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Dynamics of Localized Structures*†

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Abstract. We describe qualitative behaviour of solutions of the Gross-Pitaevskii equation in 2D in terms of motion of vortices and radiation. To this end we introduce the notion of the intervortex energy. We develop a rather general adiabatic theory of motion of well separated vortices and present the method of effective action which gives a fairly straightforward justification of this theory. Finally we mention briefly two special situations where we are able to obtain rather detailed picture of the vortex dynamics. Our approach is rather general and is applicable to a wide class of evolution nonlinear equation which exhibit localized, stable static solutions. It yields description of general time-dependent solutions in terms of dynamics of those static solutions "glued" together.

Introduction

Often solutions of nonlinear equations can be described in terms of dynamics of *stable, localized, particle-like structures* and radiation. The localized of structures mentioned above appear as "glued" together special static solutions. Our goal is to find a general description of this phenomenon. We illustrate our approach on a time-dependent version of the Ginzburg-Landau equation:

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + (|\psi|^2-1)\psi \ , \qquad \text{(SE)}$$

$$|\psi| \to 1 \quad \text{as} \quad |x| \to \infty \ ,$$

$$\psi : \mathbb{R}^{d+1} \to \mathbb{R}^m \ .$$

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($\mathbb{R}^m$ is assumed to possess a complex structure.) This equation comes up in condensed matter physics and nonlinear optics and is also known as the Gross-Pitaevskii or Ginzburg-Pitaevskii equation.

There is also a host of related equations to which our techniques are applicable:

$$-\frac{\partial\psi}{\partial t} = -\Delta\psi + (|\psi|^2 - 1)\psi \ , \tag{HE}$$

$$-\frac{\partial^2\psi}{\partial t^2} = -\Delta\psi + (|\psi|^2 - 1)\psi \ , \tag{WE}$$

Cahn-Hilliard, Allen-Cahn, Swift-Hohenberg, etc.

In what follows, I outline a general picture, using a wide brushstrock. In the supplement, I present our results on standing localized solutions – optical solitons.

Topology of ψ and localized structures

First we point out the connection between the topology of the function ψ (the *order parameter*) and the type of localized structure involved:

$\psi : \mathbb{R}^{d+1} \to \mathbb{R}^d \iff$ vortices/defects, monopoles, instantons, $\psi : \mathbb{R}^{3+1} \to \mathbb{R}^2 \iff$ line vortices, (cosmic)strings, $\psi : \mathbb{R}^{3+1} \to \mathbb{R} \iff$ kinks, domain walls.

Static solutions

The localized objects I refered to, are solutions of the corresponding stationary equation - in our case, the proper Ginzburg-Landau equation

$$-\Delta\psi + (|\psi|^2 - 1)\psi = 0 \ . \tag{GLE}$$

Thus we want to classify solutions of this equation. For simplicity we assume in what follows that

$$m = d \ .$$

Topological classification

With each $\psi : \mathbb{R}^d \to \mathbb{R}^d$ we associate the map

$$\hat{\psi} := \frac{\psi}{|\psi|}\Big|_{|x|=R} : S^{d-1} \to S^{d-1} \ .$$

Using a standard definition of the degree (see e.g. [52]), we set

$$\deg \psi := \deg \hat{\psi} \in \mathbb{Z} \ .$$

All solutions to the (GLE) are classified according to this topological invariant. This, in particular, leads to the topological conservation law for the corresponding time-dependent equations.

Group-theoretical classification

Now we want to isolate symmetric solutions. The symmetry group of (GLE) is

$$G_{\mathrm{sym}} = O(d) \times T(d) \times O(d)$$

(the group of rigid motions of the underlying physical space times the gauge group). The most symmetric solutions are

(a) translationally invariant solutions ψ_0:

$\psi_0 =$ constant unit vector (a ground state)

(b) "spherically symmetric" solutions ψ_0:

$\exists$ homomorphism $\rho : SO(d) \to SO(d)$ s.t.

$$\rho(g)\psi_0(g^{-1}x) = \psi_0(x) \quad \forall g \in O(d) \ .$$

The homotopy class of ρ's (preserving a fixed vector if $d \geq 3$) determines $\deg \psi$.

Depending on the dimension d spherically symmetric solutions have the following names:

$$d = 2 \Rightarrow \text{ vortices}$$

$$d = 3 \Rightarrow \text{ monopoles}$$

$$d = 4 \Rightarrow \text{ instantons}$$

Depending on the degree we have e.g. for $d = 2$

$$\deg \psi_0 = \pm 1 \Rightarrow \text{ vortex/antivortex}$$

$$\deg \psi_0 = n \Rightarrow \ n\text{-vortex} \ ,$$

etc.

Existence and stability

Let L_{ψ_0} be the linearized operator for (GLE) at a solution ψ_0. We use the following definition of the (linearized) stability:

Definition: A solution ψ_0 is said to be *stable* iff

$$\mathrm{spec} L_{\psi_0} \subset \overline{\mathbb{R}^+} \quad \text{and} \quad \mathrm{Null} L_{\psi_0} = \mathfrak{g}_{\mathrm{sym}} \psi_0 \ .$$

Here $\mathfrak{g}_{\mathrm{sym}}$ in the Lie algebra of the group G_{sym}. Note that $\mathfrak{g}_{\mathrm{sym}}\psi_0 \subseteq \mathrm{Null}L_{\psi_0}$ always.

Theorem.

(a) *Vortices:* $\forall\, n \in \mathbb{Z}\ \exists$ *a unique (modulo symmetry tranformations) vortex;* $|n| \leq 1$ *vortices are stable and* $|n| > 1$*, unstable*

(b) *monopoles, etc.: spherically symmetric solutions exist only for* $n = \pm 1$*, they are unique and stable.*

References:

(a) *Existence*: Hervé and Hervé [28], Chen, Elliott and Qui [16], Fife and Peletier [19], Ovchinnikov and Sigal [42].

Stability: Lieb and Loss [34] and Mironescu [37] for disc and $|n| \leq 1$. Ovchinnikov and Sigal [42] for $\mathbb{R}^2$ and all n.

(b) S. Gustafson [25].

The part (a) of the above theorem was generalized for the order parameter ψ coupled to a magnetic field (i.e. to magnetic or Abrikosov vortices) in [26].

From now on we set

$$d = m = 2 \ .$$

Rough idea of the proof of the stability resulty [42]

The outline below, though formally incorrect, gives a fairly good impression of our approach. Let ψ_0 be the 1-vortex. Since it breaks translation symmetry ($\psi_0(x) \neq \psi_0(x + h) \forall\, h \neq 0$),

$$\partial_{x_j}\psi_0 \text{ are zero modes of } L_{\psi_0} \ ,$$

the linearized operator. (In fact $\partial_{x_1}\psi_0$ and $\partial_{x_2}\psi_0$ are "proportional" to each other, so we can consider just $\partial_{x_1}\psi_0$.)

We find a positivity (open) cone $\Gamma \subset L^2(\mathbb{R}^2, \mathbb{R}^2)$ s.t.

(i) $\partial_{x_1}\psi_0 \in \Gamma$,

(ii) $\exp(-tL_{\psi_0}) : \overline{\Gamma} \to \Gamma$ (i.e. it is positively improving w.r.t. Γ).

Then the Perron-Frobenius theory implies that $\sigma(L_{\psi_0}) \subset [0, \infty)$ and 0 is a non-degenerate eigenvalue.

The reason that the argument above is incorrect is that the property (ii) does not quite hold. For $|n| \leq 1$, this hole can be patched up problem can be circumvented, while for $|n| > 1$, not. In the latter case we construct a test function ξ s.t.

$$\langle \xi, L_{\psi_0}(\xi) \rangle < 0 \ ,$$

which shows that L_{ψ_0} has a negative eigenvalue. □

Renormalized Energy

(GLE) is the equation for critical points of the celebrated Ginzburg-Landau functional

$$\mathcal{E}(\psi) = \frac{1}{2} \int \{ |\nabla\psi|^2 + \frac{1}{2}(|\psi|^2 - 1)^2 \} d^2x \ .$$

There is one problem with this functional though

Theorem. *Let ψ be a C^1 vector field on $\mathbb{R}^2$ s.t. $|\psi| \to 1$ as $|x| \to \infty$. If $\deg\psi \neq 0$, then $\mathcal{E}(\psi) = \infty$.*

Thus if we want to use the variational calculus and in general the notion of energy for vortices we have to modify $\mathcal{E}(\psi)$. We introduce the renormalized energy functional as follows

$$\mathcal{E}_{\text{ren}}(\psi) = \frac{1}{2} \int \{ |\nabla\psi|^2 - \frac{(\deg\psi)^2}{r^2}\chi + \frac{1}{2}(|\psi|^2 - 1)^2 \} d^2x \ ,$$

where $r = |x|$ and χ is a smooth cut-off function, $= 0$ for $r \leq 1$ and $= 1$ for $r \geq 2$.

In order to introduce our next key notion, we need the following notation and definition. Let $\mathbf{c} = (\mathbf{z}, \mathbf{n})$, where

$$\mathbf{z} = (z_1, \ldots, z_k) \ , \ z_j \in \mathbb{R}^2 \ ,$$

$$\mathbf{n} = (n_1, \ldots, n_k) \ , \ n_j \in \mathbb{Z} \ .$$

Definition: ψ has a *configuration* $\mathbf{c}$, conf $\psi = \mathbf{c}$, iff ψ has zeros only at $z_1 \dots z_k$ with local indices $n_1, \dots, n_k$.

Now we introduce *intervortex energy* as

$$E(\mathbf{c}) := \inf\{\mathcal{E}_{\text{ren}}(\psi) | \text{conf}\psi = \mathbf{c}\} \ . \tag{$*$}$$

We argue ([43]) that

$$(*) \text{ has a minimizer}$$

$$\longleftrightarrow \nabla_{\mathbf{z}} E(\mathbf{c}) = 0$$

and show that for intervortex dist $\gg 1$,

$$\nabla_{\mathbf{z}} E(\mathbf{c}) \neq 0 \quad \text{always.}$$

Hence for large intervortex distances there are *no stationary vortex configurations.*

For intervortex distances of order $O(1)$ stationary configuration do exist, e.g. (see [45]) (see Fig. 1).

Pinning

Introduce impurities in order to nail the vortices down:

$$\mathcal{E}_\lambda(\psi) = \mathcal{E}_{\text{ren}}(\psi) + \sum_{j=1}^{K} \frac{\lambda_j}{2} \int \delta_{b_j} |\psi|^2 \ ,$$

where $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\delta_b(x) = \frac{1}{2\pi r_0}\delta(|x-b| - r_0)$, (see Fig. 2).

We argue (see [43]) that if $\lambda_j \geq \text{const}|\nabla_{z_j} E(\mathbf{c})| \forall j$, then $\mathcal{E}_\lambda(\psi)$ has a minimizer in the class $\{\text{conf}\psi = \mathbf{c}\}$.

Asymptotics of $E(\mathbf{c})$

Let $R(\mathbf{c})$ be the intervortex distance. We show (see [43]) that as $R(\mathbf{c}) \to \infty$,

$$E(\mathbf{c}) = \sum_{i=1}^{K} E_{n_j} + H(\mathbf{c}) + O(R(\mathbf{c})^{-1}) \ , \tag{AS}$$

where E_n is the (proper) energy of the n-vortex and $H(\mathbf{c})$ is the Kirchhoff-Onsager Hamiltonian:

$$H(\mathbf{c}) = -\pi \sum_{i \neq j} n_i n_j \ln |z_i - z_j| \ .$$

The idea of a demonstration of (AS) is as follows. The upper bound, $E(\mathbf{c}) \leq$ r.h.s.(AS), is obtained by choosing an appropriate test function and performing a rather delicate many-body geometrical analysis. To prove the lower bound, $E(\mathbf{c}) \geq$ r.h.s.(AS), we use the pinning energy functional with $\lambda = O(R(\mathbf{c})^{-1})$. This gives

$$E(\mathbf{c}) \geq \inf\{\mathcal{E}_\lambda(\psi) | \mathrm{conf}\psi = \mathbf{c}\} - CR(\mathbf{c})^{-1} \ .$$

The minimization problem on the r.h.s. has a minimizer. The latter satisfies the Euler-Lagrange equation

$$-\Delta\psi + (|\psi|^2 - 1)\psi = -\sum \delta_{b_j}\psi \ .$$

This equation allows us to produce estimates on the minimizer in question which show that it is of the same form as the aforementioned test function and therefore

$$\inf\{\mathcal{E}_\lambda(\psi) | \mathrm{conf}\psi = \mathbf{c}\} = \text{ r.h.s. (AS)} \ .$$

The last two relations produce the desired lower bound which, together with the upper bound mentioned above, yields (AS).

An expansion related to (AS) is derived in [7].

Now we proceed to *dynamic* properties of vortices.

Topological solitons

The Schrödinger equation is invariant under the Galilean group. In particular

if $\psi_0(x)$ is a static solution, then

$$\psi_{\alpha,v}(x,t) := e^{i\alpha} e^{i(\frac{1}{2}x\cdot v - \frac{1}{4}v^2 t)} \psi_0(x - vt) \tag{TS}$$

solves (SE). (TS) is a *topological soliton* or a *moving solitary wave.* (There is also a related topological soliton of the form $e^{i\alpha} e^{\frac{i}{2}x\cdot v} \sqrt{1 - \frac{v^2}{4}}\psi_0\left(\sqrt{1 - \frac{v^2}{4}}(x - vt)\right)$.) However, $\psi_{\alpha,v}(x,t)$ has a phase which grows at infinity and therefore it is not in the class of functions we consider.

Asymptotic stability of vortices

Problem: For (HE), (WE) or (SE) consider solutions with initial conditions close to the stationary vortex ($n = 1$). Show that the global solutions exist and converge as $t \to \pm\infty$ to a vortex (centered possibly at a different point).

The only known results are for the *heat equation* with radially symmetric initial conditions: [24] and [58].

For the Schrödinger case the problem is completely open.

We expect that if $\psi(x,t)$ is a solution to (SE) with an initial condition close to the vortex solution ψ_0, then there are $\alpha(t)$ and $v(t)$ s.t.

$$\psi(x,t) = \psi_{\alpha(t),v(t)} + \text{ dispersive wave}$$

and

$$\alpha(t) \to \alpha_\pm \text{ and } v(t) \to v_\pm$$

as $t \to \pm\infty$, i.e. $G_{\rm sym}\psi_0$ is a stable manifold for the Schrödinger dynamics (see Fig. 3). (Note that $v_\pm = 0$ if the phase of the initial condition is bounded.) In the context of the standard nonlinear Schrödinger equation (i.e. $|\psi| \to 0$ as $|x| \to \infty$), such results were obtained by Soffer and Weinstein [54] and [55] and Buslaev and Perel'man [12].

Break up and creation of vortices

We have ([47]) the following two results: (a) description of the dynamics of vortex break-up and (b) proof that there is no energy gap for vortex-pair creation. The second result shows that there are topological fluctuations of the vacuum and around single vortices (see Fig. 4).

Multivortex dynamics

Problem: Consider (SE) with an initial condition corresponding to several vortices at large distances from each other. Describe the *dynamics of the vortex centers* corresponding to the solution.

Nonlinear adiabatic theory

Let ψ be a solution of (SE) with an initial condition of a configuration $\mathbf{c}$ and low energy, say of order $E(\mathbf{c})+O(1)$. To describe this solution, we proceed as follows (in what follows $\mathbf{n}=(n_1,\ldots,n_k)$ is fixed and is not displayed in the notation):

(i) Pick a "minimizer", $\psi_{\mathbf{z}}$ of $\mathcal{E}_{\text{ren}}(\psi)$ in $\{\text{conf}\psi=\mathbf{c}\}$, $\mathbf{c}=(\mathbf{z},\mathbf{n})$.

(ii) Define the intervortex energy $E(\mathbf{z}):=\mathcal{E}(\psi_{\mathbf{z}})$ and write the Hamiltonian equation

$$\dot{\mathbf{z}}=J\nabla E(\mathbf{z}), \tag{$*$}$$

where J is a "symplectic" matrix on $\bigoplus_{i=1}^{k}\mathbb{R}^2$:

$$J=\text{diag}\left(\frac{1}{\pi n_j}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right).$$

(iii) Insert the solution, $\mathbf{z}(t)$, of the Hamiltonian system above with appropriate initial conditions into $\psi_{\mathbf{z}}$. This gives the adiabatic order parameter as $\psi_{\mathbf{z}(t)}$.

(iv) We expect that the solution ψ is of the form

$$\psi=e^{i\alpha(t)}\psi_{\mathbf{z}(t)}+\psi_{\text{disp}}\ ,$$

where $\alpha(t)$ is some slowly varying real function of t and ψ_{disp} is a radiation to ∞, the latter of the order $O(R(\mathbf{z})^{-2})$.

Effective Action Method

Now we explain the origin of the nonlinear adiabatic theory (see [44] for more details). Let $S(\psi)$ be the action functional for Eqn (SE):

$$S(\psi)\ =\ \int\Big\{-\int\frac{1}{2}\text{Im}(\psi\dot{\overline{\psi}})d^2x+\mathcal{E}_{\text{ren}}(\psi)\Big\}dt\,,$$

where $\mathcal{E}_{\text{ren}}(\psi)$ is the renormalized Ginzburg-Landau functional introduced above. First we find an approximate minimizer, $\psi_{\mathbf{z}}$, of $\mathcal{E}_{\text{ren}}(\psi)$ under the constraint that the vortices are fixed at positions $z_1,\ldots,z_k$, $(z_1,\ldots,z_k)=\mathbf{z}$. Next, we allow $\mathbf{z}$ to depend on time and plug $\psi_{\mathbf{z}(t)}$ into $S(\psi)$. The resulting action functional,

$$S_{\text{eff}}(\mathbf{z})\ \equiv\ S(\psi_{\mathbf{z}})\,,$$

describes the dynamics of the vortex centers in the leading approximation; it is equal modulo $\int O(\ln R(\mathbf{z}) \cdot R(\mathbf{z})^{-2})dt$ to the action functional

$$S_{\rm vort}(\mathbf{z}) \;=\; \int \Big\{ -\frac{\pi}{2}\sum_{j=1}^{k} z_j \wedge \dot{z}_j - E(\mathbf{z})\Big\} dt\,,$$

whose critical points satisfy Eqn $(*)$.

To go beyond this theory we write $\psi = \psi_{\mathbf{z}} + \alpha$, where α is supposed to be a small fluctuation field around $\psi_{\mathbf{z}}$ and expand $S(\psi)$ in α up to the second order. Critical points of the resulting functional satisfy the system of coupled equations

$$\partial_{\mathbf{z}} S_{\rm eff}(\mathbf{z}) \;=\; -\nabla_{\mathbf{z}}{\rm Re}\int \overline{\alpha}\partial_{\overline{\psi}} S(\psi_{\mathbf{z}}) \tag{$**$}$$

$$S''(\psi_{\mathbf{z}})\alpha \;=\; -\partial_{\overline{\psi}} S(\psi_{\mathbf{z}})\,, \tag{$***$}$$

where ∂_ε stands for the variational derivative w.r. to ε and $S''(\psi)$ is the Hessian of S at ψ, and where we dropped the higher order term $\nabla_{\mathbf{z}}\frac{1}{2}{\rm Re}\int\int \overline{\alpha}S''(\psi_{\mathbf{z}})\alpha$. We demonstrate that provided $\mathbf{z}$ satisfies $(*)$, one can perturb $\psi_{\mathbf{z}}$ slightly in such a way that Eqn $(***)$ has a solution of the order $\alpha = O(R(\mathbf{z})^{-1})$, provided $t \le a^p$ for some $p \ge 0$. To this end we decompose the space $\mathbb{R}^2$ into several regions determined by the configurations $\mathbf{z}$ and estimate Eqn $(***)$ separately in each region. We call this method, the method of geometric solvability.

Finally, we observe that Eqns $(**)$-$(***)$ stripped of inessential terms read

$$\dot{\mathbf{z}} = J\nabla_{\mathbf{z}}E(\mathbf{z}) - \int \ddot{\chi}\nabla_{\mathbf{z}}\varphi_0 d^2x \;, \tag{CEa}$$

$$(\partial_t^2 - 2\Delta)\chi = -\ddot{\varphi}_0 \;, \tag{CEb}$$

where $\varphi_0(x) = \sum_{j=1}^{k} n_j\theta(x - z_j)$ and χ = phase of α. Here $\theta(x)$ is the polar angle of $x \in \mathbb{R}^2$. This systems represents finite dimensional Hamiltonian equations for the vortex centers $\mathbf{z}$ coupled to the wave equation for the phase fluctuation χ.

Special case: Two simple vortices

An initial condition for (SE): two simple vortices at the distance R from each other.

Two vortices of the same charge: the vortices rotate around each other with the frequency $\omega = \frac{1}{R^2}$ (see Fig. 5) and radiate at the same time, so that the distance between them grows as

$$R(t) = (3\pi t)^{\frac{1}{6}}$$

modulo lower order terms.

Two vortices of opposite charges: there is a critical distance $R_{\rm cr}$ s.t. for $R > R_{\rm cr}$ there exists a travelling wave solution corresponding to the vortices moving parallel to each other (see Fig. 6) while for $R < R_{\rm cr}$, the vortices, as they move parallel to each other, emit a shock wave (Cherenkov radiation) and eventually collapse onto each other (see Fig. 6).

History. The Hamiltonian dynamics of vortices was first suggested by Onsager [40] and then elaborated by Gross [23] and Cheswick and Morrison [17] on the basis of analogy with the motion of an incompressible fluid. Indeed $u = -\nabla(\arg\psi)$ satisfies the Euler equation

$$\dot{u} = (u \cdot \nabla)u + \nabla p \ ,$$

where $p = \frac{\nabla|\psi|}{|\psi|} - |\psi|^2 + 1$. It was derived using multiscale expansion by Neu [39] and using the nonlinear adiabatic theory by Ovchinnikov and Sigal [44]. The rigorous proof that the vortices indeed are well defined for "low energy" solutions ψ and that their centers are governed by the Hamiltonian system mentioned was given by Lin and Xin [35].

The radiation phenomena was found in [44] and [46], where the coupled equations for the vortex motion and radiation, Eqns (CE), were derived.

The special case of two vortices of the same charge was analyzed by Ovchinnikov and Sigal [46]. The existence of a solitary wave for two vortices of opposite charge at a large distance from each other was predicted by Jones and Roberts [32] (see also [29], [31], [33] and [49] and references therein) and was rigorously proven by Bethuel and Saut [8]. The appearance of the shock wave at small distances was suggested by Ovchinnikov and Sigal [46].

Related problems:

(a) Magnetic vortices: ψ coupled to the magnetic (or, in general, gauge) field.

(b) Quantized vortices.

Here one would like to describe dynamics of quantized vortices. In particular, an important problem is that of metastable states due to tunneling through a potential barrier. The probability of decay due to tunneling and its dependence on the temperature were computed in the quasiclassical regime in [41].

(c) Vortices in random potentials and at finite temperatures.

Related papers:

A systematic approach to variational problems with topological constraints was prepared in [20]. Dynamics of localized objects for Cahn-Hilliard and Allen-Cahn equations was analyzed in [1], [5], [9], [13] and [57]. The Ginzburg-Landau equation coupled to the Young-Mills field was considered in [4] and [56].

Dynamics of line vortices was investigated in [50].

The wealth of information about static vortices can be found in [7] and [50] with recent results due to Chanillo and Kiesling [15]) and Mironescu [38].

Conclusion

In this talk we described topological and group-theoretical classification of localized structures for evolution equations of the Ginzburg-Landau type. These structures are stable, "spherically-symmetric", static solutions. In order to describe the dynamics of several such structures, we introduced the notion of the interstructure energy and developed a general adiabatic theory (similar to the Born-Oppenheimer theory for molecules). This theory is justified by the technique of effective action functional. Finally, we considered two examples where the general theory is applied.

REFERENCES

[1] N. Alikakos, P.W. Bates and G. Fusco (1991), Slow motion for Cahn-Hilliard equation in one space dimension, *J. Diff. Equations* **90**, 81–135.

[2] N. Alikakos and G. Fusco (1993), Slow dynamics for the Cahn-Hilliard equation in higher spatial dimensions, Part 1: Special Estimates, Carr Reports in Mathematical Physics, University of Rome.

[3] N. Alikakos and G. Fusco (1993), Slow dynamics for the Cahn-Hilliard equation in higher spatial dimensions, Part 2: The motion of bubbles, preprint.

[4] M. Atiyah and N. Hitchin (1988), *The Geometry and Dynamics of Magnetic Monopoles,* Princeton Univ. Press.

[5] P.W. Bates, E.N. Dancer and J. Shi (1997), Multi-spike stationary solutions of the Cahn-Hilliard equation in higher-dimension and instability, preprint.

[6] P.W. Bates and J.P. Xun (1994), Metastable patterns for the Cahn-Hilliard equation, *J. Different Eqnations* **111**, 421–457.

[7] F. Bethuel, H. Brezis and F. Hélein (1994), *Ginzburg-Landau Vortices,* Birkhäuser, Basel.

[8] F. Bethuel and J.C. Saut (1998), Travelling waves for the Gross-Pitrevskii equation, preprint.

[9] L. Bronsard and D. Hilhorst (1992), On the slow dynamics for the Cahn-Hilliard Equation in One space dimension, *Proc. Roy. Soc. London Ser. A* **439**, 669–682.

[10] L. Bronsard and R.V. Kohn (1991), Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics, *J. Diff. Eqn.* **90**, 211–237.

[11] L. Bronsard and B. Stoth (1997), Volume preserving mean curvature flow as a limit of a nonlocal Ginzburg-Landau equation, *SIAM J. Math. Anal.* **28**, 769–807.

[12] V.S. Buslaev and G.S. Perel'man (1993), Scattering for the nonlinear Schrödinger equation: states close to a soliton, *St. Petersburg Math. J* **4** (6), 1111-1142.

[13] J. Carr and R. Pego (1987), Very slow phase separation in one dimension. In: Conf. on Phase Transitions M. Rasele (ed.), Nice.

[14] J. Carr and R. Pego (1989), Metastable patterns in solutions of $u_t = \varepsilon^2 u_{xx} - f(u)$, *Comm. Pure Appl. Math.* **42**, 523-576.

[15] S. Chanillo and M. Kiesling (1995), Symmetry of solutions of Ginzburg-Landau equations, *Compt. Rend. Acad. Sci. Paris,* t. 327, Série I, 1023–1026.

[16] Y. Chen, C. Elliot and T. Qui (1994), Shooting method for vortex solutions of a complex-valued Ginzburg-Landau equation, *Proc. Royal Soc. Edinburgh* **124A**, 1068–1075.

[17] J. Creswick and N. Morrison (1980), On the dynamics of quantum vortices, *Phys. Lett. A* **76**, 267.

[18] W. E (1994), Dynamics of vortices in Ginzburg-Landau theories with applications to superconductivity, *Physica D* **77**, 383-404.

[19] P. Fife and L.A. Peletier (1996), On the location of defects in stationary solutions of the Ginzburg-Landau equations on R^2, *Quart. Appl. Math.* **54**, 85-104.

[20] J. Fröhlich and M. Struwe (1990), Variational problems on vector bundles, *Commun. Math. Phys.* **131**, 431-464.

[21] V.L. Ginzburg and L.P. Pitaevskii (1958), On the theory of superfluidity, *Sov. Phys. JETP* **7**, 585.

[22] E.P. Gross (1961), *Nuovo Cimento A* **20**, 454.

[23] E. Gross (1966), Dynamics of interacting bosons, in *Physics of Many Particle Systems*, ed. E. Meeron, Gordon and Breach, NY, 268.

[24] S. Gustafson (1997a), Stability of vortex solutions of the Ginzburg-Landau heat equation, in PDE's and their applications (L. Seco et al, eds), Proceeding of Conference in PDE's, Toronto June 1995.

[25] S. Gustafson (1997b), Symmetric solutions of hte Ginzburg-Landau in all dimensions, *IMRN*, **16**, 807-816.

[26] S. Gustafson and I.M. Sigal (1998), Existence and stability of magnetic vortices. preprint (Toronto).

[27] P. Hagan (1982), Spiral waves in reaction diffusion equations, *SIAM J. Applied Math.* **42**, 762–786.

[28] M. Hervé, R. Hervé (1994), Étude qualitative des solutions réeles d'une équation différentielle liée a l'équation de Ginzburg-Landau, *Ann. Inst. Henri Poincaré, Analyse non linéaire* **11**, 427–440.

[29] S.V. Iordanskii and A.V. Smirnov (1978), *JETP Lett.* **27**, 535.

[30] A. Jaffe and C. Taubes (1980), *Vortices and Monopoles,* Birkhäuser.

[31] C. Jones, S.J. Putterman and P.M. Roberts (1986), Motion of Bose condensation V, *J. Phys. A* **19**, 2991–3011.

[32] C.A. Jones and P.M. Roberts (1982), *J. Phys. A: Math. Gen.* **15**, 2599–2619.

[33] E.A. Kuznetzov and J.J. Rasmussen (1995), Instability of two dimensional solitons and vortices in defocusing media, *Phys. Rev. E* **51, 5**, 4479–4484.

[34] E.M. Lieb and M. Loss (1994), Symmetry of the Ginzburg-Landau minimizers in a disc, *Math. Res. Lett.* **1**, 701–715.

[35] F.-H. Lin and J.X. Xin (1998), On the incompressible fluid limit and the vortex motion law of the nonlinear Schrödinger equation.

[36] N.S. Manton (1981), A remark on scattering of BPS monopoles, *Phys. Letters* **110**B, N1, 54–56.

[37] P. Mironescu (1995), On the stability of radial solutions of the Ginzburg-Landau equation, *J. Funct. Anal.* **130**, 334–344.

[38] P. Mironescu (1996), Les minimiseurs locaux pour l'équation de Ginzburg-Landau sont à symmétrie radiale, preprint.

[39] J. Neu (1990), Vortices in complex scalar fields, *Physica D* **43**, 385–406.

[40] L. Onsager (1949), Statistical hydrodynamics, *Nuovo Cimento* V-VI, Suppl. 2, 279.

[41] Yu. N. Ovchinnikov and I.M. Sigal (1993), Decay of vortex states in superconductors, *Phys. Rev. B* **48**, N2, 1085-1096.

[42] Yu. N. Ovchinnikov and I.M. Sigal (1997a), Ginzburg-Landau equation I. General discussion, in *P.D.E.'s and their Applications* (L. Seco et al., eds.), Proceedings of Conference in PDE's, Toronto, June 1995.

[43] Yu. N. Ovchinnikov and I.M. Sigal (1997b), The Ginzburg-Landau equation II. The energy of vortex configurations, preprint.

[44] Yu. N. Ovchinnikov and I.M. Sigal (1998a), The Ginzburg-Landau equation III. Vortex dynamics, Nonlinearity (to appear).

[45] Yu.N. Ovchinnikov and I.M. Sigal (1998b), Symmetry breaking in the Ginzburg-Landau equation, preprint.

[46] Yu.N. Ovchinnikov and I.M. Sigal (1998c), Long-time behaviour of Ginzburg-Landau vortices, *Nonlinearity* (to appear).

[47] Yu.N. Ovchinnikov and I.M. Sigal (1998d), Break up and creation of vorticies, in preparation.

[48] L.M. Pismen (1994), Structure and dynamics of defects in 2D complex vector field, *Physica D* **73**, 244-258.

[49] L.M. Pismen and A. Nepomnyashchy (1993), Stability of vortex rings in a model of superflow, *Physica D* **69**, 163–171.

[50] L.M. Pismen and J. Rubinstein (1991), Motion of vortex lines in the Ginzburg-Landau model, *Physica D* **47**, 353–360.

[51] L.P. Pitaevskii (1961), *Pis'ma Zh. Eksp. Teor. Fix.* **77**, 988 (*Sov. Phys. JETP* **13**, 451).

[52] A.S. Schwarz (1993), Topology for physicists, Springer-Verlag.

[53] I. Shafrir (1994), Remarks on solutions of $-\Delta u = (1-|u|^2)u$ in $\mathbb{R}^2$, *C.R. Acad. Sci. Paris,* t. 318, Série I, 327–331.

[54] A. Soffer and M.I. Weinstein (1990), Multichannel nonlinear scattering for nonintegrable equations, *Commun. Math. Phys.* **133**, 119-146.

[55] A. Soffer and M.I. Weinstein (1992), Multichannel nonlinear scattering for nonintegrable equations II. The case of anisotropic potentials and data, *Journal of Differential Equations* **98** (2), 376-390.

[56] D. Stuart (1994), *Comm. Math. Phys.* **159**, 51.

[57] M. Ward (1996), Metastable bubble solutions for the Allen-Cahn equation with mass conservation, *SIAM J. Appl. Math.* **56** (5), 1247–1279.

[58] M. Weinstein and J. Xin (1996), Dynamic stability of the vortex solutions of the Ginzburg-Landau and nonlinear Schrödinger equations, *Comm. Math. Phys.* **180**, 389–428.

Supplement: Optical Solitons

Consider the problem of propagation of electro-magnetic waves in a nonlinear medium. The principle of minimal action for the action functional

$$S(\vec{A}) = \frac{1}{2}\int\int [f(|\vec{E}|^2, x) - |\vec{B}|^2] d^3x \; dt$$

(in the Coulomb gauge) gives the equations

$$\partial_t^2(\varepsilon \cdot \vec{E}) = \Delta \vec{E} \ ,$$

where $\varepsilon = \frac{\partial f}{\partial(|\vec{E}|^2)} = \varepsilon(|\vec{E}|^2, x)$ is a dielectric constant.

Consider the simplest case described in Figure 9

Problem:

Find the transmission coefficient, $|T|^2$, as a function of an amplitude, A, of the incident wave.

Important results on this problem were derived in [1], [2], [3], [4], [5], [6], [9] and [12], (see [7] and [8] for reviews). In [10] we introduce the variational principle above and obtain analytically the following picture, which was suggested in the papers mentioned above (see Fig. 10).

Moreover, we derive the following estimate for the number of solitons at a given amplitude A for $\mu|A|^3$ large:

$$\# \text{ solutions} \approx \tfrac{9\mu n|\vec{k}|}{16\pi}|A|^2 \cdot L,$$

where $\vec{k}$ is the wave vector of the incident wave.

This picture depends crucially on the fact that the linear problem has a series of resonances at the following complex values, k of $|\vec{k}|$:

$$k = \frac{\pi m}{n \cdot L} - \frac{i}{n \cdot L} \ln \frac{1 + \frac{1}{n}}{1 - \frac{1}{n}} \ , \ m = 0, \pm 1, \ldots \ .$$

The linear theory gives

$$|\vec{k}| = \frac{\pi m}{nL} \Rightarrow |T^{\text{lin}}|^2 = 1$$

$$|\vec{k}| = \frac{\pi}{nL}(m + \frac{1}{2}) \Rightarrow |T^{\text{lin}}|^2 = \frac{2}{n^2 + 1} \approx 0 \ .$$

The nonlinearity leads to an effective wave vector which, depending on A, takes a discrete set of values, some of which are on the resonances and some off; i.e. some correspond to a *transparent* medium and some to almost *opaque*.

The resonance structure above plays a key rôle also in our stability analysis.

REFERENCES

[1] A.B. Aceves and S. Wabnitz (1989), *Phys. Lett. A* **141**, N 1,2, 37–42.

[2] W. Chen and D.L. Mills (1987), *Physical Review B* No. 2 **35**, 524–532.

[3] W. Chen and D.L. Mills (1987), *Physical Review B* **36**, 6269.

[4] W. Chen and D.L. Mills (1987), *Physical Review Letters* No. 2 **58**, 160-163.

[5] D.N. Christodoulides and R.I. Joseph (1989), *J. of Quantum Electronics* **25**, 273–279.

[6] C.M. de Sterke and J.E. Sipe (1988), *Phys. Rev. A* **38**, 5149.

[7] C.M. de Sterke and J.E. Sipe (1994), Gap solitons, Progress in Optics XXX111 (E. Wolf, ed.).

[8] D.L. Mills (1991), Nonlinear Optics, Springer.

[9] D.L. Mills and S.E. Trullinger (1987), Gap solitons in nonlinear periodic structures, *Physical Review B*, No. 2 **36**, 947–952.

[10] Yu. N. Ovchinnikov and I.M. Sigal, Optical Bistability, preprint, 35 pages.

[11] P.A. Shah (1994), Vortex scattering at near critical coupling, *Nucl. Phys. B* **429**, 259–276.

[12] J.E. Sipe and H.G. Winful (1988), Opt. Lett 13, 132.

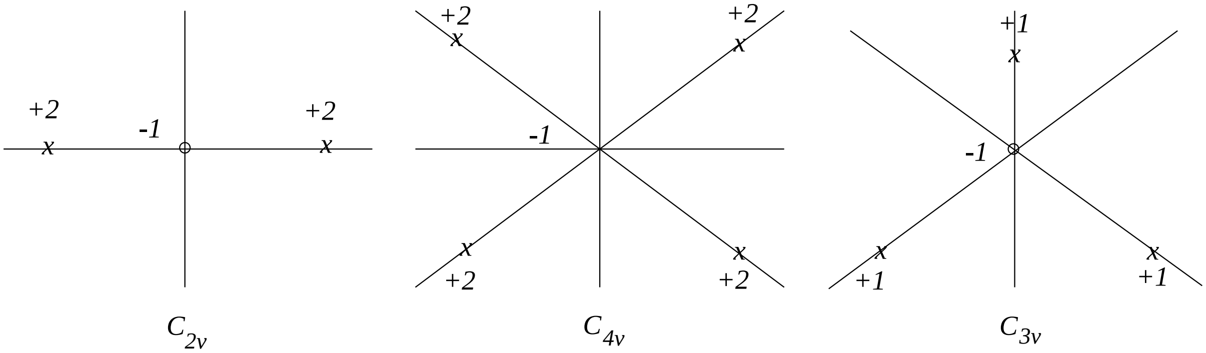


Fig. 1. Static vortex configurations

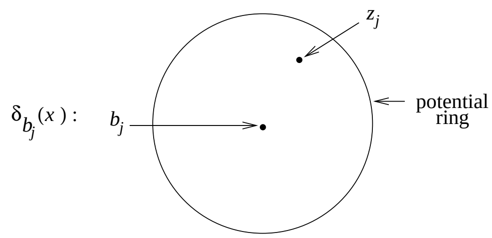


Fig. 2. Impurity potential

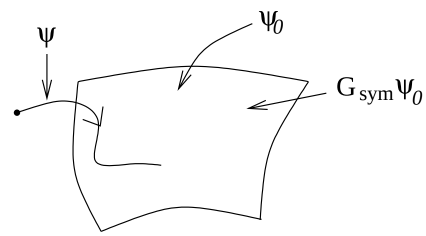


Fig. 3. Manifold of solitary waves

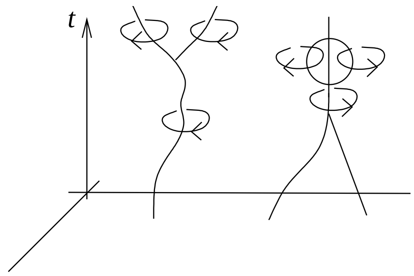


Fig. 4. Break up and recombination of vortices

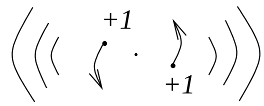


Fig. 5. Motion of two 1-vortices

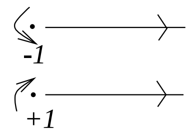


Fig. 6. Motion of a vortex-antivortex pair

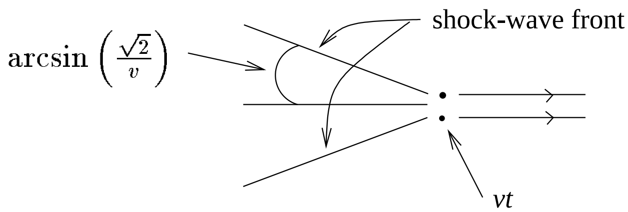


Fig. 7. Shock wave (Cherenkov radiation) by vortex-antivortex pair

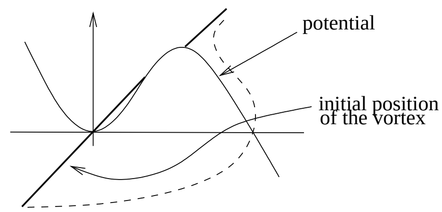
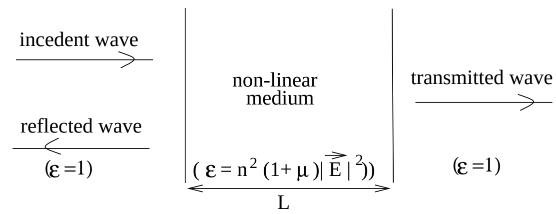


Fig. 8. Tunneling of a vortex



n is the refractive index ($n > 1$).

Fig. 9. Passing of a wave through a nonlinear medium

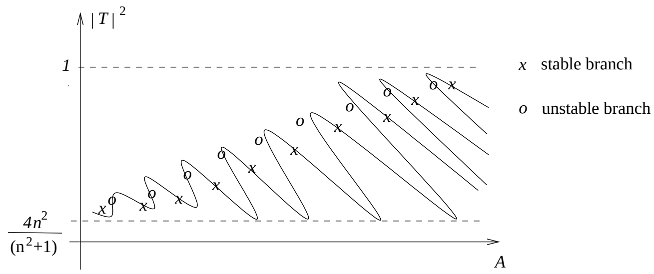


Fig. 10. Solutions and their stability parameterized by the amplitude A of the incident wave