpha_0, \alpha_1, ..., \alpha_{k-1}) < f(\alpha_1, \alpha_2, ..., \alpha_k) \text{ for all } \alpha_0 < \alpha_1 < \cdots < \alpha_k \text{ in } A.$$

Such a function $f : [A]^k \to W$ is *min-dependent* whenever for every $s, t \in [A]^k$,

$$f(s) = f(t) \text{ implies } \min s = \min t.$$

In this context it is instructive to recall the Erdös-Rado canonical Ramsey theorem ([11]) stating that for every positive integer k and every mapping $f : [\omega]^k \to \omega$ there is an infinite set $A \subseteq \omega$ and $I \subseteq \{0, 1, ..., k-1\}$ such that for $\alpha_0 < \cdots < \alpha_{k-1}$ and $\beta_0 < \cdots < \beta_{k-1}$ in A,

$$f(\alpha_0, ..., \alpha_{k-1}) = f(\beta_0, ..., \beta_{k-1}) \text{ iff } \{\alpha_i : i \in I\} = \{\beta_i : i \in I\}.$$

Thus, if $0 \in I$ the restriction of f on $[A]^k$ is min-depedent. Note however, that for functions on $[A]^k$ for large sets of ordinals A, which is our case here, min-dependence is actually quite rare. Interestingly, the ρ-functions give us such functions that are *densely often* shift-preserving and min-dependent. More precisely, we have the following fact.

Lemma 7.3 ([30]). (1) *For every $A \subseteq [\omega_n]^\omega$ there is $B \in [A]^\omega$ such that for every $i \leq n$, the restriction $f_i^{(n)} \restriction [B]^{i+1}$ is shift-increasing .*

(2) *For every $A \subseteq [\omega_n]^\omega$ there is $B \in [A]^\omega$ such that for every $i \leq n$, the restriction $f_i^{(n)} \restriction [B]^{i+1}$ is* min-*dependant .*

This fact leads us to the definition of the dense vertex-set $\mathbf{V}_n \subseteq [\omega_n]^{<\omega}$ that spans our countably chromatic positional graph.

Definition 7.4. Let $\mathbf{V}_n$ be the set of all finite subsets v of ω_n such that:

(a) $f_n \restriction [v]^{n+1}$ is min-dependent,

(b) $f_i^{(n)} \restriction [v]^{i+1}$ is shift-increasing for every $i < n$.

It follows from Lemma 7.3 that $\mathbf{V}_n$ is dense in $[\omega_n]^{<\omega}$. Define

$$c : \mathbf{V} \to \mathrm{HF}$$

by letting $c(v)$ be the transitive collapse of the structure $(v, f_n \restriction [v]^{n+1})$, the natural structure of the form $(\{0, 1, ..., |v| - 1\}, f)$ isomorphic to it. The following fact verifies that the positional graph $\mathcal{G}_{2n-1}(\mathbf{V}_n)$ is countably chromatic.

Lemma 7.5 ([30]). *For $u, v \in \mathbf{V}_n$, $c(u) = c(v)$ implies that u and v are in the $\Delta(2n-1)$-position.*

This gives us the following result about Problem 1.4 (2).

Theorem 7.6 ([30]). *For every non-negative integer n there is a normalized weakly null sequence $(e_\gamma)_{\gamma<\omega_n}$ without infinite unconditional basic subsequence.*

In view of this result and Theorems 1.1 and 1.2, the following question shows naturally.

Problem 7.7. Is there, for every non-negative integer n, a reflexive space of density ω_n without infinite unconditional basic sequence?

8. References

[1] S.A. Argyros, J. Lopez-Abad and S.Todorcevic, A class of Banach spaces with few non-strictly singular operators. *J. Functional Analysis* **222** (2005), 306–384.

[2] S.A. Argyros, P. Dodos and V. Kanellopoulos, Unconditional families in Banach spaces. *Math. Annalen* **341** (2008), 15–38.

[3] A.D. Arvanitakis, Weakly null sequences with an unconditional subsequence. *Proc. Amer. Math. Soc.* **134** (2006), 67–74.

[4] M. Bačák and P. Hájek, Mazur intersection property for Asplund spaces. *J. Functional Analysis,* **255** (2008), 2090–2094.

[5] C. Bessaga and Pelczyński, On Bases and unconditional convergence of series in Banach spaces, *Studia Math.,* **27** (1958), 152–163.

[6] S.J. Dilworth, E. Odell, T. Schlumprecht and A. Zsák, Partial unconditionality. *Houston J. Math.,* **35** (2009), 1251–1311.

[7] C.A. DiPrisco and S.Todorcevic, A cardinal defined by a polarized partition relation. *Israel J. Math.,* **109**(1999), 41-52.

[8] P.Dodos, J. Lopez-Abad and S.Todorcevic, Unconditional basic sequences in spaces of large density. *Advances in Math.,* **226** (2011), 3297-3308.

[9] P.Dodos, J. Lopez-Abad and S.Todorcevic, Partition ideals below $\aleph_\omega$. *Fund. Math.,***217** (2012) , 21–34.

[10] J. Elton, Weakly null normalized sequences in Banach spaces, Ph. D. Thesis, Yale University 1978.

[11] P. Erdös and R. Rado, A combinatorial theorem. *J. London Math. Soc.,* **26** (1950), 249–255.

[12] P. Erdös and A. Hajnal, On a problem of B. Jonsson, *Bull. de l'Acad. Polon. Sci.* **14** (1966) 19–23.

[13] F. Galvin and K. Prikry, Borel sets and Ramsey's theorem. *J. Symbolic Logic,* **38** (1973), 193-198.

[14] I. Gasparis, E.Odell and B. Wahl, Weakly null sequences in Banach spaces $C(K)$. In *Methods in Banach space theory,* 97–131, Cambridge Univ. Press, Cambridge, 2006.

[15] W.T. Gowers, Lipschitz functions on classical spaces. *European J. Math.,* **13** (1992), 141–151

[16] W.T. Gowers, An infinite Ramsey theorem and some Banach-space dichotomies. *Ann. Math.* **156** (2002), 797–833.

[17] W.T. Gowers, Ramsey methods in Banach spaces. In *Handbook of the geometry of Banach spaces,* (Edited by W.B. Johnson and J. Linenstrauss,) Elsevier, 2003.

[18] W.T. Gowers and B. Maurey, The unconditional basic sequence problem. *J. Amer. Math. Soc* **6** (1993), 851–874.

[19] J.N. Hagler and W.B. Johnson, On Banach spaces whose dual balls are not weak-star sequentially compact. *Israel J. Math.,* **28** (1977), 325–330.

[20] A.W. Hales and R.I. Jewett, Regularity of positional games. *Trans. Amer. Math. Soc.,* **106** (1963), 222-229.

[21] J.D. Halpern and H. Läuchli, A partition theorem. *Trans. Amer. Math. Soc.,* **124** (1966), 360–367.

[22] N. Hindman, Finite sums from sequences within cells of a partition of N. *J. Combinatorial Theory,* textbf17 (1974), 1-11.

[23] M. Jimenénez Sevila and J.P. Moreno, Renorming Banach spaces with the Mazur intersection property, *J. Functional Analysis*, **144** (1997), 486–504.

[24] W.B. Johnson and H.P. Rosenthal, On weak-star basic sequences and their applications. *Studia Math.,* **43** (1972), 77–92.

[25] B. Josefson, Weak sequential convergence in dual Banach space does not imply norm convergence. *Ark. Math.,* **13** (1975), 79–89.

[26] P. Koepke, The consistency strength of the free subset property at ω_ω. *J. Symbolic Logic* **49** (1984), 1198–1204.

[27] P. Koszmider, On a problem of Rolewicz about Banach spaces that admit support sets. *J. Funct. Analysis* **257** (2009), 2723-2741.

[28] J. Lopez-Abad and S. Todorcevic, Partial unconditionality of weakly null sequences. *RACSAM Rev. R. Acad. Cien. Serie A. Mat.,* **100** (2006), 237–277.

[29] J. Lopez-Abad and S. Todorcevic, Generic Banach spaces and generic simplexes. *J. Functional Analysis* **261** (2011), 300–386.

[30] J. Lopez-Abad and S. Todorcevic, Positional graphs and conditional structure of weakly null sequences. arXiv:1111.5150v1, 22 Nov. 2011

[31] B. Maurey and H.P. Rosenthal, Normalized weakly null sequence without unconditionall subsequence. *Studia Math.,* **61** (1977), 77–98.

[32] S. Mazur, Uber schwache Konvergentz in en Raumen L^p. *Studia Math.,* **4** (1933) 128–133.

[33] K.R. Milliken, Ramsey's theorem with sums and unions. *J. Combinatorial theory,* **18** (1975), 276-290.

[34] K.R. Milliken, A partition theorem for finite subtrees of a tree. *Trans. Amer. Math. Soc.,* **263** (1981), 137-148.

[35] J. Mycielski, Independent sets in topological algebras. *Fund. Math.,* **55** (1964), 165–169.

[36] C.St. J. A. Nash-Williams, On well-quasi-ordering transfinite sequences. *Proc. Cambridge Phil. Soc.,* **61** (1965,) 33–39.

[37] A. Nissenzweig, w^* sequential convergence. *Israel J. Math.,* **22** (1975, 266–272.

[38] E. Odell, On Schreier unconditional sequences, *Contemp. Math.,* **144** (1993), 197–201.

[39] R. Pol, Note on compact sets of first Baire class functions. *Proc. Amer. Math. Soc.,* **96** (1986), 152–154.

[40] P. Pudlák and V. Rödl, Partition theorems for systems of finite subsets of integers. *Discrete Math.,* **39** (1982), 67–73.

[41] S. Rolewicz, On convex sets containing only points of support. *Comment. Math., Prace Math. Special issue* **1** (1978), 279–281.

[42] S. Shelah, Independence of strong partition relation for small cardinals, and the free-subset problem. *J. Symbolic Logic* **45** (1980), 505–509.

[43] S.Todorcevic, *Partition problems in topology* Americal Mathematical Society, Providence 1989.

[44] S. Todorcevic, Compact subsets of the first Baire class. *J. Amer. Math. Soc.,* **12** (1999), 1179–1212.

[45] S. Todorcevic, High-dimensional Ramsey theory and Banach space Geometry. In *Ramsey Mathods in Analysis*, Advances Courses in Mathematics, CRM Barcelona, Birkhäuser 2005.

[46] S. Todorcevic, Biorthogonal systems and quotient spaces via Baire category methods. *Math. Ann.* **335** (2006), 687–715.

[47] S. Todorcevic, *Walks on ordinals and their characteristics.* Birkhäuser 2007.

[48] S. Todorcevic, *Introduction to Ramsey spaces,* Annals of Mathematics Studies No 74, Princeton University Press, Princeton 2010.

[49] S.Todorcevic, Combinatorial dichotomies in set theory. *J. Symbolic Logic* **17** (2011), 1-72.

[50] S. Todorcevic and K. Tyros, Subsets of products of finite sets of positive upper density. J. Combin. Theory Ser. A 120 (2013), 183193..