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Asymptotic Completeness of N Particle Long Range Scattering [1] [2]

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Abstract

We prove asymptotic completeness for N-particle long-range system with potentials vanishing as $O(|x|^{-\mu})$, where $\mu \geq 1 - 2^{-N-2}$, at infinity.

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1. Introduction

In this paper we prove asymptotic completeness for N-particle long-range systems with potentials vanishing at infinity as $O(|x|^{-\mu})$ with $\mu \geq 1-2^{-N-2}$. Previously asymptotic completeness for long-range quantum systems for 2 particles and various classes of potentials was proven by several authors (see [Sig2] for references) and for 3 particles with pair interactions vanishing at ∞ as $O(|x|^{-\mu})$ with $\mu > \sqrt{3}-1$, in [En2] (see also [SigSof5] for a different proof and see [Mo] for another approach). Asymptotic completeness for N particle Schrödinger operators with two generations of thresholds (this, in particular, includes general four particle operators) and interacting via pair potentials vanishing at ∞ as $O(|x|^{-1})$ (Coulomb-type potentials) was proven in [SigSof6].

Our approach consists of two steps. On the first step we use asymptotic clustering to reduce the problem involving an $(N+1)$-particle Schrödinger operator to that involving a time-dependent operator of the form $H(t) = H + W(x,t)$, where H is an N-particle Schrödinger operator and $W(x,t)$ is a real potential obeying $|\partial^\alpha W(x,t)| \leq C_\alpha(1+|x|+|t|)^{-\mu-|\alpha|}$. The asymptotic clustering states roughly that as $t \to \pm\infty$ a system in question disintegrates into non-interacting, and therefore freely moving, subsystems. In the short-range case $(\mu > 1)$, W is in $L^1(dt)$ and therefore can be dropped. Using this one shows that asymptotic clustering for non-threshold energies implies asymptotic completeness. In the long-range case $W(x,t)$ cannot be dropped. Since the energy is not conserved for $H(t)$, dealing with it poses a number of subtle problems. $H(t)$ is handled on the second step. On this step we use a fine microlocal analysis on ranges of singular asymptotic projections in order to reduce the problem of controlling the evolution $U(t)$ generated by $H(t)$ to the one of controlling the evolution generated by H.

To explain this approach in more precise terms we introduce some notation. For a family $B = \{B_t, t > 0\}$ of self-adjoint operators and a Borel set Δ with $|\Delta| > 0$, we

define

$$F^+_\Delta(B) \;=\; s-\lim_{t\to+\infty} U(t)^* F_\Delta(B_t)U(t)\;,$$

where $F_\Delta(\lambda)$ is a smoothed out characteristic function of Δ (see the end of this introduction). For $|\Delta|=0$ we define $F^+_\Delta(B)$ by a limiting procedure (see section 4). Denote $v=\left\{\frac{|x|}{t}\right\}$. Let τ be the threshold set of H. τ and $\{0\}$ are the singular, from the point of view of propagation, sets for H and $\frac{|x|}{t}$, respectively, and $\mathrm{Ran}\, F^+_{\{0\}}(v)\subset \mathrm{Ran}\, F^+_\tau(H)$. We show that $U(t)$ is asymptotically clustering on $(\mathcal{R}^+_0)^\perp$, where $\mathcal{R}^+_0=\mathrm{Ran}\, F^+_{\{0\}}(v)$, i.e. it is reduced to the evolutions generated by (time-dependent) Hamiltonians of non-interacting subsystems. This yields the reduction in the number of particles. Furthermore, we show that on $\mathcal{R}^+_0$, $U(t)$, modulo uniformly in t small and asymptotically clustering terms, concentrates in the set $\{|x|\le t^\alpha\}$ with $\alpha<\mu$. This allows one to pass from $U(t)$ to the evolution $U_0(t)$ generated by $H_0(t)=H+W(0,t)$, i.e. one is back to a time-independent Hamiltonian. The case $t\to-\infty$ is treated similarly.

To sum up, on the second step one characterizes subspaces of balistic and subbalistic propagation in terms of singular sets for (or the spectra of) appropriate asymptotic observables.

The method described above is a close extension of the method of [SigSof6]. Asymptotic cut-offs $F^+_\Delta(H)$ were introduced in [SigSof5], while cut-offs related to $F^+_\Delta(v)$, in [Gr]. In the present context $F^+_\Delta(v)$ can be replaced by asymptotic projections $F^+_\Delta(K)$ for $K=\left\{\frac{A}{t}\right\}$ (see e.g. sect. 7 and supplement of [SigSof6]). In fact, $F^+_{\{0\}}(v)=F^+_{\{0\}}(K)$. $F^+_\Delta(K)$ were used implicitly in [SigSof6] (see Introduction and sect. 5, especially eqns (5.10) and (5.13)). Asymptotic observables in the context of the QM scattering theory were first used in [En1]. For recent important results see [Der2, En3]. Our treatment of asymptotic clustering follows [SigSof6, GerDer], which extend earlier results of [SigSof3]. A beautiful proof of key estimates of [SigSof3] was given in [Gr]. Important ideas introduced in this proof are used in the present paper (see Appendix). Our estimates on $\mathcal{R}^+_0$ are essentially a simplified version of estimates of [SigSof6]. The latter paper proves the asymptotic clus-

tering on $(\mathcal{R}^+_{\tau,\varepsilon})^\perp$ $\forall \varepsilon > 0$ and the reduction to a time-independent Schrödinger operator, on $\mathcal{R}^+_{\tau,\varepsilon}$ for $\varepsilon > 0$ sufficiently small. Here $\mathcal{R}^+_{\tau,\varepsilon} = \operatorname{Ran} F^+_\tau(H) \cap \operatorname{Ran} F^+_{[-\varepsilon,\varepsilon]}(K)$. An important step of taking $\varepsilon = 0$, i.e. restricting not only to the singular subset τ of H, but also to the singular subset $\{0\}$ of $\frac{A}{t}$ (or of $\frac{|x|}{t}$), and proving the sharp localization of x on this subspace directly (rather than using the sharp localization of H as was done in [SigSof6]), was suggested by [Der3]. These ideas inspired the result of theorem 5.1, extending theorem 13.2 of [SigSof6].

In this paper we use the following convention for cut-off functions. In general, $F(\lambda \in \Omega)$, for a Borel set $\Omega \subset \mathbb{R}$, stands for a smoothed out characteristic function of Ω. More precisely, let ω be a smooth function on $\mathbb{R}$, supported in $|\lambda| \leq 2$ and satisfying $\omega \geq 0$, $\int \omega = 1$ and $\omega(0) = 1$. Let $\omega_\delta(\lambda) = \delta^{-1}\omega\Big(\frac{\lambda}{\delta}\Big)$. If $|\Omega| > 0$, then we set

$$F(\lambda \in \Omega) \;=\; \chi_\Omega * \omega_\delta \;,$$

where χ_Ω is the characteristic function of Ω and $\delta \leq |\Omega|$. Thus we have

$$F(\lambda \in \Omega) + F\big(\lambda \in (\mathbb{R}\backslash\Omega)\big) \;=\; 1 \;.$$

Moreover, we set for any $a \in \mathbb{R}$

$$\begin{aligned} F(\lambda = a) \;&=\; \delta F'(\lambda \geq a) \\ &=\; \omega\Big(\frac{\lambda - a}{\delta}\Big) \;. \end{aligned}$$

The parameter δ^{-1} will be called the sharpness of the cut-off function in question. Let $|\Omega|$ denote the Lebesgue measure of $\Omega \subset \mathbb{R}$. Then for $F(\lambda \geq \lambda_0)$ the sharpness is reciprocal to $\delta = |\operatorname{supp} F'(\lambda \geq \lambda_0)|$ and similarly for $F(\lambda \leq \lambda_0)$. Symbol $\langle x \rangle$ stands for a smooth function equal to $|x|$ for $|x| \geq 1$ and not less than $\frac{1}{2}$ for $|x| \leq 1$. Note that $\langle x \rangle$ is positive homogeneous of degree 1 for $|x| \geq 1$ and is invertible. The main statements are formulated for $t \to \pm\infty$, while all the other statements as well as the proofs are given for $t \to \infty$ only. $O_i(R^{-1})$, where R is either $\langle x \rangle^\alpha$ or $\langle t \rangle^\alpha$, stands for an operator s.t.

for every $s \in [0,1]$ and for every $k = 0, \ldots, i$, $R^s(-\Delta+1)^{-k}O_i(R^{-1})(-\Delta+1)^{-i+k}R^{1-s}$ extends to a bounded operator.

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2. Hamiltonians and Kinematics

Consider an N-body system in the physical space R^ν. The configuration space in the center-of-mass frame is ([SS])

$$X = \{x \in R^{\nu N} | \Sigma m_i x_i = 0\} \tag{2.1}$$

with the inner product $< x, y >= 2\Sigma m_i x_i \cdot y_i$. Here $m_i > 0$ are masses of the particles in question. The *Schrödinger operator* of such a system is

$$H = -\Delta + V(x) \qquad \text{on} \qquad L^2(X).$$

Here Δ is the Laplacian on X and $V(x) = \Sigma V_{ij}(x_i - x_j)$, where (ij) runs through all the pairs satisfying $i < j$. We assume that the potentials V_{ij} are real and obey: $V_{ij}(y)$ are Δ_y-compact. Under this condition Kato theorem applies and H is self-adjoint on $D(H) = D(\Delta)$ (see e.g. [CFKS]). Moreover, if V_{ij} are Kato potentials, i.e. $V_{ij} \in L^r(R^\nu) + \left(L^\infty(R^\nu)\right)_\varepsilon$, where $r > \frac{\nu}{2}$, if $\nu \geq 4$ and $r = 2$ if $\nu \leq 3$, and the subscript ε indicates that the L^∞-component can be taken arbitrarily small, then V_{ij} are Laplacian compact (see e.g. [CFKS]).

We introduce the conditions on the potentials which are used in this paper.

(A) $|y|^{|\beta|}|\partial^\beta V_{ij}(y)|$ are Δ_y-compact for $|\beta| = 0, 1, 2$.

(B) There is $R > 0$ s.t. for $|y| > R$, $V_{ij} \in C^2$ and obey $|\partial^\beta V(y)| \leq C|y|^{-\mu-|\beta|}$ with $|\beta| = 1, 2$ and $\mu > 0$.

In our analysis we will use an induction in number of particles. We will see later that already at the first inductive step one arrives at time-dependent Schrödinger operators of the form

$$H(t) = H + W(t) \ , \tag{2.2}$$

where H is a self-adjoint N particle Schrödinger operator satisfying restrictions (A) and (B) and $W(t)$ is the multiplication operator by a real and smooth function $W(x,t)$ obeying

$$|\partial^\alpha_{(x,t)} W(x,t)| \ \leq \ C_\alpha (1+|t|)^{-\mu-|\alpha|} \tag{2.3}$$

with $\mu > 0$ the same, though it is not necessary, as in condition (B). Thus, guided by the consideration of induction convenience, we consider in this paper time-dependent Schrödinger operators described above. By $U(t)$ we denote the evolution operator generated by $H(t)$, i.e. the solution to the Cauchy problem

$$i\frac{\partial}{\partial t} U(t) \ = \ H(t)U(t) \qquad \text{and} \quad U(0) = \mathrm{id} \ . \tag{2.4}$$

We denote $\psi_t = U(t)\psi$.

The momentum space X', which is dual to X, is identified with

$$X' \ = \ \{k \in R^{\nu N} | \Sigma k_i = 0\} \text{ with the inner product } < k, u >= \Sigma \frac{1}{2m_i} k_i \cdot u_i. \tag{2.5}$$

Thus $|k|^2$ is the symbol of $-\Delta$ and $-\Delta = |p|^2$. Using the natural bilinear form on $X \times X'$, $< x, k >= \Sigma x_i \cdot k_i$, we introduce self-adjoint operators playing a prominent role in our analysis:

$$A = \frac{1}{2}(\langle p, x\rangle + \langle x, p\rangle) \ , \tag{2.6}$$

the generator of dilations, and

$$\gamma = \frac{1}{2}(\langle p, \hat{x}\rangle + \langle \hat{x}, p\rangle) \ , \tag{2.7}$$

the operator associated with the angle between the velocity and coordinate. Here $\hat{x} = \nabla(\langle x \rangle)$. Note that

$$\gamma = \frac{1}{2} i[H, \langle x \rangle] \ . \tag{2.8}$$

Finally, we denote $E_\Delta = F(H \in \Delta)$ for a Borel set $\Delta \subset \mathbb{R}$.

3. Time-Dependent Observables

In this section we outline some technical ideas used in this paper. By the time-dependent observable we usually mean a norm-differentiable family, $\phi(t)$, of self-adjoint bounded operators, which map $D(H)$ into itself. In those few cases when we use unbounded operators it is clear from the construction how to manipulate them. Define

$$D\phi(t) = \frac{\partial \phi(t)}{\partial t} + i[H(t), \phi(t)].$$

We call this operation the *Heisenberg derivative*. (It is similar to the Lagrange derivative in Classical Mechanics.) Its property which warranted its introduction is $\frac{d}{dt}\langle \phi(t) \rangle_t = \langle D\phi(t) \rangle_t$, where $\langle \phi(t) \rangle_t = \langle \phi(t)\psi_t, \psi_t \rangle$. We will use the following refinement of this relation

$$\frac{d}{dt}\langle E_\Delta \phi(t) E_\Delta \rangle_t = \langle E_\Delta D\phi(t) E_\Delta \rangle_t + O(t^{-1-\mu}) \ . \tag{3.1}$$

It follows if we observe that estimate (2.3) on $W(x,t)$ and an elementary commutator estimate of [SigSof4, lemma A.1(i)] imply that

$$\big[E_\Delta, W(x,t)\big] = O(\langle t \rangle^{-1-\mu}) \ . \tag{3.2}$$

We will seek estimates of the form

$$E_\Delta D\phi(t) E_\Delta \geq \theta E_\Delta F(t)^2 E_\Delta - \sum F_i(t)^2, \tag{3.3}$$

where $\theta > 0$ and F and F_i are time-dependent observables with F_i satisfying

$$\int_0^\infty \|F_i(t)\psi_t\|^2 dt \leq C\|\psi\|^2 \quad \text{for all} \quad i \tag{3.4}$$

and for all $\psi \in \operatorname{Ran} E_\Delta$. Then the desired propagation estimate is given in the following:

Lemma 3.1. *Let (3.3)-(3.4) hold. Let either (a) $E_\Delta \phi(t) E_\Delta$ be bounded uniformly in t or (b) $E_\Delta \phi(t) E_\Delta \leq 0$. Then either*

$$\text{(a)} \int_0^\infty \|F(t) E_\Delta \psi_t\|^2 dt \leq C\|\psi\|^2 \text{ or (b)} \int_0^\infty \|F(t) E_\Delta \psi_t\|^2 dt \leq -\langle \phi(0) E_\Delta \psi, E_\Delta \psi \rangle. \tag{3.5}$$

for all $\psi \in L^2$.

Proof. Integrating eqn (3.1) in t from 0 to T, we obtain

$$\int_0^T \langle E_\Delta D\phi(t) E_\Delta \rangle_t dt = \langle E_\Delta \phi(T) E_\Delta \rangle_T - \langle E_\Delta \phi(0) E_\Delta \rangle_0 \tag{3.6}$$

Using inequalities (3.3) and (3.4) to estimate the l.h.s. below and using the conditions of this lemma to estimate the r.h.s. above, we arrive at (3.5). □

The most difficult task in this enterprise is to choose appropriate observables $\phi(t)$ and to prove for them estimates of the form (3.3)–(3.4). In this task we are aided by the following technical devices borrowed from [SigSof4] (see also [SigSof3,6]). We consider smooth functions f obeying

$$\int_{-\infty}^{\infty} |s|^n |\hat{f}(s)| ds < \infty \tag{3.7}$$

for some positive integer n. It is shown in [SigSof4, lemma 1.1] (see also [SigSof1,3,6]) that for f obeying (3.7) with $n = 2$ and a self-adjoint operator B s.t. $B, [B,H]$ and $\big[B,[B,H]\big]$ are H-bounded and $\big[B,[B,H]\big] = O_1(R^{-2})$, where R is either $\langle x \rangle^s$ or t^s, the following relation holds

$$\big[H, f(B)\big] = [H,B] f'(B) + O_1(R^{-2}) \tag{3.8}$$

Moreover, let $\phi(x)$ obey $|\partial^\alpha \phi(x)| \leq C_\alpha R^{-|\alpha|-\lambda}$ for $|\alpha| \leq 2$, where $\lambda \geq 0$. Then (see [SigSof1, lemmas A.4 and A.5] and also [SigSof3,4])

$$\Big[f(\gamma), \phi(x)\Big] = O(R^{-1-\lambda}), \tag{3.9}$$

$$\sigma[f(\gamma), g(H/\sigma)] \;=\; O(\langle x\rangle^{-1}) \tag{3.10}$$

and

$$\sigma\big[\phi(x), g(H/\sigma)\big] \;=\; O(R^{-1-\lambda})\;, \tag{3.11}$$

where the r.h.s.'s are uniform in σ.

Below we will use products of cut-off functions in γ, $\frac{x}{t}$ and H. Relations (3.9)-(3.11) render the order of the factors in such products immaterial for our purposes. Moreover, any such product differs by $O(t^{-s})$ or by $O(\langle x\rangle^{-s})$ with $s > 0$ from its self-adjoint symmetric rearrangement, e.g. by placing square roots of cut-off functions symmetrically as in

$$f(\gamma)g\big(\frac{\langle x\rangle}{t}\big) \;=\; f(\gamma)^{\frac{1}{2}}g\big(\frac{\langle x\rangle}{t}\big)f(\gamma)^{\frac{1}{2}} + O(t^{-1})\;,$$

etc. Having this in mind we will manipulate with such products as if they are symmetrically rearranged. Thus, e.g. an inequality between two such products means, in fact, the inequality between their symmetrical rearrangements modulo $O(t^{-s})$ or $O(\langle x\rangle^{-s})$, $s > 0$.

For a linear subspace R of $L^2(X)$ and a family $\phi(t)$ of bounded operators we define

$$\phi(t) \;\doteq\; 0 \quad \text{on } R$$

iff the following inequality

$$\int_1^\infty |\langle \phi(t)\psi_t, \psi_t\rangle| \frac{dt}{t} \;\le\; C\|\psi\|^2$$

holds for every $\psi \in R$. Here C is independent of individual ψ's (but is dependent on R) and, recall, $\psi_t = U(t)\psi$.

4. Asymptotic Observables

In this section we introduce asymptotic observables of interest and singular projections associated with them. We begin with some definitions. Let τ be the threshold set of H (see section 8 for the precise definition). An important fact about it is that it is a closure of a discrete and bounded subset of $\mathbb{R}$. In fact, it has a finite number $(\leq N-1)$ of generations of accumulation points. For $\mathcal{A} \subset \mathbb{R}$ we set

$$\Sigma_{\mathcal{A}} \;=\; \{(E-\lambda)^{\frac{1}{2}} \;|\; \lambda \in \tau,\; E \in \mathcal{A},\; \lambda \leq E\}\;. \tag{4.1}$$

For any $K \geq 0$, let P^K be the orthogonal projection on the span of K eigenfunctions of H obeying

$$P^K \;\to\; P \qquad \text{as} \quad K \to \infty\;, \tag{4.2}$$

strongly. Here P is the orthogonal projection on the pure point spectrum subspace of H. It is shown in [SigSof1, Prop. 7.2 and its proof] with extensions formulated and discussed in [SigSof6, section 7] that for any discrete and compact set $\mathcal{A} \subset \mathbb{R}$ and any $\delta \notin \Sigma_{\mathcal{A}}$ there are $\delta_0 > 0$, $K > 0$ and $\theta > 0$ s.t.

$$\begin{aligned} & E_\Delta\, i\Big[H, F\Big(\frac{\gamma}{\delta} \geq 1\Big)\Big] E_\Delta \\ \geq\; & \frac{\theta}{\delta} E_\Delta \frac{1}{\sqrt{\langle x\rangle}} F'\Big(\frac{\gamma}{\delta} \geq 1\Big) \frac{1}{\sqrt{\langle x\rangle}} E_\Delta \\ & - C\delta^{-1} E_\Delta \frac{1}{\sqrt{\langle x\rangle}} R \frac{1}{\sqrt{\langle x\rangle}} E_\Delta - C(\delta\langle x\rangle)^{-2} \end{aligned} \tag{4.3}$$

for any $\Delta \supset \mathcal{A}$ with $|\Delta| \leq \delta_0$ and any $F(\gamma/\delta \geq 1)$ satisfying $|\mathrm{supp}\, F'\Big(\frac{s}{\delta} \geq 1\Big)| \leq \delta_0 |\delta|^{-1}$. Here C is independent of δ and $\langle x\rangle$ and

$$R \;=\; F'\Big(\frac{\gamma}{\delta} \geq 1\Big)^{\frac{1}{2}} P^K F'\Big(\frac{\gamma}{\delta} \geq 1\Big)^{\frac{1}{2}}\;. \tag{4.4}$$

In [SigSof1, prop. 7.2] eqn (4.3) is formulated for $\mathcal{A} = \{E\}$ away from the thresholds and eigenvalues of H, however the proof given there holds in the framework of (4.3). We sketch here its simplified version. [SigSof1, thm 6.1 (the channel expansion theorem) and

lemma 6.5] imply that for any $\varepsilon > 0$ and any discrete and compact $\mathcal{A} \subset \mathbb{R}$ there are $\delta_0 > 0$ and $K \geq 0$ s.t. for any $\delta > 0$

$$E_\Delta i[H,A]E_\Delta \;\geq\; \big(\theta(\mathcal{A},\delta) - \varepsilon\big)(E_\Delta)^2 \\ - C E_\Delta F\Big(|\gamma| \leq \delta + \frac{2|\Delta|}{|\delta|}\Big) E_\Delta - C E_\Delta P^K E_\Delta$$

for any interval $\Delta \supset \mathcal{A}$ with $|\Delta| \leq \delta_0$ (see also [SigSof6, section 7]). Here

$$\theta(\mathcal{A},\delta) \;=\; 2\min\{E - \lambda \;\big|\; \lambda \in \tau,\; E \in \mathcal{A},\; E - \lambda \geq \delta^2\}\;.$$

If $\delta \notin \Sigma_\mathcal{A}$, then, since $\Sigma_\mathcal{A}$ is a discrete set,

$$\theta(\mathcal{A},\delta) \;>\; 2\delta^2\;.$$

Combining the last inequalities with (3.8) (with $B = \gamma$ and $R = \langle x\rangle$) and with

$$i[H,\gamma] \;=\; \frac{1}{\sqrt{\langle x\rangle}}(i[H,A] - 2\gamma^2)\frac{1}{\sqrt{\langle x\rangle}} + O(\langle x\rangle^{-2})$$

and using commutation relations (3.10) and the localization property of $F'(\gamma/\delta \geq 1)$, one arrives at (4.3).

We begin with a result parallel to that of [SigSof4, thm 4.1] on existence of asymptotic energy cut-offs:

Theorem 4.1. *Let $\mathcal{A}$ be a discrete and compact subset of $\mathbb{R}$, $\frac{\varepsilon}{2} \notin \Sigma_\mathcal{A}$ and $\varepsilon > 0$. Then there are $\delta_0, \delta_1 > 0$ s.t. the following strong limits exist*

$$F^\pm_\varepsilon \;=\; s - \lim_{t\to\pm\infty} U(t)^* E_\Delta F\Big(\frac{\langle x\rangle}{\varepsilon|t|} \leq 1\Big) E_\Delta U(t)\;, \tag{4.5}$$

provided $\Delta \supset \mathcal{A}$ and $|\Delta| \leq \delta_0$ and $|\mathrm{supp}\left(F'\left(\frac{\lambda}{\varepsilon} \leq 1\right)\right)| \leq \delta_1$. Moreover, if $\varepsilon < \varepsilon'$ and $\Delta \subset \Delta'$ are s.t.

$$F\Big(\frac{\lambda}{\varepsilon'} \leq 1\Big)F\Big(\frac{\lambda}{\varepsilon} \leq 1\Big) \;=\; F\Big(\frac{\lambda}{\varepsilon} \leq 1\Big) \tag{4.6}$$

and

$$E_{\Delta'}(\lambda)E_\Delta(\lambda) \;=\; E_\Delta(\lambda)\;, \tag{4.7}$$

then

$$F^\pm_{\varepsilon'}F^\pm_\varepsilon \;=\; F^\pm_\varepsilon\;. \tag{4.8}$$

Proof. As usual we consider only the case $t \to +\infty$. We begin with

Lemma 4.2. *Let* $\delta \notin \Sigma_{\mathcal{A}}$ *and* $\varepsilon > 0$. *Then there are* $\delta_0, \delta_1 > 0$ *s.t.*

$$F\Big(\frac{\langle x\rangle}{\varepsilon t} \geq 1\Big) F\Big(\frac{\langle x\rangle}{\sigma t} \leq 1\Big) F\Big(\frac{\gamma}{\delta} \leq 1\Big) \doteq 0 \,, \tag{4.9}$$

provided $2\delta < \varepsilon$,

$$F\Big(\frac{\langle x\rangle}{\varepsilon t} = 1\Big) F\Big(\frac{\langle x\rangle}{\sigma t} \leq 1\Big) F\Big(\frac{\gamma}{\delta} \geq 1\Big) \doteq 0 \,, \tag{4.10a}$$

provided $2\delta > \varepsilon$, *and*

$$F\Big(\frac{\langle x\rangle}{\varepsilon t} \geq 1\Big) F\Big(\frac{\langle x\rangle}{\sigma t} \leq 1\Big) F\Big(\frac{\gamma}{\delta} = 1\Big) \doteq 0 \tag{4.10b}$$

on $\mathrm{Ran}\, E_\Delta$, *for any* $\sigma \geq 1$ *and for any* Δ *with* $\Delta \supset \mathcal{A}$ *and* $|\Delta| \leq \delta_0$, *provided* $|\mathrm{supp}\, F\Big(\frac{\lambda}{\delta} = 1\Big)| + |\mathrm{supp}\, F'\Big(\frac{\lambda}{\varepsilon} \geq 1\Big)| \leq \delta_1$.

Proof. Since the proof of the first relation is a simplified version of the proof of the other two plus a covering argument (for the domain $\big\{\varepsilon \leq \frac{\langle x\rangle}{t} \leq \sigma\big\}$ by sets $\big\{\big|\frac{\langle x\rangle}{t} - \nu\big| \leq \delta_1\big\}$ with $\varepsilon \leq \nu \leq \sigma$), we do here only the latter. Pick up $\sigma \geq 1$ so large that the maximal velocity bound

$$F\Big(\frac{\langle x\rangle}{\sigma t} = 1\Big) \doteq 0 \tag{4.11}$$

holds on $\mathrm{Ran}\, E_\Omega$, where Ω is a fixed bounded interval containing $\mathcal{A}$ (see [SigSof3, thm 4.3] and [SigSof4, thm 5.2]). Let first $2\delta > \varepsilon$. We consider the propagation observable

$$\phi = F_1\Big(\frac{\langle x\rangle}{\varepsilon t} \geq 1\Big) F_2\Big(\frac{\langle x\rangle}{\sigma t} \leq 1\Big) F_3\Big(\frac{\gamma}{\delta} \geq 1\Big) \,. \tag{4.12}$$

We labeled the cut-off functions in order to simplify keeping track of them. Using (2.8) and (3.8), we compute its Heisenberg derivative

$$\begin{aligned} D\phi \;=\; & \frac{1}{\varepsilon t}\Big(2\gamma - \frac{\langle x\rangle}{t}\Big) F_1' F_2 F_3 \\ & + F_1 \frac{1}{\sigma t}\Big(2\gamma - \frac{\langle x\rangle}{t}\Big) F_2' F_3 \\ & + F_1 F_2 i[H, F_3] + O(t^{-2}) \,. \end{aligned} \tag{4.13}$$

Using that $2\delta > \varepsilon$ and picking $|\mathrm{supp}\, F'\Big(\frac{s}{\delta} \geq 1\Big)| + |\mathrm{supp}\, F'\Big(\frac{s}{\varepsilon} \geq 1\Big)| < \delta - \frac{\varepsilon}{2}$, we conclude that the first term on the r.h.s. is

$$\geq \theta_1 t^{-1} F_1' F_2 F_3 - Ct^{-2} \quad \text{with} \quad \theta_1 = \Big(\delta - \frac{\varepsilon}{2}\Big)\varepsilon^{-1} > 0 \ .$$

Next, equation (4.3) implies that there are δ_0, δ_1 and K s.t. the third term on the r.h.s. sandwiched by E_Δ on each side is $\geq \theta_2 t^{-1} E_\Delta F_1 F_2 F_3' E_\Delta - Ct^{-1} E_\Delta R_1 E_\Delta$ for some $\theta_2 > 0$ and for any Δ with $\Delta \supset \mathcal{A}$ and $|\Delta| \leq \delta_0$, provided $|\mathrm{supp}\, F'\Big(\frac{\gamma}{\delta} \geq 1\Big)| \leq \delta_1$. Here

$$R_1 \ = \ F_1^{\frac{1}{2}} F_2^{\frac{1}{2}} (F_3')^{\frac{1}{2}} P^K (F_3')^{\frac{1}{2}} F_2^{\frac{1}{2}} F_1^{\frac{1}{2}} \ . \tag{4.14}$$

These two conclusions together with the fact that γ is H-bounded yield, under the above restrictions, that

$$\begin{aligned} E_\Delta D\phi E_\Delta \geq & \frac{\theta_1}{t} E_\Delta F_1' F_2 F_3 E_\Delta + \frac{\theta_2}{t} E_\Delta F_1 F_2 F_3' E_\Delta \\ & - \frac{C}{t} E_\Delta F_1 F_2' F_3 E_\Delta - \frac{C}{t} E_\Delta R_1 E_\Delta - \frac{C}{t^2} \ . \end{aligned} \tag{4.15}$$

In order to estimate the contribution by R_1 we use (3.9) to commute the $F_3^{'\frac{1}{2}}$'s on the r.h.s. of (4.14) to the outside positions: $R_1 = R_2 + O(t^{-1})$ with

$$R_2 \ = \ (F_3')^{\frac{1}{2}} F_1^{\frac{1}{2}} F_2^{\frac{1}{2}} P^K F_2^{\frac{1}{2}} F_1^{\frac{1}{2}} (F_3')^{\frac{1}{2}} \ . \tag{4.16}$$

Now, denoting $\varphi(t) = (F_3')^{\frac{1}{2}} E_\Delta \psi_t$, we rewrite

$$\begin{aligned} & \int_1^\infty |\langle R_2 \psi_t, \psi_t \rangle| \frac{dt}{t} \\ = & \int_1^\infty \| P^K F_2^{\frac{1}{2}} F_1^{\frac{1}{2}} \varphi(t) \|^2 \frac{dt}{t} \ . \end{aligned}$$

Write $P^K f = \sum_{\text{finite}} \psi^\sigma \langle \psi^\sigma, f \rangle$. Then

$$\begin{aligned} & \int_1^\infty |\langle R_2 \psi_t, \psi_t \rangle| \frac{dt}{t} \\ = & \sum_{\text{finite}} \int_1^\infty |\langle \psi^\sigma, F_2^{\frac{1}{2}} F_1^{\frac{1}{2}} \varphi(t) \rangle|^2 \frac{dt}{t} \\ \leq & C \|\psi\|^2 \sum_{\text{finite}} \int_1^\infty \| F_1^{\frac{1}{2}} F_2^{\frac{1}{2}} \psi^\sigma \|^2 \frac{dt}{t} \ . \end{aligned}$$

Since

$$\begin{aligned} & \int_1^\infty \|F_1^{\frac{1}{2}} F_2^{\frac{1}{2}} \psi^\sigma\|^2 \frac{dt}{t} \\ \leq & \int\!\!\int_{\substack{\frac{\varepsilon}{2} t \leq \langle x \rangle \leq 2\sigma t \\ t \geq 1}} |\psi^\sigma(x)|^2 \frac{dx dt}{t} \\ = & \ln\Big(\frac{4\sigma}{\varepsilon}\Big) \|\psi^\sigma\|^2 \ , \end{aligned} \tag{4.17}$$

we conclude that

$$\int_1^\infty |\langle R_1 \psi_t, \psi_t \rangle| \frac{dt}{t} \leq C \|\psi\|^2 \ . \tag{4.18}$$

Integrating differential inequality (4.15) and using (3.1), (4.11) and (4.18), we conclude that (4.10a) and (4.10b) hold on $\operatorname{Ran} E_\Delta$, provided $2\delta > \varepsilon$. Since for any $\varepsilon_1 > 0$ there is $\varepsilon < \varepsilon_1$ satisfying $0 < \varepsilon < 2\delta$ and

$$\begin{aligned} & F\Big(\frac{\langle x \rangle}{\varepsilon_1 t} \geq 1\Big) \\ = & F\Big(\frac{\langle x \rangle}{\varepsilon_1 t} \geq 1\Big) F\Big(\frac{\langle x \rangle}{\varepsilon t} \geq 1\Big) \ , \end{aligned}$$

we conclude that (4.10b) holds for any $\varepsilon > 0$ and any $\delta \notin \Sigma_{\mathcal{A}}$. $\square$

This lemma implies readily

Corollary 4.3. *Let $\mathcal{A}$ be a discrete and compact subset of $\mathbb{R}$ and $\frac{\varepsilon}{2} \notin \Sigma_{\mathcal{A}}$. Then there are $\delta_0, \delta_1 > 0$, s.t.*

$$F\Big(\frac{\langle x \rangle}{\varepsilon t} = 1\Big) \doteq 0 \tag{4.19}$$

on $\operatorname{Ran} E_\Delta$, for any Δ with $\Delta \supset \mathcal{A}$ and $|\Delta| \leq \delta_0$ and for any cut-off function obeying $|\operatorname{supp} F\Big(\frac{\lambda}{\varepsilon} = 1\Big)| \leq \delta_1$.

Now we return to the proof of theorem 4.1. Using (3.1), we obtain

$$\begin{aligned} & \frac{d}{dt} U(t)^* E_\Delta F\Big(\frac{\langle x \rangle}{\varepsilon t} \leq 1\Big) E_\Delta U(t) \\ = & U(t) E_\Delta DF\Big(\frac{\langle x \rangle}{\varepsilon t} \leq 1\Big) E_\Delta U(t) + O(t^{-1-\mu}) \ . \end{aligned} \tag{4.20}$$

Moreover, using, as before, (2.8) and (3.8), we find

$$DF\Big(\frac{\langle x \rangle}{\varepsilon t} \leq 1\Big) = -\frac{1}{\varepsilon t}\Big(2\gamma - \frac{\langle x \rangle}{\varepsilon t}\Big) F'\Big(\frac{\langle x \rangle}{\varepsilon t} \leq 1\Big) + O\big((\varepsilon t)^{-2}\big) \ .$$

Using that $F'\Big(\frac{\lambda}{\varepsilon} \le 1\Big) = cF\Big(\frac{\lambda}{\varepsilon} = 1\Big)$, where $c > 0$, using that γ is H-bounded and using (4.19), we arrive at

$$\begin{aligned} &\int_1^T \Big|\frac{d}{dt}\Big\langle v, U(t)^* E_\Delta \Big(DF\Big(\frac{\langle x\rangle}{\varepsilon t} \le 1\Big)\Big) E_\Delta U(t) u\Big\rangle\Big| dt \\ &\le\ C\|u\|\|v\| \end{aligned}$$

for any $u, v \in L^2$ and with C independent of the u's and v's. The latter inequality implies that $U(t)^* E_\Delta F\Big(\frac{\langle x\rangle}{\varepsilon t} \le 1\Big) E_\Delta U(t)$ converges strongly as $t \to +\infty$. This proves (4.5).

Now we show (4.8). By (4.5) and by (4.7)

$$\begin{aligned} &F^\pm_{\varepsilon'} F^\pm_\varepsilon \\ &= s - \lim_{t\to\pm\infty} U(t)^* E_{\Delta'} F_{\varepsilon'} E_\Delta F_\varepsilon E_\Delta U(t)\ , \end{aligned} \tag{4.21}$$

where $F_\varepsilon = F\Big(\frac{\langle x\rangle}{t} \le \varepsilon\Big)$. Let $\overline{F}_{\varepsilon'} = \mathbf{1} - F_{\varepsilon'}$. By (4.6) and by (3.11)

$$\overline{F}_{\varepsilon'} E_\Delta F_\varepsilon \ =\ O(t^{-1})\ . \tag{4.22}$$

This together with (4.7) and (4.21) yields (4.8). □

Below by $\varepsilon \to 0$ we mean that, given a discrete and compact set $\mathcal{A}$, ε approaches 0 along the set $\mathbb{R}\backslash\Sigma_{\mathcal{A}}$ (which is possible since $\Sigma_{\mathcal{A}}$ is discrete). We emphasize that in the definition of $F^\pm_\varepsilon$, eqn (4.5), the energy interval Δ depends on ε, while $\varepsilon \notin \Sigma_{\mathcal{A}}$. The next theorem introduces singular projections associated with $\langle x\rangle/t$. It is similar to [SigSof6, thm 4.1] (cf. also [SigSof4, thm 4.1 (ii)]).

Theorem 4.4. *The following bounded operators*

$$F^\pm \ =\ s - \lim_{\varepsilon\downarrow 0} F^\pm_\varepsilon \tag{4.23}$$

exist and obey

$$(F^\pm)^2 \ =\ F^\pm\ , \tag{4.24}$$

$$(F^{\pm})^* \;=\; F^{\pm} \;\geq\; 0 \tag{4.25}$$

and for any $\varepsilon > 0$

$$F_{\varepsilon}^{\pm} F^{\pm} \;=\; F^{\pm} \;. \tag{4.26}$$

Proof. Note that $F_{\varepsilon}^{\pm} \geq 0$ and are monotonically non-increasing as $\varepsilon \downarrow 0$. Hence there are $F^{\pm} \geq 0$ s.t. $F_{\varepsilon}^{\pm} \downarrow F^{\pm}$ as $\varepsilon \downarrow 0$, weakly. Picking ε', Δ', ε and Δ so that (4.6) and (4.7) hold and using (4.8), we find

$$\begin{aligned} &\|(F_{\varepsilon'}^{\pm} - F_{\varepsilon}^{\pm})u\|^2 \\ &= \|F_{\varepsilon'}^{\pm} u\|^2 + \|F_{\varepsilon}^{\pm} u\|^2 - 2\langle F_{\varepsilon}^{\pm} u, u\rangle \;. \end{aligned} \tag{4.27}$$

Since $0 \leq F_{\varepsilon}^{\pm} \leq 1$, we find furthermore

$$\begin{aligned} &\|(F_{\varepsilon'}^{\pm} - F_{\varepsilon}^{\pm})u\|^2 \\ &\leq \|(F_{\varepsilon'}^{\pm})^{\frac{1}{2}} u\|^2 + \|(F_{\varepsilon}^{\pm})^{\frac{1}{2}} u\| - 2\langle F_{\varepsilon}^{\pm} u, u\rangle \\ &= \langle F_{\varepsilon'}^{\pm} u, u\rangle - \langle F_{\varepsilon}^{\pm} u, u\rangle \;\to\; 0 \end{aligned}$$

as $\varepsilon', \varepsilon \to 0$. Hence $F_{\varepsilon}^{\pm} \to F^{\pm}$ strongly as $\varepsilon \to 0$. (4.25) is obvious and (4.24) and (4.26) follow from (4.23) and (4.8). $\square$

Next, we need the asymptotic energy cut-offs of [SigSof6]. That paper states that the following limits

$$E_{\Delta}^{\pm} \;=\; s - \lim_{t \to \pm\infty} U(t)^* E_{\Delta} U(t)$$

exist for any open Borel set Δ. Moreover, if $\mathcal{A}$ is a Borel set of measure 0, then

$$Q_{\mathcal{A}}^{\pm} \;=\; s - \lim_{\Delta \to \mathcal{A}} E_{\Delta}^{\pm} \;,$$

where the limit is taken over a sequence of open sets containing $\mathcal{A}$ and shrinking to it, exist and are independent of the sequence of Δ's taken. It is shown in [SigSof6] that $Q_{\mathcal{A}}^{\pm}$ are self-adjoint projections and that $Q_{\mathcal{A}}^{\pm} Q_{\mathcal{B}}^{\pm} = Q_{\mathcal{A}\cap\mathcal{B}}^{\pm}$ (this is not used in this paper) and

$$\|\psi_t - E_{\Delta}\psi_t\| \;\leq\; C|\Delta|^{-1} t^{-\mu} \tag{4.28}$$

for any Δ containing $\mathcal{A}$, provided $\psi \in \operatorname{Ran} Q_{\mathcal{A}}^{\pm}$. Here C is independent of t and of Δ (the independence of Δ is not used in this paper). Property (4.28) and theorem 4.4 yield

Theorem 4.5. *Let $\mathcal{A}$ be a discrete and compact subset of $\mathbb{R}$. Then the following limits*

$$F_\varepsilon^\pm \;=\; s-\lim_{t\to\pm\infty} U(t)^* F\Big(\frac{\langle x\rangle}{\varepsilon|t|}\le 1\Big)U(t)\;,$$

with $\varepsilon>0$ and $\varepsilon\notin\Sigma_{\mathcal{A}}$, and

$$F^\pm \;=\; s-\lim_{\varepsilon\downarrow 0} F_\varepsilon^\pm$$

exist on $\mathrm{Ran}\, Q_{\mathcal{A}}^\pm$. Moreover, these limits coincide (on $\mathrm{Ran}\, Q_{\mathcal{A}}^\pm$, of course) with (4.5) and (4.23), respectively.

We will also need the following

Theorem 4.6. *Operators E_Δ^+ and F^+ (resp. $Q_{\mathcal{A}}^+$ and F^+) commute. The statement remains true if $+$ is replaced by $-$.*

Proof. By the definition, by a property of strong convergence and by uniform boundedness of F_ε^+ and E_Δ^+,

$$[F^+,Q_{\mathcal{A}}^+] \;=\; s-\lim_{\substack{\varepsilon\to 0\\ \Delta\to\mathcal{A}}} [F_\varepsilon^+,E_\Delta^+]\;,$$

where the limits can be taken in any order. Again, by a property of strong convergence and uniform boundedness of $U(t)^*E_{\Delta_1}F\big(\frac{\langle x\rangle}{\varepsilon t}\le 1\big)E_{\Delta_1}U(t)$ and of $U(t)^*E_\Delta U(t)$,

$$[F_\varepsilon^+,E_\Delta^+] \;=\; \lim_{t\to+\infty} U(t)^*E_{\Delta_1}\big[F\big(\frac{\langle x\rangle}{\varepsilon t}\le 1\big),E_\Delta\big]E_{\Delta_1}U(t)\;.$$

By (3.11) with $R=t$, the commutator on the r.h.s. is $O(t^{-1})$. Hence $[F_\varepsilon^+,E_\Delta^+]=0$, which yields $[F^+,Q_{\mathcal{A}}^+]=0$. $\square$

We denote $Q^\pm=Q_{\mathcal{A}}^\pm$ whenever $\mathcal{A}$ is the *threshold set*, τ, of H.

Theorem 4.7. $\mathrm{Ran}\, F^\pm\subseteq\mathrm{Ran}\, Q^\pm$.

Proof. Let $\psi\in(\mathrm{Ran}\, Q^+)^\perp$. Then, by the definition of Q^+, for any $\varepsilon>0$ there is a closed set Ω disjoint from τ and s.t.

$$\|\psi-E_\Omega^+\psi\| \;\le\; \varepsilon\;. \tag{4.29}$$

By the minimal velocity estimate ([SigSof4, thm 5.1], see also [SigSof3, thm 4.2]) there is $\delta > 0$ s.t.

$$\Big\| F\Big(\frac{\langle x\rangle}{t} \leq \delta\Big) E_\Omega \psi_t \Big\| \;\to\; 0$$

as $t \to +\infty$. In terms of asymptotic cut-offs this can be rewritten as $F^+_\delta E^+_\Omega \psi = 0$, i.e. $E^+_\Omega \psi \in (\mathrm{Ran}\, F^+_\delta)^\perp \subset (\mathrm{Ran}\, F^+)^\perp$. Taking this into account and remembering that $\varepsilon > 0$ in (4.29) is arbitrary, we obtain that $\psi \in (\mathrm{Ran}\, F^+)^\perp$. $\square$

5. Microlocal analysis on $\mathrm{Ran}\, F^\pm$

In this section we show that $\langle x\rangle$ is sharply localized on $\mathrm{Ran}\, F^\pm$. We begin with

Theorem 5.1. *Let $\delta > 0$ and $\varepsilon > 0$. Then there is $\Delta \supset \tau$ s.t. for any $\alpha \in (0,1]$ and any $\psi \in \mathrm{Ran}\, F^+$ we have*

$$\|F\Big(t^{1-\alpha}\frac{\gamma}{\delta} \geq 1\Big) F\Big(\frac{\langle x\rangle}{\varepsilon t^\alpha} \geq 1\Big) E_\Delta \psi_t\|^2 \leq\; C t^{3-2\alpha-\eta}\;, \tag{5.1}$$

where $\eta = (1 - 2^{-N-1})^{-1}$.

Proof. Let γ_t be the self-adjoint operator related to γ and defined in theorem A.1. Let first $\delta \notin \Sigma_{\mathcal{A}}$ and $\varepsilon < 2\delta$. Consider the propagation observable

$$\phi \;=\; F\Big(\frac{\langle x\rangle}{\varepsilon t} \geq 1\Big) F\Big(\frac{\gamma_t}{\delta} \geq 1\Big) F\Big(\frac{\langle x\rangle}{\varepsilon t} \geq 1\Big)\;. \tag{5.2}$$

In the proof below $O(\cdots)$ and $O_1(\cdots)$ stand for an estimate uniform in ε and δ. Using eqn (3.8), we compute its Heisenberg derivative

$$\begin{aligned} D\phi \;&= (\varepsilon t)^{-1}\Big(2\gamma - \frac{\langle x\rangle}{t}\Big) F_1' F_2 F_1 + \text{ adjoint} \\ &\quad + F_1 D F_2 F_1 + O\big((\varepsilon t)^{-2}\big)\;, \end{aligned} \tag{5.3}$$

where we have used our standard abbreviations for cut-off functions and where we have used that

$$\langle x\rangle^{-s} F\Big(\frac{\langle x\rangle}{\varepsilon t} \geq 1\Big) \;=\; O\big((\varepsilon t)^{-s}\big)\;. \tag{5.4}$$

Since $2\delta > \varepsilon$, we can choose the cut-off functions so that

$$2\inf\operatorname{supp} F_2\Big(\frac{\lambda}{\delta} \geq 1\Big) - \sup\operatorname{supp} F_1'\Big(\frac{\lambda}{\varepsilon} \geq 1\Big) > 0 \ . \tag{5.5}$$

Thus, due to (A.1), the first two terms on the r.h.s. are

$$\geq \ \frac{\theta_1}{\varepsilon t}\mathrm{Re}(F_1' F_2 F_1) - C(\varepsilon t)^{-2} \tag{5.6}$$

for some $\theta_1 > 0$. Next, using first (3.8), with $B = \gamma_t$ and $R = t^{-\frac{\eta}{2}}$, then symmetrizing with the help of estimates of the type of (3.10) and (A.2), we obtain

$$DF_2 \ = \ \frac{1}{\delta}(F_2')^{\frac{1}{2}} D\gamma_t (F_2')^{\frac{1}{2}} + O_1\big(\delta^{-2}t^{-\eta}\big) \ .$$

Now using (A.3), we derive

$$DF_2 \ \geq \ O_1(\delta^{-2}t^{-\eta}) \ . \tag{5.7}$$

This together with (5.3) and (5.6) yields

$$D\phi \ \geq \ O_1(\varepsilon^{-2}t^{-\eta}) \ . \tag{5.8}$$

Now we introduce, for Δ the same as in lemma 4.2,

$$\phi_\Delta^t \ = \ U(t)^* E_\Delta \phi E_\Delta U(t) \ . \tag{5.9}$$

Combining this with a more precise version, due to (3.11), of (3.1), we obtain

$$\frac{d}{dt}\phi_\Delta^t \ = \ U(t)^* E_\Delta D\phi E_\Delta U(t) + O(|\Delta|^{-1}t^{-1-\mu}) \tag{5.10}$$

Due to eqn (A.1) we can replace in the statement of lemma 4.2 γ by γ_t. This and eqn (5.3) yield that for any $\sigma > 0$

$$\int_1^\infty |\langle E_\Delta u_t, (D\phi)F\big(\frac{\langle x\rangle}{\sigma t} \leq 1\big)E_\Delta v_t\rangle| dt \ \leq \ C\|u\|\|v\| \ .$$

Next we need the maximal velocity estimate of [SigSof3, thm 4.3] (or [SigSof4, thm 3.3]): for any bounded Ω, for $\sigma > 0$ sufficiently large (depending on Ω) and for any $\psi \in L^2$

$$\int_1^\infty \|F\big(\frac{\langle x\rangle}{\sigma t} \geq 1\big) E_\Omega \psi_t\|^2 t^{-\rho} dt \leq C \|\langle x\rangle^{\frac{1}{2}} \psi\|^2 \ , \tag{5.11}$$

where $\rho > \frac{\text{const}}{\sigma}$ with the constant positive and depending only on Ω. The last three estimates and the fact that $D\phi = O_{\frac{1}{2}}(t^{-1})$, yield

$$\int_1^\infty |\frac{d}{dt}\langle \phi_\Delta^t \psi, u\rangle| dt \ \leq \ C \|\langle x\rangle^{\frac{1}{2}} \psi\| \|u\| \ ,$$

which implies that

$$\phi^\infty \ = \ s - \lim_{t\to+\infty} \phi_\Delta^t$$

exist on $\langle x\rangle^{-\frac{1}{2}} L^2$ and therefore on L^2. Let now

$$\phi^t \ = \ U(t)^* \phi U(t) \ .$$

By (4.28) and (3.10)–(3.11) (with $R = \varepsilon t$)

$$\phi^t \ = \ \phi_\Delta^t + O\big((|\Delta|\delta\varepsilon t)^{-1}\big) + O(|\Delta|^{-1} t^{-\mu})$$

on $\operatorname{Ran} Q^+$. Hence

$$\phi^\infty \ = \ s - \lim_{t\to+\infty} \phi^t \tag{5.12}$$

exists on $\operatorname{Ran} Q^+$. Now combining

$$\phi^t - \phi^\infty \ = \ -\int_t^\infty \frac{d}{ds} \phi^s ds \tag{5.13}$$

with (5.8) and with the relation $\frac{d}{ds}\phi^s = U(s)^* D\phi U(s)$, we obtain that on $\operatorname{Ran} Q^+$

$$\phi^t \ \leq \ \phi^\infty + C\varepsilon^{-2} t^{1-\eta} \ . \tag{5.14}$$

Note now that by the definition of F^+ and a property of strong convergence

$$\phi^\infty \psi \ = \ 0 \qquad \text{for any } \psi \in \operatorname{Ran} F^+ \ .$$

The last two relations and the definition of ϕ^t yield

$$\|F\Big(\frac{\gamma}{\delta} \geq 1\Big) F\Big(\frac{\langle x \rangle}{\varepsilon t} \geq 1\Big) \psi_t\|^2 \;\leq\; C\big(\min(\varepsilon, \delta)\big)^{-2} t^{1-\eta} \;, \tag{5.15}$$

provided $\psi \in \mathrm{Ran}\, F^+$ ($\subset \mathrm{Ran}\, Q^+$), $\delta \notin \Sigma_\tau$ and $\varepsilon < 2\delta$. Clearly, the last two restrictions on δ and ε can be dropped. Replacing in (5.15) ε by $\varepsilon t^{\alpha-1}$ and δ by $\delta t^{\alpha-1}$, we arrive at (5.1). □

Now we consider the remaining part of the phase-space. The results which follow are closely related to results of [SigSof6,sect. 12].

Theorem 5.2. *Let* $\alpha > \frac{3-\eta}{2}$, $\varepsilon > 0$ *and* $\delta < \frac{\alpha\varepsilon}{2}$. *Then*

$$F\Big(\frac{\langle x \rangle}{t^\alpha} = \varepsilon\Big) F(t^{1-\alpha}\gamma \leq \delta) \;\doteq\; 0 \tag{5.16}$$

on $\mathrm{Ran}\, E_\Omega$ *for any bounded* Ω.

Proof. Consider the propagation observable

$$\phi \;=\; F\Big(\frac{\langle x \rangle}{t^\alpha} \geq \varepsilon\Big) F(t^{1-\alpha}\gamma_t \leq \delta) \tag{5.17}$$

with the parameters specified in the theorem. Using (3.8) and (A.2), we compute its Heisenberg derivative

$$\begin{aligned} D\phi \;&=\; \frac{1}{t}\Big(t^{1-\alpha} 2\gamma - \alpha \frac{\langle x \rangle}{t^\alpha}\Big) F'F \\ &\quad + F t^{1-\alpha} D\gamma_t F' + O(t^{-2\alpha}) + O_1(t^{1-\alpha-\eta}) \;. \end{aligned}$$

Using (A.1) (and the fact that $\eta_1 > 1-\alpha$), the support properties of the cut-off functions and choosing these functions sufficiently sharp, we conclude that the first term on the r.h.s. is $\leq -\theta t^{-1} F'F$ for some $\theta > 0$. Symmetrizing the second term on the r.h.s. and using (A.2) and (A.3), we find that it is $\leq O_1(t^{2-2\alpha-\eta})$. Thus

$$D\phi \;\leq\; -\theta t^{-1} F'F + C t^{-2\alpha} + O_1(t^{2-2\alpha-\eta}) \;. \tag{5.18}$$

By the condition on α the second and third terms on the r.h.s. are integrable. Applying the first part of lemma 3.1 to this differential inequality and using that $F'\Big(\frac{\langle x\rangle}{t^\alpha} \geq 1\Big) = \text{const} \cdot F\Big(\frac{\langle x\rangle}{t^\alpha} = 1\Big)$ with a positive constant, we arrive at $F\Big(\frac{\langle x\rangle}{t^\alpha} = \varepsilon\Big)F(t^{1-\alpha}\gamma_t \leq \delta) \doteq 0$ on Ran E_Ω for any bounded Ω. Finally, using (A.1) we derive readily that

$$F(t^{1-\alpha}\gamma_t \leq \delta) \;=\; F(t^{1-\alpha}\gamma \leq \delta) + O(t^{1-\alpha-\eta_1}) \;, \tag{5.19}$$

where $\eta_1 = 2^{-N-1}$. Since $\eta_1 > 1-\alpha$, this allows us to pass from the last statement to (5.16). $\square$

Theorem 5.3. *Let* $\frac{3-\eta}{2} < \alpha \leq 1$, $\varepsilon > 0$, $\delta < \frac{\alpha\varepsilon}{2}$ *and* $0 \leq \rho < \min(2\alpha+\eta-3, \alpha+\eta_1-1)$. *Let* Ω *be bounded. Then*

$$\int_1^\infty \|F\Big(\frac{\langle x\rangle}{t^\alpha} \geq \varepsilon\Big)F(t^{1-\alpha}\gamma \leq \delta)E_\Omega\psi_t\|^2 \frac{dt}{t^{1-\rho}} \;\leq\; C\|\langle x\rangle^{\frac{1}{2}}\psi\|^2 \tag{5.20}$$

for any $\psi \in L^2$ *and with* C *independent of the* ψ*'s.*

Proof. For given Ω we pick up $\sigma > 0$ so large that (5.11) holds. Let $f(\lambda) = \lambda F(\lambda \geq \varepsilon)$. With the parameters defined in the lemma we introduce the observable

$$\phi \;=\; -t^\rho f\big(\frac{\langle x\rangle}{t^\alpha}\big)F\big(\frac{\langle x\rangle}{t} \leq \sigma\big)F(t^{1-\alpha}\gamma_t \leq \delta) \;, \tag{5.21}$$

where, recall, γ_t is given in theorem A.1. Compute the Heisenberg derivative

$$\begin{aligned} D\phi \;&= t^{\rho-1}\big(\alpha\frac{\langle x\rangle}{t^\alpha} - \frac{2\gamma}{t^{\alpha-1}}\big)F'FF - \rho t^{\rho-1}fFF \\ &\quad - t^\rho fDFF - t^\rho fFt^{1-\alpha}(D\gamma_t)F' + O_1(t^{\rho+1-\alpha-\eta}) + O(t^{\rho-2\alpha}) \;. \end{aligned} \tag{5.22}$$

Symmetrizing and using (A.2) and (A.3), we estimate the fourth term on the r.h.s. from below by $O_1(t^{\rho+2-2\alpha-\eta})$. Next, using support properties of the first and third cut-off functions and using that $f(\lambda) \leq \lambda f'(\lambda)$, we estimate the first two terms on the r.h.s. from below by $t^{-1}\big((\alpha-\rho)\varepsilon - 2\delta - \delta_1\big)f'FF$, where $\delta_1 = |\text{supp}\, F'(\lambda \geq \varepsilon)| + |\text{supp}\, F'(\lambda \leq \delta)|$. Picking up $\delta_1 \leq \frac{1}{2}\big((\alpha-\rho)\varepsilon - 2\delta\big) \equiv \eta$ and using that $f(\lambda) \geq \varepsilon F(\lambda \geq \varepsilon)$, we conclude that

$$\begin{aligned} D\phi \;&\geq \eta t^{\rho-1}FFF + fDFF \\ &\quad + O_1(t^{\rho+2-2\alpha-\eta}) \;. \end{aligned} \tag{5.23}$$

Next we have

$$DF\big(\frac{\langle x\rangle}{t}\le\sigma\big) \;=\; t^{-1}\big(2\gamma-\frac{\langle x\rangle}{t}\big)F'\big(\frac{\langle x\rangle}{t}\le\sigma\big)+O(t^{-2})\;. \tag{5.24}$$

Using now that $F'\big(\frac{\langle x\rangle}{t}\le\sigma\big)=-F\big(\frac{\langle x\rangle}{t}=\sigma\big)$, times a positive constant, and that γ is $\sqrt{-\Delta}$-bounded and using that $\phi\le 0$ and following the second option in lemma 3.1, we derive from (5.23), (5.24) and (5.11) that

$$\begin{aligned}\int_1^\infty \|F\big(\frac{\langle x\rangle}{t^\alpha}\ge\varepsilon\big)F\big(\frac{\langle x\rangle}{t}\le\sigma\big)F(t^{1-\alpha}\gamma_t\le\rho)E_\Omega\psi_t\|^2\frac{dt}{t^{1-\rho}}\\ \le C\|\langle x\rangle^{\frac12}\psi\|^2\;.\end{aligned} \tag{5.25}$$

Combining this with (5.11) and using (5.19) in order to pass from γ_t to γ, we obtain (5.20). □

Theorem 5.4. *Let $\alpha>\frac{3-\eta}{2}$, $\varepsilon>0$ and $\delta<\frac{\alpha\varepsilon}{2}$. Let Ω be bounded. Then for any $\psi\in L^2$ and as $t\to+\infty$*

$$F\Big(\frac{\langle x\rangle}{t^\alpha}\ge\varepsilon\Big)F\Big(t^{1-\alpha}\gamma\le\delta)E_\Omega\psi_t \;\to\; 0\;. \tag{5.26}$$

Proof. Let ϕ_t be given by (5.17). Equations (5.20) and (5.19) imply that there is a sequence t_n s.t. $t_n\to\infty$ and

$$\phi_{t_n}\psi_{t_n} \;\to\; 0 \tag{5.27}$$

as $n\to\infty$ for any $\psi\in L^2$. Consider now the Heisenberg family

$$\phi^t \;=\; U(t)^*E_\Omega\phi_t^*\phi_tE_\Omega U(t)\;. \tag{5.28}$$

Observe that, due to (3.9) with $R=t^\alpha$ and $\lambda=0$, $\phi_t^*\phi_t$ is, modulo $O(t^{1-2\alpha})$, of the form (5.17). In fact, an estimate of the type of (5.18) can be shown directly for $\phi_t^*\phi_t$. Hence, by (3.1) and (5.18)

$$\frac{d}{dt}\phi^t \;\le\; Ct^{-1-\mu}+O_1(t^{2-2\alpha-\eta})\;. \tag{5.29}$$

Since, by our assumptions on α, the r.h.s. is integrable and since $\phi^t\ge 0$, we obtain that $\langle\phi^t\psi,\psi\rangle=\|\phi_tE_\Omega\psi_t\|^2$ converges as $t\to+\infty$. This together with (5.27) yields that $\phi_tE_\Omega\psi_t\to 0$ as $t\to\infty$. Finally, using (5.19), we can pass in this relation from γ_t to γ to arrive at (5.26). □

Corollary 5.5. *Let* $\alpha > \frac{3-\eta}{2}$. *Then for any* $\psi \in \operatorname{Ran} F^+$

$$F\Big(\frac{\langle x\rangle}{t^\alpha} \geq 1\Big)\psi_t \;\to\; 0 \qquad \text{as } t \to +\infty \;. \tag{5.30}$$

Proof. Combining theorems 5.1 and 5.4, we obtain

$$F\Big(\frac{\langle x\rangle}{t^\alpha} \geq 1\Big)E_\Omega\psi_t \;\to\; 0$$

as $t \to +\infty$, for any bounded Ω . On the other hand by [SigSof4, thm 4.1 (ii)] for any $\delta > 0$ there is a bounded interval Ω s.t.

$$\sup \|(1 - E_\Omega)\psi_t\| \;\leq\; \delta \;.$$

The last two relations yield (5.30). □

6. Reduction

In this section we show that ψ_t , for any $\psi \in \operatorname{Ran} F^+$, approaches as $t \to +\infty$ an orbit generated, modulo a time-dependent factor, by the time-independent Hamiltonian H . Indeed, if

$$U_0(t) \;=\; e^{-iHt}e^{-i\int_0^t W(0,s)ds} \;, \tag{6.1}$$

then the difference between the generators of $U(t)$ and $U_0(t)$, i.e. $W(x,t) - W(0,t)$, is $O(t^{-1-\mu+\alpha})$ in the region $|x| \leq Ct^\alpha$ in which ψ_t localized as $t \to +\infty$. Therefore it is integrable for $\alpha < \mu$. In this section we follow closely the arguments of [SigSof6, sect. 5].

To cast the discussion above into rigorous terms we introduce the wave operators

$$W_0^\pm \;=\; s - \lim_{t\to\pm\infty} U_0(t)^* F\Big(\frac{\langle x\rangle}{|t|^\alpha} \leq 1\Big)U(t) \tag{6.2}$$

whenever they exist.

Theorem 6.1. *Let* $\mu > \alpha > \frac{3-\eta}{2}$. *Then* $W_0^\pm$ *exist on* $\operatorname{Ran} F^\pm$.

Proof. Differentiating the family $W_0(s) \equiv U_0(s)^* F\Big(\frac{\langle x\rangle}{s^\alpha} \leq 1\Big)U(s)$, we find

$$\frac{d}{ds}W_0(s) \;=\; iU_0(s)^*(A + B)U(s) \;, \tag{6.3}$$

where

$$A \;=\; i\big(W(0,s)-W(x,s)\big)F\Big(\frac{\langle x\rangle}{s^\alpha}\le 1\Big) \tag{6.4}$$

and

$$B \;=\; DF\big(\frac{\langle x\rangle}{s^\alpha}\le 1\big)\;. \tag{6.5}$$

By (2.3), $A = O(s^{\alpha-1-\mu})$ and therefore contributes an integrable term into $\frac{d}{ds}W(s)$. Next, let $\delta < \frac{\alpha}{2}$. Using that

$$\begin{aligned} DF\Big(\frac{\langle x\rangle}{s^\alpha}\le 1\Big) \;&=\; \frac{1}{s}\Big(2s^{1-\alpha}\gamma-\alpha\frac{\langle x\rangle}{s^\alpha}\Big)F'\Big(\frac{\langle x\rangle}{s^\alpha}\le 1\Big) \\ &\quad + O(s^{-2\alpha}) \end{aligned} \tag{6.6}$$

and that $F'\Big(\frac{\langle x\rangle}{s^\alpha}\ge 1\Big) = \text{(positive constant)}\cdot F\Big(\frac{\langle x\rangle}{s^\alpha}=1\Big)$, we find that due to theorem 5.2

$$F(s^{1-\alpha}\gamma\le\delta)DF\Big(\frac{\langle x\rangle}{s^\alpha}\le 1\Big) \;\doteq\; 0 \tag{6.7}$$

on $\mathrm{Ran}\, E_\Omega$. On the other hand, by theorem 5.1 and $\sqrt{-\Delta}$-boundedness of γ we have that

$$F(s^{1-\alpha}\gamma\ge\delta)DF\Big(\frac{\langle x\rangle}{s^\alpha}\le 1\Big) \;=\; O_{\frac{1}{2}}(s^{-2\alpha-\eta+2}) \tag{6.8}$$

on $\mathrm{Ran}\, F^+$. Collecting the last three statements and remembering (4.28) and theorem 4.6 we conclude that

$$\int_0^t \Big|\frac{d}{ds}\langle W_0(s)\psi,u\rangle\Big|ds \;\le\; C\|\psi\|\|u\|$$

for any $\psi\in\mathrm{Ran}\, F^+$ for any $u\in L^2$ and with C independent of the ψ's and u's. This inequality shows that $W_0(t) = W_0(1)+\int_1^t \frac{d}{ds}W_0(s)ds$ converges strongly as $t\to\infty$. $\square$

Theorem 6.1 implies that for any $g\in\mathrm{Ran}\,(F^\pm Q^\pm)$ there is θ s.t.

$$\Big\|F\Big(\frac{\langle x\rangle}{t^\alpha}\le 1\Big)g_t - U_0(t)\theta\Big\| \;\to\; 0$$

as $t\to+\infty$, provided $\mu>\alpha>\frac{3-\eta}{2}$. Combining this with (5.30), we obtain that, if $\mu>\frac{3-\eta}{2}$, then for any $g\in\mathrm{Ran}\,(F^+Q^+)$ there is θ s.t.

$$\|g_t - U_0(t)\theta\| \;\to\; 0 \tag{6.9}$$

as $t\to+\infty$.

Theorem 6.2. *Let* $\mu > \frac{3-\eta}{2}$. *For any* $\psi \in L^2$ *and for any* $\delta > 0$ *there are a finite union,* Ω, *of compact intervals disjoint from the threshold set of* H, $\varepsilon > 0$ *and* $T > 0$ *and there is* θ *s.t.*

$$\|\psi_t - F\Big(\frac{\langle x\rangle}{t} \geq \varepsilon\Big)\varphi_t - e^{-iHt}e^{-i\int_0^t W(0,s)ds}\theta\| \;\leq\; \delta \tag{6.10}$$

for $t > T$. *Here* φ *is the projection of* ψ *onto* $(\mathrm{Ran}\, F^+)^\perp$.

Proof. We can decompose any $\psi \in L^2$ as $\psi = \varphi + g$ with

$$\varphi \perp \mathrm{Ran}\, F^+ \quad \text{and} \quad g \in \mathrm{Ran}\, F^+ \;. \tag{6.11}$$

We claim that for any $\delta > 0$ there are $\varepsilon > 0$ and $T \geq 0$ s.t.

$$\|F\Big(\frac{\langle x\rangle}{t} \leq \varepsilon\Big)\varphi_t\| \;\leq\; \delta \qquad \text{for any } t \geq T \;. \tag{6.12}$$

Indeed, denote $F^t_\varepsilon = U(t)^* F\Big(\frac{\langle x\rangle}{t} \leq \varepsilon\Big)U(t)$. Using that $F^+\varphi = 0$, we obtain

$$\begin{aligned} &\|F\Big(\frac{\langle x\rangle}{t} \leq \varepsilon\Big)\varphi_t\| \\ &\leq \|(F^t_\varepsilon - F^+_\varepsilon)\varphi\| + \|(F^+_\varepsilon - F^+)\varphi\| \;. \end{aligned} \tag{6.13}$$

First for any $\delta > 0$ we pick up $\varepsilon > 0$ so that

$$\|(F^+_\varepsilon - F^+)\varphi\| \;\leq\; \frac{\delta}{3} \tag{6.14}$$

and then for $\delta > 0$ and $\varepsilon > 0$ we pick $T = T(\delta, \varepsilon)$ s.t.

$$\|(F^t_\varepsilon - F^+_\varepsilon)\varphi\| \;\leq\; \frac{\delta}{3} \qquad \text{for any } t \geq T(\delta, \varepsilon) \;. \tag{6.15}$$

The last two relations yield (6.12).

Equations (6.9), (6.11) and (6.12) yield (6.10). □

7. Asymptotic Clustering Revisited

In this section we present results on break up of the system in question due to propagation of fragments away from each other. We begin with definitions pertaining to such break ups. Denote by $a, b, \ldots$ *partitions* of the set $\{1, \ldots, N\}$ into non-empty disjoint subsets, called clusters. The relation $b < a$ means that b is a refinement of a and $b \neq a$. Then $a \vee b$, $a_{\min}$ and $a_{\max}$ are defined in a standard way. $|a|$ denotes the number of clusters in a. We also identify pairs $\ell = (ij)$ with partitions having $N-1$ clusters: $(ij) \leftrightarrow \{(ij)(1)\cdots(i)\cdots(j)\cdots(N)\}$. We emphasize that the relation $\ell \not\leq a$ (resp. $\ell \leq a$) with $\ell = (ij)$ is equivalent to saying that i and j belong to different clusters (resp. to same cluster) of a.

We define the *intercluster interaction* for a partition a as $I_a =$ sum of all potentials linking different clusters in a, i.e. $I_a = \sum_{\ell \not\leq a} V_\ell$. For each a we introduce the truncated Hamiltonian:

$$H_a \;=\; H - I_a. \tag{7.1}$$

This operator is clearly self-adjoint. It describes the motion of the original system broken into non-interacting clusters of particles.

For each cluster decomposition a, define the configuration space of relative motion of the clusters in a:

$$X_a \;=\; \{x \in X | x_i = x_j \text{ if } i \text{ and } j \text{ belong to same cluster of } a\}$$

and the configuration space of the internal motion within those clusters:

$$X^a \;=\; \{x \in X | \sum_{j \in C_i} m_j x_j = 0 \text{ for all } C_i \in a\}.$$

Clearly X_a and X^a are orthogonal (in our inner product) and they span X:

$$X \;=\; X^a \oplus X_a \;. \tag{7.2}$$

Given generic vector $x \in X$ (resp., $k \in X'$) its projections on X^a and X_a (resp., $(X^a)'$ and X'_a) will be denoted by x^a and x_a (resp., k^a and k_a). The momentum operators canonically conjugate to x_a and x^a and corresponding to k_a and k^a will be denoted by p_a and p^a, respectively. Let $v_a = i[|p_a|^2, x_a]$, the cluster velocity operator.

For long-range potentials, even as clusters move away from each other, the intercluster interaction cannot be entirely ignored. Its effect remains in the form of time-dependent modification of the internal potentials of the clusters. We proceed to the definition of this modification. We introduce the following cut-off function

$$F_a(x) = \prod_{\ell \not\subseteq a} F\left[\langle x^\ell \rangle > 1\right] \ . \tag{7.3}$$

Note that this function is supported in the region of the configuration space in which the distance between clusters in a is at least 1. Now we are prepared to define the modified Hamiltonian of independent clusters:

$$H_a(t) = H_a + W_a(t), \tag{7.4}$$

where $W_a(t)$ is the family of pseudodifferential operators written in detail as

$$W_a(t) = W_{a,t}(x^a, v_a t) \tag{7.5}$$

(note here that x^a and v_a commute). Here (note the identification $x = (x^a, x_a)$)

$$W_{a,t}(x) = W_a(x) F\left(\frac{\langle x_a \rangle}{|t|+1} \geq \lambda\right) + W(x,t) \quad \text{and} \quad W_a(x) = I_a(x) F_a\left(\frac{x}{\delta \langle x \rangle}\right) \ , \tag{7.6}$$

where $\delta > 0$ is a geometrical constant depending only on N and the m_i's and $\lambda > 0$ is sufficiently small and is specified in appropriate places. Due to condition (B) the effective potentials $W_{a,t}$ obey the estimates

$$|\partial_x^\alpha \partial_t^\beta W_{a,t}(x)| \leq C_{\alpha,\beta}(1 + |x| + |t|)^{-\mu - |\alpha| - \beta} \tag{7.7}$$

for $|\alpha| + |\beta| \leq 2$. These inequalities hold strictly speaking only for t sufficiently large so that $\delta\lambda t > R$. However, in order not to complicate the notation we ignore this qualification.

Since H_a is self-adjoint and $W_a(t)$ is uniformly (in t) bounded, one concludes that $H_a(t)$ generates the evolution $U_a(t)$ (see e.g. [RSII]). More precisely, there is a family $U_a(t)$ of unitary operators (with a certain group property) which solves uniquely the Schrödinger Cauchy problem:

$$i\frac{\partial U_a(t)}{\partial t} = H_a(t)U_a(t), \quad U_a(0) = id \tag{7.8}$$

We introduce $U_{a,\lambda}(t) = F(|p_a| \geq \lambda)U_a(t)$, where λ is the same as in (7.6). Of course $F(|p_a| \geq \lambda)$ commutes with $U_a(t)$. Observe also that $|v_a| = 2|p_a|$, where the l.h.s. norm is in X, while the r.h.s., in X'.

Theorem 7.1. *Assume conditions (A) and (B) with $\mu > \frac{1}{2}$. Let ϕ be either $E_\Delta(1-P)$ with Δ a bounded interval disjoint from the thresholds of H or $F\Big(\frac{\langle x\rangle}{t} \geq \varepsilon\Big)E_\Delta$ with $\varepsilon > 0$ and Δ bounded. Then there are $\psi_a^\pm \in L^2$ and $\lambda > 0$ s.t.*

$$\|\phi\psi_t - \sum_{|a|\geq 2} U_{a,\lambda}(t)\psi_a^\pm\| \;\to\; 0 \tag{7.9}$$

as $t \to +\infty$. Here λ belongs to $(0, \delta\varepsilon_1)$, where $\delta > 0$ depends on N, the m_i's and Δ only and where $\varepsilon_1 = \min\limits_{\lambda<E,\lambda\in\tau}(E-\lambda)^{\frac{1}{2}}$ with $E = \inf\Delta$ in the first case and $\varepsilon_1 = \varepsilon$, in the second case.

When (7.9) holds we say that $\phi\psi_t$ is *asymptotically clustering.*

This theorem for $\mu = 1$ is formulated in [SigSof6, thm 8.3] and its proof is essentially contained in [SigSof1,3] (a simpler version is given in [Der1]) with minor adjustments discussed in [SigSof6]. To obtain the result for $\mu > \frac{1}{2}$ one combines the proof of $\mu = 1$ case with an observation of [GerDer]. The latter consists of proving the existence of the Deift-Simon wave operators in two steps, first, by comparing $U(t)$ with the evolution generated

by $H_a + W_{a,t}(x)$ and then by comparing the latter evolution with $U_a(t)$ (generated by $H_a(t) = H_a + W_{a,t}\big(x^a, v_a t\big)\big)$. One can also derive theorem 7.1 directly from the proof of [GerDer].

8. Asymptotic Completeness

In this section we combined results of sections 5–7 in order to prove asymptotic completeness. To begin with, we cast the intuitive notion of asymptotic completeness into precise mathematical terms. To this end we need the notion of channel. If i and j belong to some cluster of a, then $x_i - x_j = (\pi^a x)_i - (\pi^a x)_j$, where π^a is the orthogonal projection in X on X^a. This elementary fact and the fact that $-\Delta = \langle p, p\rangle$ with $p = -i\nabla_x$ (see equation (2.5) and the sentence after it) yield the following decomposition:

$$H_a = H^a \otimes 1 + 1 \otimes T_a \text{ on } L^2(X) \;=\; L^2(X^a) \otimes L^2(X_a). \tag{8.1}$$

Here H^a is the Hamiltonian of the non-interacting a-clusters with their centers-of-mass fixed at the origin, acting on $L^2(X^a)$, and $T_a = -(\text{Laplacian on } X_a)$, the kinetic energy of the center-of-mass motion of those clusters.

The eigenvalues of H^a, whenever they exist, will be denoted by ε^α, where $\alpha = (a, m)$ with m, the number of the eigenvalue in question *counting the multiplicity*. For $a = a_{\min}$, we set $\varepsilon^\alpha = 0$. The set $\tau = \{\varepsilon^\alpha, \text{ all } \alpha\}$ is called the *threshold set* of H and ε^α are called the *thresholds* of H. For $\alpha = (a, m)$ we denote $|\alpha| = |a|$ and $a(\alpha) = a$.

Pairs $\alpha = (a, m)$ are called the channels. Each pair can be identified with an asymptotic state of the system in question as $t \to +\infty$ or $t \to -\infty$: a specifies a decomposition of the system into subsystems and m specifies a stable motion (i.e. a bound state) within each subsystem of this decomposition. Next we define the channel Hamiltonians for the long-range scattering as

$$H_\alpha(t) = \varepsilon^\alpha + |p_a|^2 + W_a(0, v_a t)$$

on $L^2(X_a)$. Denote now by $U_\alpha(t)$ the evolution on $L^2(X_a)$ generated by $H_\alpha(t)$:

$$i\frac{\partial}{\partial t}U_\alpha(t) = H_\alpha(t)U_\alpha(t) \quad \text{and} \quad U_\alpha(0) = id \tag{8.2}$$

$U_\alpha(t)$ are the channel evolution operators. We now can formulate the property of asymptotic completeness. We say that a system obeying (B) (see section 2) with $\mu > \frac{1}{2}$ is *asymptotically complete* if for any $\psi \in L^2(X)$

$$\|\psi_t - \sum_\alpha \psi^\alpha \otimes U_\alpha(t) u_\alpha^\pm\| \to 0 \tag{8.3}$$

as $t \to \pm\infty$ for some $u_\alpha^\pm$. Here ψ^α are the eigenfunctions of H^a corresponding to eigenvalues ε^α and the sum extends over all channels including a trivial one with $|a(\alpha)| = 1$.

This definition is equivalent to a standard one in terms of the channel wave operators (provided the latter exist). The smoothness restriction in condition (B) is immaterial, it can be replaced by any other suitable condition. However, the restriction on the decay is essential (see [Sig1]).

We formulate the main result of this paper.

Theorem 8.1. *Assume conditions (A) – (B) hold with* $\mu > \frac{3-\eta}{2}$, *where* $\eta = (1 - 2^{-N-1})^{-1}$. *Then asymptotic completeness holds.*

Proof. First of all theorem 6.2 states that for any $\psi \in L^2$, for any $\delta > 0$ there are $\varepsilon > 0$, $T > 0$ and $\varphi, \theta \in L^2$ s.t.

$$\|\psi_t - F\Big(\frac{\langle x\rangle}{t} \geq \varepsilon\Big)\varphi_t - e^{-iHt} e^{-i\int_0^t W(0,s)ds}\theta\| \;\leq\; \delta \tag{8.4}$$

for $t \geq T$. For $\delta > 0$ we pick up a neighbourhood, Ω', of $\tau \cup \{\infty\}$, where, recall, τ is the threshold set of H, so that

$$\|E_{\Omega'}\theta\| \;\leq\; \delta\ . \tag{8.5}$$

Let Ω be a compact set disjoint from the thresholds of H and s.t.

$$E_\Omega + E_{\Omega'} \;=\; \mathbf{1}\ . \tag{8.6}$$

Then, by theorem 7.1, $E_\Omega \overline{P} e^{-iHt}\theta$, where $\overline{P} = \mathbf{1} - P$, is asymptotically clustering, i.e. there are $\lambda > 0$ and θ_a s.t.

$$\| e^{-iHt} e^{-i\int_0^t W(0,s)ds} E_\Omega \overline{P}\theta - \sum_{|a|\geq 2} U_{a,\lambda}(t)\theta_a \| \ \to \ 0 \tag{8.7}$$

as $t \to +\infty$. Here $U_{a,\lambda}(t)$ are the evolution operators described in section 7 (after eqn (7.8). For the discrete part we have

$$\begin{aligned} & e^{-iHt} e^{-i\int_0^t W(0,s)} E_\Omega P\theta \\ = & \sum e^{-i(\varepsilon^\alpha t + \int_0^t W(0,s)ds)} P_\alpha E_\Omega \theta \ , \end{aligned} \tag{8.8}$$

where the sum is taken over all α's s.t. $|a(\alpha)| = 1$ and $\varepsilon^\alpha \in \Omega$. Since Ω is disjoint from the thresholds of H, the sum on the r.h.s. is finite.

By [SigSof4, thm 4.1 (ii)] for any $\delta > 0$ there is a bounded set Ω_1 s.t.

$$\sup_t \| (1 - E_{\Omega_1})\varphi_t \| \ \leq \ \delta \ . \tag{8.9}$$

Next, by theorem 7.1, $F\Big(\frac{\langle x\rangle}{t} \geq \varepsilon\Big) E_{\Omega_1}\varphi_t$ is asymptotically clustering, i.e. there are $\lambda > 0$ and $\varphi_a \in L^2$ s.t.

$$\| F\Big(\frac{\langle x\rangle}{t} \geq \varepsilon\Big) E_{\Omega_1}\varphi_t - \sum_{|a|\geq 2} U_{a,\lambda}(t)\varphi_a \| \ \to \ 0 \tag{8.10}$$

as $t \to +\infty$. Combining eqns (8.4)–(8.10) yields that for any $\delta > 0$ there are $\lambda > 0$, a finite subset, $\mathcal{A}$, of the α's with $|a(\alpha)| = 1$, $\psi_a \in L^2$ and $\theta \in L^2$ s.t.

$$\begin{aligned} \sup_t \| \psi_t - & \sum_{\alpha\in\mathcal{A}} U_\alpha(t) P_\alpha \theta \\ & - \sum_{|a|\geq 2} U_{a,\lambda}(t)\psi_a \| \ \leq \ \delta \ . \end{aligned}$$

Applying this statement inductively and taking into account that $P_\alpha U_{a,\lambda}(t) = U_\alpha(t) P_\alpha F(|p_a| \geq \lambda)$ for $a(\alpha) = a$, we conclude that for any $\psi \in L^2$ and $\delta > 0$ we can find $\varphi_{\alpha,\delta} \in L^2$ and a finite subset, $\mathcal{B}$, of the α's s.t.

$$\sup_t \| \psi_t - \sum_{\alpha\in\mathcal{B}} U_\alpha(t) P_\alpha \varphi_{\alpha,\delta} \| \ \leq \ \delta \ .$$

On the other by a standard argument the wave operators

$$s - \lim U(t)^* P_\alpha U_\alpha(t)$$

exist. The last two results yield (8.3). $\square$

Appendix: Operator γ_t .

The analysis of this supplement is motivated by [SigSof1,3] and, to a larger extent, by [Gr]. Some of the analysis of section 7 simplifies if we use the operator γ_t , constructed in this supplement, instead of γ . An important fact is that though γ_t is close to γ , its Heisenberg derivative has a useful lower bound globally. The main result of this section is

Theorem A.1. *There is a family, γ_t , of self-adjoint operators s.t. $(-\Delta + 1)^{-s}\gamma_t(-\Delta + 1)^{-\frac{1}{2}+s}$ is bounded and norm differentiable for any $s \in [0, \frac{1}{2}]$ and s.t.*

$$\gamma_t \;=\; \gamma + O(t^{-\eta_1}) \;, \tag{A.1}$$

$$\big[\gamma_t, [\gamma_t, H]\big] \;=\; O_1(t^{-\eta_2}) \tag{A.2}$$

and

$$D\gamma_t \;\geq\; -Ct^{-\eta} \tag{A.3}$$

with $\eta_1 > \frac{\eta-1}{2} > 0$ and $\eta_2 > \eta > 1$, depending entirely on N and μ , and with the constant independent of t . If $\mu \geq \frac{2}{3}$, we can choose then $\eta_1 = 2^{-N-1}$, $\eta_2 = (1-2^{-N})^{-1}$ and $\eta = (1-2^{-N-1})^{-1}$.

Below we construct γ_t explicitly and prove an estimate stronger than (A.3) and which implies positivity of $D\gamma_t$ sandwiched by appropriate energy cut-offs. Before proceeding to the proof we introduce some definitions and notation. We introduce self-adjoint operators

$$\gamma^a \;=\; \frac{1}{2}(\langle p^a, \hat{x}^a\rangle + \langle \hat{x}^a, p^a\rangle) \;,$$

and

$$\gamma_a \;=\; \frac{1}{2}(\langle p_a, \hat{x}_a\rangle + \langle \hat{x}_a, p_a\rangle) \;, \tag{A.4}$$

related to the internal motion of the clusters and to the motion of the centers-of-mass of the clusters.

Let numbers $\delta_a \in \left(\max\left(\frac{1}{2}, \frac{1}{1+\mu}\right), 1\right)$ satisfy $\delta_b < \delta_a$ if $b < a$. Moreover, we assume that $2\min_{b>a} \delta_b > \delta_a + 1$ for any a with $|a| \geq 2$. Let $t \geq 1$ and $t_a = t^{\delta_a}$. [SigSof6] have constructed the following partition of unity (see also [SigSof1] and, especially, [Gr]): $0 \leq j_{a,t}(x) \leq 1$, $j_{a,t}(x)$ are smooth in x and t and obey

$$\sum j_{a,t}(x)^2 \;=\; 1 \;, \tag{A.5}$$

$$\operatorname{supp} j_{a,t} \;\subset\; \{|x^a| \leq t_a,\; |x|_a \geq c\bar{t}_a\} \tag{A.6}$$

for some geometrical constant c dependent only on N and the m_i's and for

$$\bar{t}_a \;=\; \min_{b>a} t_b \;, \tag{A.7}$$

$$\partial_x^\beta j_{a,t}(x) \;=\; O(t_a^{-|\beta|}) \;, \tag{A.8}$$

for any β, and

$$\nabla_x j_{a,t} \;=\; -\frac{\hat{x}^a}{t_a}\chi_a + O(\bar{t}_a^{-1}) \;, \tag{A.9}$$

where $0 \leq \chi_a \leq 1$ is a smooth function supported in $\operatorname{supp} j_{a,t}$ and obeying $\partial^\beta \chi_a = O(t_a^{-|\beta|})$. Moreover,

$$\partial_t j_{a,t} \;=\; O(t^{-1}) \;. \tag{A.10}$$

Thus $\{j_{a,t}\}$ is a smooth partition of unity describing various break-ups of the system.

Remarks A.2. (a) [SigSof6] defines $j_{a,t}$ for $|a| \geq 2$ satisfying $\sum_{|a|\geq 2} j_{a,t} = F\left(\frac{\langle x\rangle}{t_{a\,\max}} \geq 1\right)$. Thus adding to this collection $j_{a,t} = F\left(\frac{\langle x\rangle}{t_a} \leq 1\right)$ for $a = a_{\max}$ and renormalizing produces the desired partition of unity. (b) In fact, the partition in [SigSof6] is defined for arbitrary scales t_a satisfying $t_b < (10M)^{-1}t_a$ if $b < a$ with M given by $M = \max_{a,b} \sup_x \left(|x^{a\vee b}|(|x^a| + |x^b|)^{-1}\right) < \infty$.

Equations (A.8), (A.9) and (A.10) and a simple computation show that

$$\begin{aligned} Dj_{a,t} &= -2t_a^{-1}\gamma^a\chi_a \\ &\quad + O_1(\bar{t}_a^{-1}) + O(t_a^{-2}) \ . \end{aligned} \tag{A.11}$$

Now we define

$$\gamma_t = \sum j_{a,t}\gamma_a j_{a,t} \ . \tag{A.12}$$

Proof of theorem A.1. Using that by (A.8)

$$\begin{aligned} [\gamma_a, j_{a,t}] &= -i\hat{x}_a(\nabla_a j_{a,t}) \\ &= O(t_a^{-1}) \ , \end{aligned} \tag{A.13}$$

we obtain

$$\gamma_t = \sum (j_{a,t})^2\gamma_a + O(\max t_a^{-1}) \ .$$

This together with (A.7) yields

$$\gamma_t = \gamma + R + O(\max t_a^{-1}) \ , \tag{A.14}$$

where

$$R = \sum (j_{a,t})^2(\gamma_a - \gamma) \ . \tag{A.15}$$

It remains to estimate R. Ignoring higher order terms, we assume $\widehat{x} = x\langle x\rangle^{-1}$, etc. Using that

$$\begin{aligned} \widehat{x}_a \cdot p_a &= \langle x_a\rangle^{-1} x_a \cdot p_a = \langle x_a\rangle^{-1}(x \cdot p - x^a \cdot p^a) \\ &= \frac{\langle x\rangle}{\langle x_a\rangle}\widehat{x} \cdot p - \frac{\langle x^a\rangle}{\langle x_a\rangle}\widehat{x}^a \cdot p^a \ , \end{aligned} \tag{A.16}$$

we find

$$\gamma_a - \gamma = \Big(\frac{\langle x\rangle}{\langle x_a\rangle} - 1\Big)\gamma - \frac{\langle x^a\rangle}{\langle x_a\rangle}\gamma^a + O(\langle x_a\rangle^{-1}) \ . \tag{A.17}$$

Next, using that

$$\begin{aligned} \langle x\rangle\langle x_a\rangle^{-1} &= \frac{\langle x\rangle - \langle x_a\rangle}{\langle x_a\rangle} \\ &= \frac{\langle x^a\rangle^2}{(\langle x\rangle + \langle x_a\rangle)\langle x_a\rangle} + O(\langle x_a\rangle^{-2}) \ , \end{aligned} \tag{A.18}$$

we obtain

$$\gamma_a - \gamma \;=\; \frac{\langle x^a\rangle^2}{(\langle x\rangle + \langle x_a\rangle)\langle x_a\rangle}\gamma - \frac{\langle x^a\rangle}{\langle x_a\rangle}\gamma^a + O(\langle x_a\rangle^{-1}) \;. \tag{A.19}$$

Using this relation, the definition of R, using (A.6) and using that γ and γ^a are $\sqrt{-\Delta}$-bounded, we obtain

$$\begin{aligned} R \;&=\; O_{\frac{1}{2}}(\max_a t_a^2 \cdot \bar{t}_a^{-2}) + O_{\frac{1}{2}}(\max_a t_a \cdot \bar{t}_a^{-1}) + O(\max_a \bar{t}_a^{-1}) \\ &=\; O_{\frac{1}{2}}(\max_a t_a \cdot \bar{t}_a^{-1}) \;. \end{aligned}$$

This together with (A.14) and the relations $t_a = t^{\delta_a}$ and $\bar{t}_a = \min_{b>a} t_b$ yields (A.1) with $\eta_1 = \min_a(\min_{b>a}\delta_b - \delta_a) > 0$.

(A.2) with $\eta_2 = 2\min_a \delta_a > 1$ results from a straightforward computation.

Now we prove (A.3). Using that

$$\begin{aligned} i[H,\gamma_a] \;&= 4\frac{1}{\sqrt{\langle x_a\rangle}}(p_a^2 - \gamma_a^2)\frac{1}{\sqrt{\langle x_a\rangle}} \\ &\quad + O(\langle x_a\rangle^{-2}) + O(\nabla_a I_a) \end{aligned}$$

and that $\nabla I_a = O(|x|_a)^{-1-\mu})$ and (A.6), we find

$$\begin{aligned} D\gamma_t \;&= 2\sum j_{a,t}\frac{1}{\sqrt{\langle x_a\rangle}}(p_a^2 - \gamma_a^2)\frac{1}{\sqrt{\langle x_a\rangle}} j_{a,t} \\ &\quad + R_1 + O(\max_a t_a^{-1-\mu}) \;, \end{aligned} \tag{A.20}$$

where we have assumed that $\mu \le 1$ and where

$$R_1 \;=\; 4\sum (Dj_{a,t})\gamma_a j_{a,t} + \text{adjoint} \;.$$

Using (A.19) and the relation $\sum D(j_{a,t}^2) = 0$, and using (A.6) and (A.8), we get

$$\begin{aligned} R_1 \;&=\; -\sum Dj_{a,t}\frac{\langle x^a\rangle}{\langle x_a\rangle}\gamma^a j_{a,t} \\ &\quad + O_{\frac{1}{2}}(\max t_a^{-2}) + O_1(\max t_a^{-1}\cdot t_a^2 \cdot \bar{t}_a^{-2}) + O_{\frac{1}{2}}(\max t_a^{-1}\cdot \bar{t}_a^{-1}) \;. \end{aligned}$$

Now remembering (A.11), we find

$$\begin{aligned} R_1 \;&= 2\sum t_a^{-1}\gamma^a \chi_a \frac{\langle x^a\rangle}{\langle x_a\rangle} j_{a,t}\gamma^a \\ &\quad + O_{\frac{1}{2}}(\max t_a^{-2}) + O_1(\max t_a \cdot \bar{t}_a^{-2}) \;. \end{aligned} \tag{A.21}$$

Combining (A.20) and (A.21), we obtain

$$\begin{aligned} D\gamma_t \;&= 2\sum j_{a,t}\frac{1}{\sqrt{\langle x_a\rangle}}(p_a^2-\gamma_a^2)\frac{1}{\sqrt{\langle x_a\rangle}}j_{a,t} \\ &\quad + 2\sum \gamma^a\frac{\chi_a\langle x^a\rangle}{t_a\langle x_a\rangle}\gamma^a \\ &\quad + O_{\frac{1}{2}}(\max t_a^{-2}) + O_1(\max t_a\cdot \bar{t}_a^{-2}) + O(\max t_a^{-1-\mu})\ . \end{aligned} \tag{A.22}$$

Since each of the first two terms on the r.h.s. is positive $(p_a^2 \geq \gamma_a^2)$, this yields (A.3) with $\eta = \min_a \min(2\min_{b>a}\delta_b - \delta_a, (1+\mu)\delta_a)$. By our choice of the δ_a's, $\eta > 1$ and $\eta_2 > \eta$.

Now take $\delta_a = \delta_{|a|}$ with $(1+\mu)^{-1} < \delta_N < \cdots < \delta_1 < 1$. Pick up $\delta_j = 2^{j-1}\delta_1 - (2^{j-1}-1)b$ and $b = \delta_1\left[1 - \frac{\mu}{1+\mu}2^{1-N}\right]^{-1}$. Taking δ_1 very close to 1 and observing that, if $\mu > \frac{2}{3}$, then $b > (1-2^{-N-1})^{-1}$, we conclude that $\eta_1 > 2^{-N-1}$, $\eta_2 > (1-2^{-N})^{-1}$ and $\eta > (1-2^{-N-1})^{-1}$. Thus (A.1)–(A.3) are obeyed for our choice of the η's. □

Note here that eqn (A.22) and the first part of lemma 3.1 imply

Theorem A.2. *Let $j_{a,t}$ be as above. Then*

$$\int_1^\infty \|(p_a^2-\gamma_a^2)^{\frac{1}{2}}\frac{1}{\sqrt{\langle x_a\rangle}}j_{a,t}(H+i)^{-1}\psi_t\|^2 dt \;\leq\; C\|\psi\|^2 \tag{A.23}$$

for any $\psi \in L^2$ and with C independent of the ψ's.

Observe that $p_a^2 - \gamma_a^2 \geq 0$ and

$$p_a^2 - \gamma_a^2 \;=\; p_a\cdot[p_a - \frac{1}{2}(\gamma_a\widehat{x}_a + \widehat{x}_a\gamma_a)] + O_{\frac{1}{2}}(\langle x_a\rangle^{-1})\ .$$

Hence (A.23) contains a rather strong statement about propagation: ψ_t "vanishes" on $\{k_a/|k_a| = x_a/|x_a|\} \cap \text{supp } j_{a,t}$. It can be used as one of the key ingredients in the proof of asymptotic clustering (theorem 7.1). This approach is similar to that of [Gr].

References

[CFKS] H. Cycon, R. Froese, W. Kirsch and B. Simon, Schrödinger Operators, Springer-Verlag, 1987.

[Der1] J. Derezinski, A new proof of the propagation theorem for N-body quantum systems, Comm. Math. Phys. **122** (1989), 203–231.

[Der2] J. Derezinski, Algebraic approach to the N-body long-range scattering, Reviews in Math. Phys. **3** (1991), 1–62.

[Der3] J. Derezinski, Private communication, July 1991.

[En1] V. Enss, Asymptotic observables on scattering states, Comm. Math. Phys. **89** (1983), 245–268.

[En2] V. Enss, Quantum scattering theory for two- and three-body systems with potentials of short- and long-range, in "Schrödinger Operators", S. Graffi (ed.), Springer-Verlag Lecture Notes in Math. N1159 (1985).

[En3] V. Enss, Two- and three-body quantum scattering ..., in the Proceedings Teubner Texte zur Mathematics, Leipzig (M. Demuth and B.-W. Schultze, eds.).

[GerDer] C. Gérard and J. Derezinski, A remark on asymptotic clustering for N-particle quantum systems, preprint, Ecole Poly., Paris (1991).

[Gr] G.-M. Graf, Asymptotic completeness for N-body short-range quantum systems: a new proof, Comm. Math. Phys. **132** (1990), 73–101.

[Mo] E. Mourre, Opérateurs conjugués et propriétés de propagation, Comm. Math. Phys. **91** (1983), 279–300.

[RS] M. Reed and B. Simon, Methods of Modern Mathematical Physics II, Academic Press, 1979.

[Sig1] I.M. Sigal, Mathematical questions of quantum many-body problem, Sem. equat. deriv. part. 1986-1987, École Poly, Exposé No. XXIII.

[Sig2] I.M. Sigal, On the long-range scattering, Duke Math. J. **60,** N2 (1990).

[SigSof1] I.M. Sigal and A. Soffer, The N -particle scattering problem: asymptotic completeness for short-range systems, Anal. Math. **125** (1987).

[SigSof2] I.M. Sigal and A. Soffer, Asymptotic completeness of multiparticle scattering, in "Differential Equations and Mathematical Physics", Knowles and Saito (eds.), Lecture Notes in Math. N1285 (1987).

[SigSof3] I.M. Sigal and A. Soffer, Long-range many-body scattering. Asymptotic clustering for Coulomb-type potential, Invent. Math. **99** (1990), 115-143.

[SigSof4] I.M. Sigal and A. Soffer, Local decay and velocity bounds, preprint, Princeton (1988).

[SigSof5] I.M. Sigal and A. Soffer, Asymptotic completeness for Coulomb-type 3-body systems, preprint, Princeton (1991).

[SigSof6] I.M. Sigal and A. Soffer, Asymptotic completeness for $N \leq 4$ particle systems with the Coulomb-type interactions, Duke Math. J. **71** (1993), 243–298.

[SS] A.G. Sigalov and I.M. Sigal, Description of the spectrum of the energy operator of quantum mechanical systems ..., Theor. Math. Phys. **5** (1970), 990–1005.