Chapter 1

I. Fibre Bundles

1.1 Definitions

Definition 1.1.1 Let X be a topological space and let $\{U_j\}_{j\in J}$ be an open cover of X. A *partition of unity* relative to the cover $\{U_j\}_{j\in J}$ consists of a set of functions $f_j : X \to [0,1]$ such that:

1) $\overline{f_j^{-1}\big((0,1]\big)} \subset U_j$ for all $j \in J$;

2) $\{\overline{f_j^{-1}\big((0,1]\big)}\}_{j\in J}$ is locally finite;

3) $\sum_{j\in J} f_j(x) = 1$ for all $x \in X$.

A *numerable cover* of a topological space X is one which possesses a partition of unity.

Theorem 1.1.2 *Let X be Hausdorff. Then X is paracompact iff for every open cover $\mathcal{U}$ of X there exists a partition of unity relative to $\mathcal{U}$.*

See MAT1300 notes for a proof.

Definition 1.1.3 Let B be a topological space with chosen basepoint $*$. A (*locally trivial*) *fibre bundle* over B consists of a map $p : E \to B$ such that for all $b \in B$ there exists an open neighbourhood U of b for which there is a homeomorphism $\phi : p^{-1}(U) \to p^{-1}(*) \times U$ satisfying $\pi'' \circ \phi = p|_U$, where π'' denotes projection onto the second factor. If there is a numerable open cover of B by open sets with homeomorphisms as above then the bundle is said to be *numerable*.

If ξ is the bundle $p : E \to B$, then E and B are called respectively the *total space*, sometimes written $E(\xi)$, and *base space*, sometimes written $B(\xi)$, of ξ and $F := p^{-1}(*)$ is called the *fibre* of ξ. For $b \in B$, $F_b := p^{-1}(b)$ is called the fibre over b; the local triviality conditions imply that all the fibres are homeomorphic to $F = F_*$, probdied B is connected. We sometimes use the phrase "$F \to E \xrightarrow{p} B$ is a bundle" to mean that $p : E \to B$ is a bundle with fibre F.

If ξ and ξ' are the bundles $p : E \to B$ and $p' : E' \to B'$ then a bundle map $\phi : \xi \to \xi'$ (or *morphism of bundles*) is defined as a pair of maps $(\phi_{\text{tot}}, \phi_{\text{base}})$ such that

$$\begin{tikzcd}
E \arrow[r, "\phi_{\text{tot}}"] \arrow[d, "p"] & E' \arrow[d, "p'"] \\
B \arrow[r, "\phi_{\text{base}}"] & B'
\end{tikzcd}$$

commutes. Of course, the map ϕ_{base} is determined by ϕ_{tot} and the commutativity of the diagram, so we might sometimes write simply ϕ_{tot} for the bundle map. We say that $\phi : \xi \to \xi'$ is a *bundle map over* B, if $B(\xi) = B(\xi') = B$ and ϕ_{base} is the identity map 1_B.

A *cross-section* of a bundle $p : E \to B$ is a map $s : B \to E$ such that $p \circ s = 1_B$.

A topological space together with a (continuous) action of a group G is called a G-space. We will be particularly interested in fibre bundles which come with an action of some topological group. To this end we define:

Definition 1.1.4 Let G be a topological group and B a topological space. A *principal G-bundle* over B consists of a fibre bundle $p : E \to B$ together with an action $G \times E \to E$ such that:

1) the "shearing map" $G \times E \to E \times E$ given by $(g, x) \mapsto (x, g \cdot x)$ maps $G \times E$ homeomorphically to its image;

2) $B = E/G$ and $p : E \to E/G$ is the quotient map;

3) for all $b \in B$ there exists an open neighbourhood U of b such that $p : p^{-1}(U) \to U$ is G-bundle isomorphic to the trivial bundle $\pi'' : G \times U \to U$. That is, there exists a homeomorphism $\phi : p^{-1}(U) \to G \times U$ satisfying $p = \pi'' \circ \phi$ and $\phi(g \cdot x) = g \cdot \phi(x)$, where $g \cdot (g', u) = (gg', u)$.

The action of a group G on a set S is called *free* if for all $g \in G$ different from the identity of G, $g \cdot s \neq s$ for every $s \in S$. The action of a group G on a set S is called *effective* if for all $g \in G$ different from the identity of G there exists $s \in S$ such that $g \cdot s \neq s$.

The shearing map is injective if and only if the action is free, so by condition (1), the action of G on the total space of a principal bundle is always free. If G and E are compact, then of course, a free action suffices to satisfy condition (1). In general, a free action produces a well defined "translation map" $\tau : Q \to G$, where $Q = \{(x, g \cdot x) \in X \times X\}$ is the image of the shearing function. Condition (1) is equivalent to requiring a free action with a continuous translation function.

Recall that if $X_1 \subset X_2 \subset \ldots X_n \subset \ldots$ are inclusions of topological spaces then the direct limit, X, is given by $X := \varinjlim_n \cup_n X_n$ with the topology determined by specifying that $A \subset X$ is closed if and only if $A \cap X_n$ is closed in X_n for each n.

Lemma 1.1.5 *Let G be a topological group. Let $X_1 \subset X_2 \subset X_n \ldots \subset \ldots$ be inclusions of topological spaces and let $X_\infty = \varinjlim_n X_n = \cup_n X_n$. Let $\mu : G \times X_\infty \to X_\infty$ be a free action of G on X_∞ which restricts to an action of G on X_n for each n. (I.e. $\mu(G \times X_n) \subset X_n$ for each n.) Then the action of G on X_n is free and if the translation function for this action is continuous for all n then the translation function for μ is continuous. In particular, if G is compact and X_n is compact for all n, then the translation function for μ is continuous.*

Proof: Since the action of G on all of X_∞ is free, it is trivial that the action on X_n is free for all n. Let $Q_n \subset X_n \times X_n$ be the image of the shearing function for X_n and let $\tau_n : Q_n \to G$ be the translation function for X_n. Then as a topological space $Q_\infty = \cup_n Q_n = \varinjlim_n Q_n$ and $\tau_\infty\big|_{Q_n} = \tau_n$. Thus τ_∞ is continuous by the universal property of the direct limit. $\square$

By condition (2), the fibre of a principal G-bundle is always G. However we generalize to bundles whose fibre is some other G-space as follows.

Let G be a topological group. Let $p : E \to B$ be a principal G-bundle and let F be a G-space on which the action of G is effective. The *fibre bundle with structure group G* formed from p and F is defined as $q : (F \times E)/G \to B$ where $g \cdot (f, x) = (g \cdot f, g \cdot x)$ and $q(f, x) = p(x)$. We sometimes use the term "G-bundle" for a fibre bundle with structure group G. In the special case where $F = \mathbb{R}^n$ (respectively $\mathbb{C}^n$) and G is the orthogonal group $O(n)$ (respectively unitary group $U(n)$) acting in the standard way, and the restrictions of the trivialization maps to each fibre are linear transformations, such a fibre bundle is called an *n-dimensional real (*respectively *complex) vector bundle.* A one-dimensional vector bundle is also known as a *line bundle.*

Examples

- For any spaces F and B, there is a "trivial bundle" $\pi_2 : F \times B \to B$.
- If $X \to B$ is a covering projection, then it is a principal G-bundle where G is the group of covering transformations with the discrete topology.

- Let M be an n-dimensional differentiable manifold.

$$C^\infty(M) := \{C^\infty \text{ functions } f : M \to \mathbb{R}\}.$$

For $x \in M$, $C^\infty(x) := \{\text{germs of } C^\infty \text{functions at } x\} := \varinjlim_{x\in U} C^\infty(U)$ and

$$T_xM := \{X : C^\infty(x) \to \mathbb{R} \mid X(af+bg) = aX(f)+bX(g) \text{ and } X(fg) = gX(f)+fX(g)\}.$$

T_xM is an n-dimensional real vector space. A coordinate chart for a neighbourhood of x, $\psi : \mathbb{R}^n \to U$ yields an explicit isomorphism $\mathbb{R}^n \to T_xM$ via $v \mapsto D_v(x)(\;)$ where $D_v(x)(\;)$ denotes the directional derivative

$$D_v(x)(f) := \lim_{t\to 0} \frac{f \circ \psi\big(\psi^{-1}(x)+tv\big) - f(x)}{t}.$$

For $M = \mathbb{R}^n = \langle u^1 \dots u^n\rangle \cong U$, at each x we have a basis $\partial/\partial u^1, \dots, \partial/\partial u^n$ for $T\mathbb{R}^n$. (In this context it is customary to use upper indices u^i for the coordinates in $\mathbb{R}^n$ to fit with the Einstein summation convention.)

Define $TM := \cup_{x\in M} T_xM$ with topology defined by specifying that for each chart U of M, the bijection $U \times \mathbb{R}^n \to TU$ given by $(x,v) \mapsto D_v(x)$ be a homeomorphism. This gives TM the structure of a $2n$-dimensional manifold. The projection map $p : TM \to M$ which sends elements of T_xM to x forms a vector bundle called the *tangent bundle* of M. (TM has an obvious local trivialization over the charts of the manifold M.)

If M comes with an embedding into $\mathbb{R}^N$ for some N (the existence of such an embedding can be proved using a partition of unity,) the total space TM can be described as

$$TM = \{(x,v) \in \mathbb{R}^N \times \mathbb{R}^N \mid x \in M \text{ and } v \perp x\}.$$

A cross-section of the tangent bundle TM is called a *vector field* on M. It consists of a continuous assignment of a tangent vector to each point of M. Of particular interest are nowhere vanishing vector fields (χ such that $\chi(x) \neq 0_x$ for all x). For example, as shown in `MAT1300`, S^n possesses a nowhere vanishing vector field if and only if n is odd.

- Recall that real projective space $\mathbb{R}P^n$ is defined by $\mathbb{R}P^n := S^n/\!\sim$ where $x \sim -x$. Define the *canonical line bundle* over $\mathbb{R}P^n$, written γ_n^1, by

$$E(\gamma_n^1) := \{([x], v) \mid v \text{ is a multiple of } x\}.$$

In other words, a point $[x] \in \mathbb{R}P^n$ can be thought of as the line $L_{[x]}$ joining x to $-x$, and an element of $E(\gamma_n^1)$ consists of a line $L_{[x]} \in \mathbb{R}P^n$ together with a vector $v \in L_{[x]}$.

To see that the map $p : E(\gamma_n^1) \to \mathbb{R}P^n$ given by $p\big(([x], v)\big) = x$ is locally trivial: given $[x] \in \mathbb{R}P^n$ choose an evenly covered neighbourhood of $[x]$ with respect to the covering projection $q : S^n \to \mathbb{R}P^n$. I.e. choose U sufficiently small that $q^{-1}(U)$ does not contain any pair of antipodal points and so consists of two disjoint copies of U. Say $q^{-1}(U) = V \amalg W$. Define $h : \mathbb{R} \times U \cong p^{-1}(U)$ by $h(t, [y]) := ([y], ty)$, where y is the representative for y which lies in V. The map h is not canonical (we could have chosen to use W instead of V) but it **is** well defined since each $[y]$ has a unique representative in V.

- Recall that the unitary group $U(n)$ is defined the set of all matrices in $M_{n\times n}(\mathbb{C})$ which preserve the standard inner product on $\mathbb{C}^n$. (i.e. $T \in U(n)$ iff $\langle Tx, Ty\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{C}^n$. Equivalently $T \in U(n)$ iff $TT^* = T^*T = I$, where T^* denotes the conjugate transpose of T and I is the identity matrix, or equivalently $T \in U(n)$ iff either the rows or columns of T (and thus both the rows and the columns) form an orthonormal basis for $\mathbb{C}^n$.

  Define $q : U(n) \to S^{2n-1}$ to be the map which sends each matrix to its first column. For $x \in S^{2n-1}$, the fibre $q^{-1}(x)$ is homeomorphic to the set of orthonormal bases for the orthogonal complement $x^\perp$ of x. I.e. $q^{-1}(x) \cong U(n-1)$. To put it another way, the set of (left) cosets $U(n)/U(n-1)$ is homeomorphic to S^{2n-1}. (Except when $n < 3$, $U(n-1)$ is not normal in $U(n)$ and S^{2n-1} does not inherit a group structure.) As we will see later (Cor. 1.4.7), $q : U(n) \to S^{2n-1}$ is locally trivial and forms a principal $U(n-1)$ bundle.

$\square$

Proposition 1.1.6 *If $p : E \to B$ is a bundle then p is an open map, i.e. takes open sets to open sets.*

Proof: Let $V \subset E$ be open. To show $p(V)$ is open in B it suffices to show that each $b \in p(V)$ is an interior point. Given $b \in p(V)$, find an open neighbourhood U_b of b such that $p : p^{-1}(U_b) \to U_b$ is trivial. Since $b \in p\big(V \cap p^{-1}(U_b)\big) \subset p(V)$, to show that b is interior in $p(V)$, it suffices to show that $p\big(V \cap p^{-1}(U_b)\big)$ is open in B. By means of the homeomorphism $p^{-1}(U_b) \cong F \times U_b$ this reduces the problem to showing that the projection $\pi_2 : F \times U_b \to U_b$ is an open map, but the fact that projection maps are open is standard. $\square$

Every vector bundle has a cross-section (called the "zero cross-section") given by $s(b) = 0_b$ where 0_b denotes the zero element of the vector space F_b. In the future we might sometimes write simply 0 for 0_b if the point b is understood.

If X and Y are G-spaces, a morphism of G-spaces (or G-map) is a continuous equivariant function $f : X \to Y$, where "equivariant" means $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and $x \in X$.

A morphism of principal G-bundles is a bundle map which is also a G-map. A morphism $\phi : \xi \to \xi'$ of fibre bundles with structure group G consists of a bundle map $(\phi_{\text{tot}}, \phi_{\text{base}})$ in which ϕ_{tot} is formed from a G-morphism ϕ^G of the underlying principal G-bundles by specifying $\phi_{\text{tot}}\big((f,x)\big) = (f, \phi^G_{\text{tot}} x)$. A morphism of vector bundles consists of a morphism of fibre bundles with structure group $O(n)$ (respectively $U(n)$) in which the restriction to each fibre is a linear transformation.

Notice that because the action of G on the total space of a principal G-bundle is always free, it follows if $\phi : \xi \to \xi'$ is a morphism of principal G-bundles then the restriction of ϕ to each fibre is a bijection, which is translation by $\phi(1)$ since ϕ is equivariant. Furthermore the condition the shearing map be a homeomorphism to its image (equivalently the translation function is continous) implies that this bijection is a self-homeomorphism. Thus any morphism of principal G-bundles is an isomorphism and the definitions then imply that this statement also holds for any fibre bundle with structure group G and for vector bundles.

Proposition 1.1.7 *For any G-space F, the automorphisms of the trivial G-bundle bundle $\pi_2 : F \times B \to B$ are in $1-1$ correspondence with continuous functions $B \to G$.*

Proof: By definition, any bundle automorphism of p comes from a morphism of the underlying trivial G-principal bundle $\pi_2 : G \times B \to B$ so we may reduce to that case. If $\tau : B \to G$, then we define $\phi_\tau : G \times B \to G \times B$ by $\phi_\tau(g,b) := (\tau(b)g, b)$. Since $(g,b) = g \cdot (1,b)$, the map is completely determined by $\phi_\tau(1,b)$. Thus, conversely given a G-bundle map $\phi : G \times B \to G \times B$, we define $\tau(b)$ to be the first component of $\phi(1,b)$.

□

Let $p : X \to B$ be a principal G-bundle and let $\{U_j\}$ be a local trivialization of p, that is, an open covering of B together with G-homeomorphisms $\phi_{U_j} : p^{-1}(U_j) \to F \times U_j$. For every pair of sets U, V in our covering, the homeomorphisms ϕ_U and ϕ_V restrict to homeomorphisms $p^{-1}(U \cap V) \to F \times (U \cap V)$. The composite $F \times (U \cap V) \xrightarrow{{\phi_U}^{-1}} p^{-1}(U \cap V) \xrightarrow{\phi_V} F \times (U \cap V)$ is an automorphism of the trivial bundle $F \times (U \cap V)$ and thus, according to the proposition, determines a function $\tau_{U_j,U_i} : U_j \cap U_i \to G$. The functions $\tau_{U_j,U_i} : U_j \cap U_i \to G$ are called the *transition functions* of the bundles (with respect to the covering $\{U_j\}$). These functions are compatible in the sense that $\tau_{U,U}(b) =$ identity of G, $\tau_{U_j,U_i}(b) = \big(\tau_{U_i,U_j}(b)\big)^{-1}$, and if b lies in $U_i \cap U_j \cap U_k$ then $\tau_{U_k,U_i}(b) = \tau_{U_k,U_j}(b) \circ \tau_{U_j,U_i}(b)$.

A principal bundle can be reconstructed from its transition functions as follows. Given a covering $\{U_j\}_{j \in J}$ of a space B and a compatible collection of continuous functions $\tau_{U_j,U_i} : U_i \cap U_j \to G$, we can construct a principal G-bundle having these as transition functions by setting $X = \coprod_{j \in J} (G \times U_j)/\!\sim$ where $(g,b)_i \sim \big(g \cdot \tau_{U_j,U_i}(b), b\big)_j$ for all i, $j \in J$ and $b \in U_i \cap U_j$. Thus a principal bundle is equivalent to a compatible collection of transition functions. For an arbitrary G-bundle we also need to know the fibre to form the bundle from its associated principal bundle.

1.2 Operations on Bundles

- Pullback

$$\begin{tikzcd} E' \arrow[r, dotted, "\hat{f}"] \arrow[d, dotted, "\pi'"] & E \arrow[d, "\pi"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

  Given a bundle $\xi = \pi : E \to B$ and a map $f : B' \to B$, let

$$E' := \{(b', e) \in B' \times E \mid f(b') = \pi(e)\}$$

  with induced projection maps π' and $\hat{f}$. Then $\xi' := \pi' : E' \to B'$ is a fibre bundle with the same structure group and fibre as ξ. If ξ has a local trivialization with respect to the cover $\{U_j\}_{j\in J}$ then ξ' has a local trivialization with respect to $\{f^{-1}(U_j)_{j\in J}\}$. The bundle ξ', called the *pullback of ξ by f* is denoted $f^*(\xi)$ or $f^!(\xi)$.

  Pullback with the trivial map, $f(x) = *$ for all x, produces the trivial bundle $\pi_2 : F \times B \to B$. It is straightforward to check from the definitions that if $f : B' \to B$ and $g : B'' \to B'$ then $(f \circ g)^*(\xi) = g^*\bigl(f^*(\xi)\bigr)$.

- Cartesian Product

  Given $\xi_1 = E_1 \xrightarrow{p_1} B_1$ and $\xi_2 = E_2 \xrightarrow{p_2} B_2$ with the same structure group, define $\xi_1 \times \xi_2$ by $p_1 \times p_2 : E_1 \times E_2 \to B_1 \times B_2$. If F_1 and F_2 are the fibres of ξ_1 and ξ_2 then fibre of $\xi_1 \times \xi_2$ will be $F_1 \times F_2$. If ξ_1 has a local trivialization with respect to the cover $\{U_i\}_{i\in I}$ and ξ_2 has a local trivialization with respect to the cover $\{V_j\}_{j\in J}$ then $\xi_1 \times \xi_2$ has a local trivialization with respect to the cover $\{U_i \times V_j\}_{(i,j)\in I\times J}$. $\xi_1 \times \xi_2$ is called the *(external) Cartesian product* of ξ_1 and ξ_2.

  If ξ_1 and ξ_2 are bundles over the same base B, we can also form the internal Cartesian product of ξ_1 and ξ_2 which is the bundle over B given by $\Delta^*(\xi_1 \times \xi_2)$, where $\Delta : B \to B \times B$ is the diagonal map.

1.3 Vector Bundles

In this section we discuss some additional properties specific to vector bundles.

For a field F, let VS_F denote the category of vector spaces over F.

From the description of G-bundles in terms of transition functions, we see that any functor $T : (\mathrm{VS}_F)^n \to \mathrm{VS}_F$, (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$,) induces a functor

$$T : (\text{Vector Bundles over } B)^n \to \text{Vector Bundles over } B$$

by letting $T(\xi_1, \dots, \xi_n)$ be the bundle whose transition functions are obtained by applying T to the transition functions of $\xi_1 \dots, \xi_n$. In this way we can form, for example, the direct sum of vector bundles, written $\xi \oplus \xi'$, the tensor product of vector bundles, written $\xi \otimes \xi'$, etc. The direct sum of vector bundles is also called their *Whitney sum* and is the same as the internal Cartesian product.

Recall that if V is a vector space over a field F, then to any symmetric bilinear function $f : V \times V \to F$ we can associate a function $q : V \to F$, called its associated quadratic form, by setting $q(v) := f(v, v)$. If the characteristic of F is not 2 we can recover f from q by $f(v, w) = \big(q(v+w) - q(v) - q(w)\big)/2$. In particular, if $\mathbb{F} = \mathbb{R}$, specifying an inner product on V is equivalent to specifying a positive definite quadratic form $q : V \to \mathbb{R}$ (i.e. a quadratic form such that $q(0) = 0$ and $q(v) > 0$ for $q \neq 0$).

Definition 1.3.1 A *Euclidean metric* on a real vector bundle ξ consists of a continuous function $q : E(\xi) \to \mathbb{R}$ whose restriction to each fibre of $E(\xi)$ is a positive definite quadratic form. In the special case where $\xi = TM$ for some manifold M, a Euclidean metric is known as a *Riemannian metric.*

Exercise: Using a partition of unity, show that any bundle over a paracompact base space can be given a Euclidean metric.

We will always assume that our bundles come with some chosen Euclidean metric.

Let ξ be a sub-vector-bundle of η. i.e. we have a bundle map

$$\begin{tikzcd}
E(\xi) \arrow[r, hook] \arrow[d, "p(\xi)"] & E(\eta) \arrow[d, "p(\eta)"] \\
B \arrow[r, equal] & B
\end{tikzcd}$$

We wish to find a bundle $\xi^\perp$ (the orthogonal complement of ξ in η) such that $\eta \cong \xi \oplus \xi^\perp$. Using the Euclidean metric on η, define $E(\xi^\perp) := \{v \in E(\eta) \mid v \perp w \text{ for all } w \in F_{p(v)}(\xi)\}$.

To check that $\xi^\perp$ is locally trivial, for each $b \in B$ find an open neighbourhood $U \subset B$ sufficiently small that $\xi|_U$ and $\eta|_U$ are both trivial.

It is straightforward to check that $\eta \cong \xi \oplus \xi^\perp$.

Given B, let ϵ^k denote the trivial k-dimensional vector bundle $\pi_2 : \mathbb{F}^k \times B \to B$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Proposition 1.3.2 *Let $\xi : E \to B$ be a vector bundle which has a cross-section $s : B \to E$ such that $s(b) \neq 0$ for all b (a nowhere vanishing cross-section). Then ξ has a subbundle isomorphic to the trivial bundle and thus $\xi \cong \epsilon^1 \oplus \xi'$ for some bundle ξ'.*

Proof: Define $\phi : \mathbb{F} \times B \to E$ by $\phi(f, b) = fs(b)$, where $fs(b)$ is the product within F_b of the scalar f times the vector $s(b)$. Then since s is nowhere zero, $\operatorname{Im}\phi$ is a subbundle of ξ and $\phi : \epsilon \to \operatorname{Im}\phi$ is an isomorphism. Thus $\xi \cong \epsilon^1 \oplus \xi'$, where $\xi' = (\operatorname{Im}\phi)^\perp$. □

By induction we get

Corollary 1.3.3 *Let $\xi : E \to B$ be a vector bundle which has a k linearly independent cross-sections $s_j : B \to E$ for $j = 1, \ldots, k$. (That is, for each $b \in B$, the set $\{s_1(b), \ldots, s_k(b)\}$ is linearly independent.) Then $\xi \cong \epsilon^k \oplus \xi'$ for some bundle ξ'. In particular, if a n-dimensional vector bundle has n-linearly independent cross-sections then it is isomorphism to the trivial bundle ϵ^n.*

Recall that a smooth function $j : M \subset N$ between differentiable manifolds is called an immersion if its derivative $(dj)_x : T_xM \to T_yM$ is an injection for each x. (This differs from an embedding in that the function j itself is not required to be an injection.) Let $j : M \to N$ be an immersion from an n dimensional manifold to an $n + k$ dimensional manifold. Then $dj : TM \to TN$ becomes a bundle map and induces an inclusion of TM as a subbundle of the pullback $j^*(TN)$. In this special case the orthogonal complement $TM^\perp$ of TM within $j^*(TN)$ is called the *normal bundle* to M in N. The k-dimensional bundle $TM^\perp$ is often denoted ν_M if N is understood, and called the "normal bundle to M" although of course it depends on N as well. For an immersion $j : M \to \mathbb{R}^q$ we will later see that the normal bundle has some intrinsic properties which are independent of the immersion j.

As before, let γ_n^1 denote the canonical line bundle over $\mathbb{R}P^n$. By definition, $E(\gamma_n^1) = \{(x, v) \mid x \in \mathbb{R}P^n, v \in L_x\}$ where for $x \in \mathbb{R}P^n$, L_x denotes the line in $\mathbb{R}^{n+1}$ joining the two representatives for x. Thus γ_n^1 is a one-dimensional subbundle of the trivial $n+1$ dimensional bundle $\epsilon_{n+1} := \pi_2 : \mathbb{R}^{n+1} \times \mathbb{R}P^n \to \mathbb{R}P^n$. Let $\gamma^\perp$ be the orthogonal complement of γ_1^n in ϵ^{n+1}. Thus $E(\gamma^\perp) = \{(x, v) \mid x \in \mathbb{R}P^n, v \perp L_x\}$. From the functor Hom( ), we define the bundle $\operatorname{Hom}(\gamma_n^1, \gamma^\perp)$ over $\mathbb{R}P^n$.

Lemma 1.3.4 $T(\mathbb{R}P^n) \cong \mathrm{Hom}(\gamma_n^1, \gamma^\perp)$

Proof:
Let $q : S^n \twoheadrightarrow \mathbb{R}P^n$ be the quotient map. Then q induces a surjective map of tangent bundles $T(q) : T(S^n) \twoheadrightarrow T(\mathbb{R}P^n)$ so that $T(\mathbb{R}P^n) = T(S^n)/\sim$ where $(x, v) \sim (-x, -v)$.

Points of $\mathrm{Hom}(\gamma_n^1, \gamma^\perp)$ are pairs (x, A) where $x \in \mathbb{R}P^n$ and A is a linear transformation $A : F_x(\gamma_n^1) \to F_x(\gamma^\perp)$.

Given $(x, v) \in T(\mathbb{R}P^n)$, define a linear transformation $A_{(x,v)} : L_x \to (L_x)^\perp$ by $A_{(x,v)}(x) := v$. This is well defined, since $A(-x) := -v$ defines the same linear transformation. Then $(x, v) \mapsto A_{(x,v)}$ is bundle map which on induces an isomorphism on each fibre and so is a bundle isomorphism. □

Theorem 1.3.5 $T(\mathbb{R}P^n) \oplus \epsilon^1 \cong (\gamma_n^1)^{\oplus(n+1)}$

Proof:
Observe that the one-dimensional bundle $\mathrm{Hom}(\gamma_n^1, \gamma_n^1)$ is trivial since it has the nowhere zero cross-section $x \mapsto (x, I)$ where $I : L_x \to L_x$ is the identity map. Therefore

$$\begin{aligned} T(\mathbb{R}^n) \oplus \epsilon^1 &\cong \mathrm{Hom}(\gamma_n^1, \gamma^\perp) \oplus \mathrm{Hom}(\gamma_n^1, \gamma_n^1) \cong \mathrm{Hom}(\gamma_n^1, \gamma^\perp \oplus \gamma_n^1) \cong \mathrm{Hom}(\gamma_n^1, \epsilon^{n+1}) \\ &\cong \big(\mathrm{Hom}(\gamma_n^1, \epsilon^1)\big)^{\oplus(n+1)} \cong \big((\gamma_n^1)^*\big)^{\oplus(n+1)} \cong (\gamma_n^1)^{\oplus(n+1)} \end{aligned}$$

where we have used Milnor's exercise 3-D stating that if a bundle ξ has a Euclidean metric then it is isomorphic to its dual bundle ξ^*. □

This is an example of the following concept.

Definition 1.3.6 Bundles ξ and η over the same base are called *stably isomorphic* if there exist trivial bundle ϵ_q and ϵ_r such that $\xi \oplus \epsilon_q \cong \eta \oplus \epsilon_r$.

Another example of stably trivial bundles is given in the following example.

Example 1.3.7
It is clear that the normal bundle $\nu(S_n)$ to the standard embedding of S^n in $\mathbb{R}^{n+1}$ has a nowhere zero cross-section so is a trivial one-dimensional bundle. By construction $T(S^n) \oplus \nu(S^n) \cong T(\mathbb{R}^{n+1})$ and the right hand side is also a trivial bundle. Therefore the tangent bundle $T(S^n)$ is stably trivial for all n. A manifold whose tangent bundle is trivial is called *parallelizable* so one could say that all spheres are *stably parallelizable*. Determination of the values of n for which the S^n is actually parallelizable was completed by Frank Adams in the 1960's. Our

techniques will enable us to show that if n is even then S^n is not parallelizable. It is trivial to see that S^1 is parallelizable and not too hard to show that S^3 and S^7 are parallelizable. The general case of odd spheres of dimension is difficult to resolve.

1.4 Free proper actions of Lie groups

We begin by recalling some basic properties of Lie groups and their actions.

Let M be a (smooth) manifold. Given a vector field X and a point $x \in M$, in some neighbourhood of x there is an associated *integral curve* $\phi : (-\epsilon, \epsilon) \to M$ such that $\phi(0) = x$ and $\phi'(t) = X\big(\phi(t)\big)$ for all $t \in (-\epsilon, \epsilon) \to M$, obtained by choosing a chart $\psi : \mathbb{R}^n \to M$ for some neighbourhood of x and applying ψ to the solution of the Initial Value Problem $x'(t) = D\psi^{-1}\Big(X\big(\psi(t)\big)\Big)$; $x'(0) = \psi^{-1}(x)$.

Recall that if G is a Lie group, its associated Lie algebra $\mathfrak{g}$ is defined by $\mathfrak{g} := T_eG$, where e (or sometimes 1) denotes the identity of G.

Let $\mu : G \times M \to M$ be a smooth action of Lie group G on M. For $x \in M$, let $R_x : G \to M$ by $R_x(g) := g \cdot x$. For each $X \in \mathfrak{g}$, μ induces a vector field $X^\#$ by $X^\#(x) = dR_x(X) \in T_{R_x(e)}M = T_xM$ and so to each $x \in M$ there is an associated integral curve through x which we denote by $\exp_{X,x}$. Since it is the derivative of a smooth map, the association $\mathfrak{g} \to T_xM$ is a linear transformation, and in fact is it a Lie algebra homomorphism. In other words $[X, Y]^\# = [X^\#, Y^\#]$.

Using the uniqueness of solutions to IVP's one shows

Lemma 1.4.1

1) $g \circ \exp_{X,x} = \exp_{X,g\cdot x}$ *for* $g \in G$, $X \in \mathfrak{g}$, $x \in M$.

2) $\exp_{\lambda X,x}(t) = \exp_{X,x}(\lambda t)$

Consider the special case where $M = G$ and G acts on itself by left multiplication. Since $\exp_{X,e}$ is defined in a neighbourhood of 0, the Lemma implies that it can be extended to a group homomorphism $\mathbb{R} \to G$ by setting $\exp_{X,e}(t) := \big(\exp_{X,e}(t/N)\big)^N$ for all sufficiently large N. Define the *exponential map* $\exp : \mathfrak{g} \to G$ by $\exp(X) = \exp_{X,e}(1)$.

Returning to the general case, given $X \in \mathfrak{g}$, the exponential map produces, for each $x \in M$, a curve γ_x through x given by $\gamma_x(t) := \exp(tX) \cdot x$ having the property that $\gamma_x(0) = x$ and ${\gamma_x}'(0) = X^\#(x) \in T_xM$.

A continuous function $f : X \to Y$ is called *proper* if the inverse image $f^{-1}(K)$ of every compact set $K \subset Y$ is compact. Obviously, if X is compact, then any continuous function with domain X is proper. A continuous action $G \times X \to X$ is called a *proper action* if the "shearing map" $G \times X \to X \times X$ given by $(g, x) \mapsto (g \cdot x, x)$ is proper.

Lemma 1.4.2 *Let M be a manifold and let x be a point in M. Let V be a vector subspace of T_xM. Then there exists a submanifold $N \subset M$ containing x such that the inclusions $T_xN \subset T_xM$ and $V \subset T_xM$ induce an isomorphism $T_xM \cong V \oplus T_xN$*

Proof: By picking a coordinate neighbourhood U of x in M and composing with the diffeomorphism $U \cong \mathbb{R}^n$, we may reduce to the case where $M \subset \mathbb{R}^n$ and by translation we may assume $x = 0$. By rechoosing the basis for $\mathbb{R}^N$ we may assume that V is spanned by $\{\partial/\partial x_1, \ldots, \partial/\partial x_k\}$. Let $N = \{0, \ldots, 0, x_{k+1}, \ldots, x_n\}$. $\square$

Lemma 1.4.3 *Let $\phi : X \to Y$ be a smooth injection between manifolds of the same dimension. Suppose that $d\phi(x) \neq 0$ for all $x \in X$. Then $\phi(X)$ is open in Y and $\phi : X \to \phi(X)$ is a diffeomorphism.*

Proof: Suppose $y \in \phi(X)$. Find $x \in X$ such that $\phi(x) = y$. Since $d\phi(x) \neq 0$, by the Inverse Function Theorem, $\exists$ open neighbourhoods U of x in X and V of y in Y such that the restriction $\mu : U \to V$ is a diffeomorphism. Then $V = \phi(U) \subset \phi(X)$, so y is an interior point of $\phi(X)$. It follows that $\phi(X)$ is open in Y. The Inverse Function Theorem says that the inverse map is also differentiable at each point, and since differentiability is a local property the inverse bijection $\phi^{-1} : \phi(X) \to X$ is also differentiable. $\square$

Lemma 1.4.4 *Let G be a topological group, let X be a manifold and let $(g, x) \mapsto g \cdot x)$ be an action of G on X. Then the quotient map $q : X \to X/G$ to the space of orbits X/G is open.*

Proof: Let $U \subset X$ be open. Then $q^{-1}\big(q(U)\big) = \cup_{g \in G} g \cdot U$ which is open. Thus $q(U)$ is open by definition of the quotient topology. $\square$

Lemma 1.4.5 *Let $(g, x) \mapsto g \cdot x)$ be a free action of G on M. Then for each $x \in M$, the linear transformation $dR_x : \mathfrak{g} \to T_xM$; $X \mapsto X^{\#}(x)$ is injective.*

Proof: Suppose $X^{\#}(x) = 0$. Let $\gamma(t) := \exp(tX) \cdot x$. Then

$$\gamma'(t) = \lim_{h \to 0} \exp\big((t+h)X\big)/h = \exp(tX) \lim_{h \to 0} \exp(hX)/h = \exp(tX)\gamma'(0) = \exp(tX)X^{\#}(x) = 0$$

So $\gamma(t)$ is constant and thus $\exp(tX) \cdot x = \gamma(0) = x$ for all t. Since the action is free, this implies $\exp(tX) = e$ for all t, and so differentiating gives $X = \exp(tX)'(0) = 0$. $\square$

Theorem 1.4.6 *Let G be a Lie group and let M be a manifold. Let $\mu : G \times M \to M$; $(g, x) \mapsto g \cdot x$ be a free proper action of G on M. Then the space of orbits X/G forms a smooth manifold and the quotient map $q : X \to X/G$ to the space of orbits forms a principal G-bundle.*

Proof: Suppose $x \in M$.

Since the action is free, the linear transformation $dR_x : \mathfrak{g} \to T_xM$ is injective so its image is a subspace of T_xM which is isomorphic to $\mathfrak{g}$. Choose a complementary subspace $V \subset T_xM$ so that $T_xM \cong \mathfrak{g} \oplus V$ (where we are identifying $\mathfrak{g}$ with its image $dR_x\big(\mathfrak{g}\big)$). Apply Lemma 1.4.2 to get a submanifold $N \subset M$ such that $x \in N$ and $T_xN = V$. By construction $T_xM \cong \mathfrak{g} \oplus T_xN$ so by continuity, there is a neighbourhood of $x \in N$ such that $T_yM \cong \mathfrak{g} \oplus T_yN$ for all y in the neighbourhood. Replacing N by this neighbourhood, we may assume $T_yM \cong \mathfrak{g} \oplus T_yN$ for all $y \in N$. Thus $d\mu\big((1, y)\big)$ is an isomorphism for all $y \in N$.

Differentiating $(gh) \cdot y = g \cdot (h \cdot y)$ at $(1, y)$ gives

$$d\mu\big((g, y)\big) = d\mu_g(y) \circ d\mu\big((1, y)\big)\big)$$

where $\mu_g : M \to M$ is the action of g. Since μ_g is a diffeomorphism, it follows that $d\mu\big((g, y)\big)$ is an isomorphism for all $(g, y) \in G \times N$.

We wish to show that there is an open neigbourhood N' of x in N such that $\mu\big|_{G\times N'}$ is an injection.

Note: Since $d\mu\big((1, x)\big) \neq 0$, the existence of some open neighbourhood U of $(1, x) \in G \times N$ such that $\mu\big|_U$ is an injection is immediate from the Inverse Function Theorem. The point is that we wish to show that U can be chosen to have the form $G \times N'$ for some $N' \subset N$.

Suppose there is no such neighbourhood N'. Then for every open set $W \subset N$ containing x there are pairs (g, y) and (h, z) in $G \times W$, such that $(g, y) \neq (h, z)$ but $g \cdot y = h \cdot z$. Thus there exist sequences (g_j, y_j) and (h_j, z_j) in $G \times N$ with $g_j \to x$ and $h_j \to x$ such that for all j, $(g_j, y_j) \neq (h_j, z_j)$ but $g_j \cdot y_j = h_j \cdot z_j$. Since the action is free, if $y_j = z_j$ for some j, then $g_j = h_j$ contradicting $(g_j, y_j) \neq (h_j, z_j)$. Therefore $k_j := h_j^{-1} g_j$ is never 1 for any j. Since $(k_j y_j, y_j) = (z_j, y_j)$ is convergent (converging to(x, x)), it follows that $\{(k_j y_j, y_j\}$ lies in some compact subset of $M \times M$ and therefore, since the action is proper, $\{(k_j, y_j)\}$ lies in some compact subset of $G \times M$. Thus $\{k_j\}$ contains a convergent subsequence, converging to some $k \in G$. Since $(k_j y_j, y_j) = (z_j, y_j) \to (x, x)$, we deduce that $k \cdot x = x$ and so $k = 1$. But then every neighbourhood of $(1, x)$ contains both (g_j, y_j) and (h_j, z_j) for some j, contradicting the conclusion from the Inverse Function Theorem that $(1, x)$ has a neighbourhood U such that $\mu\big|_U$ is an injection.

Thus there exists an open open neighbourhood N' of x in N such that $\mu\big|_{G\times N'}$ is an injection. Let $V_x = \mu(G \times N')$. Since we showed earlier that $d\mu\big((g, y)\big)$ is an isomorphism for all $(g, y) \in G \times N$, it follows from Lemma 1.4.3 that V_x is open in M and that $\mu : G \times N' \to V_x$ is a diffeomorphism.

We have now shown that given $[x] \in M/G$ there exists an open neighbourhood V_x of the representative $x \in M$ such that $\mu : G \times N_x \cong V_x$ for some submanifold N_x containing x. Since $q(V_x)$ is open by Lemma 1.4.4, we have constructed a local trivialization of M/G with respect to the cover $\{q(V_x)\}_{[x]\in X/G}$. Furthermore, the inverse of the shearing map $(y, g) \mapsto (y, g \cdot y)$ is given locally by $(y, g \cdot y) \mapsto (y, g, y) \mapsto (g, y)$ which is the composite

$$V_x \times V_x \xrightarrow{1_{V_x} \times \mu^{-1}} V_x \times G \times N_x \xrightarrow{\pi_2} G \times N_x$$

of differentiable maps and is, in particular, continuous.

The preceding discussion also shows that every point in X/G has a Euclidean neighbourhood. To complete the proof, it remains only to check that X/G is Hausdorff and that the transition functions between overlapping neighbourhoods are smooth, which we leave as an exercise. □

Corollary 1.4.7 *Let H be a closed subgroup of a compact group G. Then the quotient map $G \to G/H$ is a principal H-bundle.*

□

Chapter 2

Universal Bundles and Classifying Spaces

Throughout this chapter, let G be a topological group. The classification of numerable principal G-bundles has been reduced to homotopy theory. In this chapter we will show that for every topological group G, there exists a topological space BG with the property that for every space B there is a bijective correspondence between isomorphism classes of numerable principal G-bundles over B and $[B, BG]$, the homotopy classes of basepoint-preserving maps from B to BG. The bijection is given by taking pullbacks of a specific bundle $EG \to BG$ (to be described later) with maps $B \to BG$, and the first step is to show that the isomorphism class of a pullback bundle $f^*(\xi)$ depends only upon the homotopy class. This is an important theorem in its own right, and will eventually be used to prove that our bijection is well defined.

Lemma 2.0.8 *Let $\xi : E \to B \times I$ be a principal G-bundle in which the restrictions $\xi\big|_{B\times[0,1/2]}$ and $\xi\big|_{B\times[1/2,1]}$ are trivial. Then ξ is trivial.*

Proof: The trivializations ϕ_1 and ϕ_2 of $\xi\big|_{B\times[0,1/2]}$ and $\xi\big|_{B\times[1/2,1]}$ might not match up along $B \times 1/2$ but according to Prop. 1.1.7 they differ by some map $d : B \to G$. Thus replacing $\phi_2(b,t)$ by $d(b)^{-1} \cdot \phi_2(b,t)$ gives a new trivialization of $\xi\big|_{B\times[1/2,1]}$ which combines with ϕ_1 to produce a well defined trivialization of ξ. □

Theorem 2.0.9 *Let G be a topological group and let ξ be a numerable principal G-bundle over a space B. Suppose $f \simeq h : B' \to B$. Then $f^*(\xi) \cong h^*(\xi)$.*

Proof: (Sketch)

1) Let i_0 and i_1 denote the inclusions $B' \to B' \times I$ at the two ends. By definition, $\exists$ $H : B' \times I \to B$ such that $i_0 \circ H = f$ and $i_1 \circ H = g$. Since $f^*(\xi) = H^*\big(i_0^*(\xi)\big)$ and $g^*(\xi) = H^*\big(i_1^*(\xi)\big)$ we are reduced to showing that $i_0^*(\xi) \cong i_1^*(\xi)$. In other words, it suffices to consider the special case where $B = B' \times I$ and f and h are the inclusions i_0 and i_1.

2) Using Lemma 2.0.8 and compactness of I we can construct a trivialization of ξ with respect to a cover of $B' \times I$ of the form $\{U_j \times I\}_{j \in J}$. A partition of unity argument shows that the cover can be chosen to be numerable. (See [4]; Lemma 4(9.5).)

3) Use the partition of unity to construct functions $v_j : B' \to I$ for $j \in J$ having the property that v_j is 0 outside of U_j and $\max_{j \in J} v_j(b) = 1$ for all $b \in B'$.

4) Let ξ be the bundle $p : X \to B' \times I$ and let $\phi_j : U_j \times I \times G \to p^{-1}(U_j \times I)$ be a local trivialization of ξ over $U_j \times I$. For each $j \in J$ define a function $h_j : p^{-1}(U_j \times I) \to p^{-1}(U_j \times I)$ by $h_j\big(\phi_j(b,t,g)\big) = \phi_j\Big(b, \max\big(v_j(b), t\big), g\Big)$. Each h_j induces a self-homeomorphism of G on the fibres, so is a bundle isomorphism.

5) Pick a total ordering on the index set J and for each $x \in X$ define $h(x) \in X$ by composing, in order, the functions h_j for the (finitely many) j such that the first component of $p(x)$ lies in U_j. Check that $h : X \to X$ is continuous using the local finiteness property of the cover U_j. Then $\operatorname{Im} h \subset p^{-1}(B' \times 1)$ and the restriction $h|_{p^{-1}(B' \times 0)}$ is a G-bundle isomorphism $i_0^*(\xi) \to i_1^*(\xi)$. $\square$

Corollary 2.0.10 *Any bundle over a contractible base is trivial.*

Example 2.0.11 Let $\xi := p : E \to S^n$ be a bundle. Let $U^+ = S^n \setminus \{\text{South Pole}\}$ and $U^- = S^n \setminus \{\text{North Pole}\}$. Then $S^n = U^+ \cup U^-$ is an open cover of S^n by contractible sets. By the preceding corollary, the pullbacks (equivalently restrictions) of p to U^+ and U^- are trivial bundles. Thus the isomorphism class of the bundle ξ is uniquely determined by the homotopy class of the transition function $\tau : U^+ \cap U^- \to F$, or equivalently, since the inclusion $S^{n-1} \to U^+ \cap U^-$ is a homotopy equivlence, by its restriction $\tau\big|_{S^{n-1}} \to F$. This restriction is called the "clutching map" of the bundle.

The next step is to describe the bundle $EG \to BG$, called the universal G-bundle. We start by defining the properties we want this bundle to have and then show that such a bundle exists.

Definition 2.0.12 A numerable principal G-bundle γ over a pointed space $\tilde{B}$ is called a *universal* G-bundle if:

1) for any numerable principal G-bundle ξ there exists a map $f : B \to \tilde{B}$ from the base space B of ξ to the base space $\tilde{B}$ of γ such that $\xi = f^*(\gamma)$;

2) whenever f, h are two pointed maps from some space B into the base space $\tilde{B}$ of γ such that $f^*(\gamma) \cong h^*(\gamma)$ then $f \simeq h$.

In other words, a numerable principal G-bundle γ with base space $\tilde{B}$ is a universal G-bundle if, for any pointed space B, pullback induces a bijection from the homotopy classes of maps $[B, \tilde{B}]$ to isomorphism classes of numerable principal bundles over B. If $p : E \to B$ and $p' : E \to B'$ are both universal G-bundles for the same group G then the properties of universal bundles produce maps $\phi : B \to B'$ and $\psi : B' \to B$ such that $\psi \circ \phi \simeq 1_B$ and $\phi \circ \psi = 1_{B'}$. It follows that the universal G-bundle (should it exist) is unique up to homotopy equivalence.

If γ is a universal principal G-bundle and F is a G-space, then we can form the bundle $\big(F \times E(\gamma)\big)/G \to B(\gamma)$ and it is clear from the definitions that it becomes a universal bundle for fibre bundles with structure group G and fibre F.

The first construction of universal G-bundle was given by Milnor, and is known as the Milnor construction, presented below. Nowadays there are many constructions of such a bundle: different constructions give different topological spaces, but of course, they are all homotopy equivalent, according to the preceding discussion. In preparation for the Milnor construction, we discuss suspensions and joins.

2.1 Suspensions and Joins

Let A and B be a pointed topological spaces. The *wedge* of A and B, denoted $A \vee B$, is defined by $A \vee B := \{(a, b) \in A \times B \mid a = * \text{ or } b = *\}$. The *smash product* of A and B, denoted $A \wedge B$, is defined by $A \wedge B := (A \times B)/(A \vee B)$. The contractible space $A \wedge I$ is called the (*reduced*) *cone* on A, and denoted CA. The (*reduced*) *suspension* on A, denoted SA, is defined by $SA := A \wedge S^1$. In other words, $CA = (A \times I)/\big((A \times \{0\}) \cup \{*\}(\times I)\big)$ and $SA = (A \times I)/\big((A \times \{0\}) \cup (A \times \{1\}) \cup (\{*\} \times I)\big)$.

There are also unreduced versions of the cone and suspension defined by $(A \times I)/(A \times \{0\})$ and $(A \times I)/\big((A \times \{0\}) \cup (A \times \{1\})\big)$. It is clear that the unreduced suspension of S^n is homeomorphic to S^{n+1}. The reduced suspension is obtained from the unreduced by collapsing the contractible set $\{*\} \times I$. For "nice" spaces (e.g. CW-complexes), this implies that the reduced and unreduced suspensions are homotopy equivalent. (See Cor. 3.1.9.) For spheres, a stronger statement is true: they are actually homeomorphic.

Proposition 2.1.1 $SS^n \cong S^{n+1}$

Proof: Up to homeomorphism $SS^n = S\left(I^n/\partial(I^n)\right) = I^{n+1}/\sim$ for some identifications $\sim$. Examining the definition we find the identifications are precisely to set $\partial I^{n+1} \sim *$, so we get $SS^n = I^{n+1}/\partial(I^{n+1}) \cong S^{n+1}$. $\square$

For pointed topological spaces A and B, the (*reduced*) *join* of A and B, denoted $A * B$, is defined by $A * B = (A \times I \times B)/\sim$ where $(a, 0, b) \sim (a', 0, b)$, $(a, 1, b) \sim (a, 1, b')$, and $(*, t, *) \sim (*, t'*) \sim *$ for all $a, a' \in A$, $b, b' \in B$, and $t, t' \in I$. It is sometimes convenient to use the equivalent formulation

$$A * B = \{(a, s, b, t) \in A \times I \times B \times I \mid s + t = 1\}/\sim$$

where $(a, 0, b, t) \sim (a', 0, b, t)$, $(a, s, b, 0) \sim (a, s, b', 0)$, and $(*, s, *, t) \sim *$.

Proposition 2.1.2 $A * S^0 \cong SA$.

Proof: Writing $S^0 = \{*, x\}$, $S^0 * A \cong (A \times I \times S^0)/\sim = \big((A \times I) \amalg (A \times I)/\sim$. The identifications $(a, 0, *) \sim (a, 0, x)$ join the two copies of $A \times I$ along $A \times \{0\}$ to produce a space homeomorphic to $A \times [-1, 1]$, so we have $S^0 * A \cong (A \times [-1, 1])/\sim$. The remaining identifications collapse the two ends $A \times \{-1\}$ and $A \times \{1\}$, along with the line $* \times [-1, 1]$ to produce SA. $\square$

Proposition 2.1.3 $S^n * S^k \cong S^{n+k+1}$.

Proof: If $n = 0$, this follows from the previous two propositions. Suppose by induction that $S^m * S^k \cong S^{m+k+1}$ is known for $m < n$. Then

$$S^n * S^k \cong (S^0 * S^{n-1}) * S^k \cong S^0 * (S^{n-1} * S^k) \cong S^0 * S^{n+k} \cong S^{n+k+1}.$$

$\square$

It follows from the definitions that $S(A \wedge B) = (A * B)/(CA \vee CB)$ where CA includes into $A * B$ via $(a, t) \mapsto (a, t, *)$ and CB includes via $(b, t) \mapsto (*, 1 - t, b)$. Since CA and CB are contractible for "nice" spaces this implies that the quotient map $q : A * B \to S(A \wedge B)$ induces an isomorphism on homology. If the basepoints of A and B are closed subsets, for "nice" spaces it is actually a homotopy equivalence, according to [10]; Thm. 7.1.8.

The inclusion map $A \longrightarrow A * B$ is null homotopic since it factors as $A \to CA \to A * B$, and similarly $B \longrightarrow A * B$ is null homotopic.

2.2 Milnor Construction

Let G be a topological group. Let EG be the infinite join $EG = \varinjlim_n G^{*n}$. Explicitly, as a set (not as a topological space),

$$EG = \{(g_0, t_0, g_1, t_1, \dots, g_n, t_n, \dots) \in (G \times I)^\infty\}/\sim$$

such that at most finitely many t_i are nonzero, $\sum t_i = 1$, and

$$(g_0, t_0, g_1, t_1, \dots, g_n, 0, \dots) \sim (g_0, t_0, g_1, t_1, \dots, g'_n, 0, \dots)$$

. The topology is defined as the one in which a subset A is closed if and only if $A \cap G^{*n}$ is closed in G^{*n} for all n. A G-action on EG is given by

$$g \cdot (g_0, t_0, g_1, t_1, \dots, g_n, t_n, \dots) = (gg_0, t_0, gg_1, t_1, \dots, gg_n, t_n, \dots).$$

Let $BG = EG/G$. We also write $E_nG = G^{*(n+1)}$ and $B_n(G) = E_nG/G$, referring to the inclusions $B_0G \hookrightarrow B_1G \hookrightarrow B_2G \hookrightarrow \dots \hookrightarrow B_nG \hookrightarrow \dots$ as the Milnor filtration on BG.

Theorem 2.2.1 (Milnor) *For every topological group G, the quotient map $EG \to BG$ is a numerable principal G-bundle and this bundle is a universal G-bundle.*

Proof: (Sketch)

Let ζ be the bundle $p : EG \to BG$. To show that ζ is a numerable principal G-bundle, define

$$V_i = \{(g_0, t_0, g_1, t_1, \dots, g_n, t_n, \dots) \in EG \mid t_i > 0\}$$

and $U_i = V_i/G \subset BG$. Then $\{U_i\}_{i=0}^\infty$ forms an open cover of BG and one checks that it is a numerable cover. The maps $\phi : V_i = p^{-1}(U_i) \to G \times U_i$ defined by

$$\phi(g_0, t_0, g_1, t_1, \dots, g_n, t_n, \dots) = \big(g_i, p(g_0, t_0, g_1, t_1, \dots, g_n, t_n, \dots)\big)$$

give a local trivialization relative to the cover $\{U_i\}_{i=0}^\infty$ and the conditions for a principal bundle are easily checked.

Let $q : X \to B$ be any numerable principal G-bundle. To construct a map $f : B \to BG$ inducing q from $EG \to BG$ the first step is to choose an arbitrary numerable cover of B over which q is locally trivial and use its partition of unity to replace it by a countable numerable cover $\{W_i\}_{i=0}^\infty$ of B over which q is trivial. Suppose now that $\phi_i : q^{-1}(W_i) \to G \times W_i$ is a local trivialization over W_i and let $\{h_i\}_{i=0}^\infty$ be a partition of unity relative to $\{W_i\}_{i=0}^\infty$. Define $\tilde{f} : X \to EG$ by

$$\tilde{f}(z) = \big(\pi'(\phi_0 z), h_0(qz), \pi'(\phi_1 z), h_1(qz), \dots, \pi'(\phi_i z), h_i(qz), \dots\big)$$

where π' denotes projection onto the first factor. This makes sense since for any i for which $\phi_i z$ is not defined, $h_i z = 0$ and so the first element of the ith component is irrelevant. Since $\tilde{f}$ commutes with the G-action it induces a well defined map $f : B \to BG$ whose pullback with $EG \to BG$ produces the bundle q. Explicitly, the homeomorphism from X to the pullback is given by $x \mapsto \big(qx, \tilde{f}(x)\big)$.

Conversely suppose that f, $f' : B \to BG$ satisfy $f^*(\zeta) \cong f'^*(\zeta)$. Let e denote the identity of G. Let $\alpha : EG \to EG$ be the map

$$\alpha(g_0, t_0, g_1, t_1, \ldots, g_n, t_n, \ldots) = (g_0, t_0, e, 0, g_1, t_1, e, 0, \ldots, e, 0, g_n, t_n, e, 0, \ldots)$$

whose image is contained in the even components and let $\alpha' : EG \to EG$ be the map

$$\alpha'(g_0, t_0, g_1, t_1 \ldots, g_n, t_n, \ldots) = (e, 0, g_0, t_0, e, 0, g_1, t_1, \ldots, e, 0, g_n, t_n, e, 0, \ldots)$$

whose image is contained in the odd components. We construct homotopies $H_{\text{even}} : \alpha \simeq 1_{EG}$ and $H_{\text{odd}} : \alpha' \simeq 1_{EG}$ which commute with the action of G as follows. Given n, let $I_n = [1 - (1/2)^n, 1 - (1/2)^{n+1}] \subset I$ and let $L_n(s) : I_n \to I$ be the affine homeomorphism taking the left endpoint to 0 and the right endpoint to 1. (Explicitly $L_n(s) = 2^{n+1}s - 2^{n+1} + 2$.) Using $I = \cup_n I_n$, define $H_{\text{odd}} : EG \times I \to EG$ by

$$H_{\text{odd}}\big((g_0, t_0, g_1, t_1, \ldots, g_n, t_n, \ldots), s\big) := \Big(g_0, t_0, g_1, t_1, \ldots g_{n-1}, t_{n-1}, g_n, \big(1 - L_n(s)\big)t_n, g_n, L_n(s)t_n, e, 0, g_{n+1}, t_{n+1}, e, 0, \ldots\Big)$$

for $s \in I_n$. Similarly define $H_{\text{even}} : EG \times I \to EG$. Explicitly the components of H_{even} are given by

$$H_{\text{even}}\big((g_0, t_0, g_1, t_1, \ldots, g_n, t_n, \ldots), s\big) = \Big(g_0, t_0, H_{\text{odd}}\big((g_1, t_1, \ldots, g_n, t_n, \ldots), s\big)\Big).$$

Since H_{even} and H_{odd} commute with the G-action, they induce homotopies $\bar{\alpha} \simeq 1_{BG}$ and $\bar{\alpha}' \simeq 1_{BG}$, where $\bar{\alpha}$ and $\bar{\alpha}'$ are the maps on BG induced by α and α'. Let $\theta : E\big(f^*(\zeta)\big) \to E\big(f'^*(\zeta)\big)$ be the homeomorphism on total spaces coming from the G-bundle isomorphism and let $\sigma : E\big(f^*(\zeta)\big) \to EG$ and $\sigma' : E\big(f'^*(\zeta)\big) \to EG$ be the maps induced on total spaces by f and f'. Define $H : E\big(f^*(\zeta)\big) \times I \to EG$ by

$$H(z, s) = \big(g_0, st_0, g'_0, (1 - s)t'_0, g_1, st_1, g'_1, (1 - s)t'_1, g_2, st_2, g'_2, (1 - s)t'_2, \ldots\big)$$

where $\sigma(z) = (g_0, t_0, g_1, t_1, g_2, t_2, \ldots)$ and $\sigma'\big(\theta(z)\big) = (g'_0, t'_0, g'_1, t'_1, g'_2, t'_2, \ldots)$. Then H commutes with the G-action and thus induces a homotopy $\bar{\alpha}f \simeq \bar{\alpha}'f'$. Therefore $f \simeq \bar{\alpha}f \simeq \bar{\alpha}'f' \simeq f'$.

$\square$

To summarize, we have shown

Theorem 2.2.2 *Given any topological group, there exists a classifying space BG having the property that for any space B, pullback sets up a bijection between $[B, BG]$ and isomorphism classes of principal G-bundles over B.*

Examples

- Let $G = \mathbb{Z}/(2\mathbb{Z})$, with the discrete topology. Regard G as the subgroup $S^0 = \{-1, 1\} \subset \mathbb{R}\smallsetminus\{0\}$ of the multiplicative nonzero reals. Then $E_nG = (S^0)^{*n} \cong S^{n+1}$, and the action of G on E_nG becomes the antipodal action of G on S^{n+1}, so $B_nG \cong \mathbb{R}P^n$. Thus $EG \cong S^\infty$, and $BG \cong \mathbb{R}P^\infty$.

- Let $G = S^1$ regarded as the subgroup $S^1 \subset \mathbb{C}\smallsetminus\{0\}$ of the multiplicative nonzero complexes. Then $E_nG = (S^1)^{*n} \cong S^{2n+1}$; $B_nG \cong \mathbb{C}P^n$, $EG \cong S^\infty$, and $BG \cong \mathbb{C}P^\infty$.

- Let $G = S^3$ regarded as the subgroup $S^3 =\subset \mathbb{H}\smallsetminus\{0\}$ of the multiplicative nonzero quaternions. Then $E_nG = (S^1)^{*n} \cong S^{4n+1}$; $B_nG \cong \mathbb{H}P^n$, $EG \cong S^\infty$, and $BG \cong \mathbb{H}P^\infty$.

  An element of the special unitary group $SU(2)$ is uniquely determined by its first row, and this gives an isomorphism of topological groups $SU(2) \cong S^3$. So the preceding can also be regarded as a computation of $BSU(2)$.

□

The following theorem generalizes the special case $K = *$ which was shown in Thm. 2.2.1.

Theorem 2.2.3 *Let K be a subgroup of a topological group G. Then the quotient map*

$$EG/K \to EG/G = BG$$

is locally trivial with fibre homeomorphic to the space G/K of cosets.

Proof: Let $V_i \subset EG$ be as in the previous proof and set $W_i := V_i/K$ and $U_i = V_i/G$. The maps $\phi : V_i = p^{-1}(W_i) \to G/K \times W_i$ defined by

$$\phi(g_0, t_0, g_1, t_1, \ldots, g_n, t_n, \ldots) = \big([g_i], p(g_0, t_0, g_1, t_1, \ldots, g_n, t_n, \ldots)\big)$$

gives a local trivialization of $EG/K \to EG/G$ relative to the cover $\{U_i\}_{i=0}^\infty$ of EG/G. □

It is clear by naturality of the Milnor construction that a group homomorphism $G \to G'$ induces a continuous function $BG \to BG'$.

Proposition 2.2.4 *EG is contractible.*

Proof: Let $\beta : EG \to EG$ be the map

$$\beta(g_0, t_0, g_1, t_1, \ldots, g_{2n}, t_{2n}, \ldots) = (e, 0, g_0, t_0, g_1, t_1, \ldots, g_{2n}, t_{2n}, \ldots).$$

$H : EG \times I \to EG$ by

$$H\big((g_0, t_0, g_1, t_1, \ldots, g_{2n}, t_{2n}, \ldots), s\big) := \\ \Big(g_0, t_0, g_1, t_1, \ldots g_{n-1}, t_{n-1}, g_n, \big(1 - L_n(s)\big)t_n, g_{n+1}, L_n(s)t_{n+1}, g_{n+2}, t_{n+2}, g_{n+3}, t_{n+3}, \ldots\Big)$$

for $s \in I_n$, where I_n and L_n are as in the previous proof. Then $H : \beta \simeq 1_{EG}$. However as observed earlier, the inclusion $B \to A * B$ is always null homotopic and β is the inclusion $EG \to G * EG \cong EG$. Thus $1_{EG} \simeq \beta \simeq *$. □

Note: While H commutes with the action of G and thus induces a homotopy $\beta \simeq 1_{BG}$, the null homotopy of β does not, so it is not valid to conclude that BG is contractible.

Corollary 2.2.5 *Let G be a topological group and let $\xi' = (E'G \to B'G)$ be any universal principal G-bundle. Then $E'G$ is contractible.*

Proof: Let $\xi = EG \to BG$ denote Milnor's universal bundle. The universal properties give inverse homotopy equivalences $\phi : BG' \to BG$, $\psi : BG' \to BG$ such that $\phi^*(\xi) = \xi'$ and $\psi^*(\xi) = \xi'$. It follows from Thm. 2.0.9 that the induced map $E'G \to EG$ is a homotopy equivalence. Thus since EG is contractible, so is $E'G$. □

Remark 2.2.6 As we shall see (Cor. 3.1.25), the converse of this Corollary is also true.

Any element $g \in G$ yields an automorphism ϕ_g of G given by $\phi_g(x) := gxg^{-1}$. Such automorphisms are called "inner automorphism" and other autmorphisms are called "outer automormophisms".

Proposition 2.2.7 *Let ϕ_g be an inner automorphism of the topological group G. Then $B\phi_g \simeq 1_{BG} : BG \to BG$.*

Proof: Let $\xi : EG \to BG$ denote Milnor's universal bundle. By Theorem 2.2.1, to show $B\phi_g \simeq 1_{BG}$ it suffices to show that the principal G-bundles $(B\phi_g)^*(\xi)$ and ξ are G-bundle isomorphic. Write $g^*(EG)$ for the total space of $(B\phi_g)^*(\xi)$. Elements of $g^*(EG)$ are pairs (x, y) where $x = (x_0, t_0, \ldots x_n, t_n, \ldots) \in EG$ and $y = [y_0, t_0, \ldots y_n, t_n, \ldots] \in BG$ with

$$[x_0, t_0, \ldots x_n, t_n, \ldots] = [\phi_g(y_0), t_0, \ldots, \phi_g(y_n), t_n, \ldots],$$

and the action of G on $g^*(EG)$ given by $h \cdot (x, y) = (h \cdot x, y)$. Define $\psi : EG \to g^*(EG)$ by

$$\psi(x_0, t_0, \ldots x_n, t_n, \ldots) := \big((x_0 g, t_0, \ldots, x_n g, t_n, \ldots), [x_0, t_0, \ldots, x_n, t_n, \ldots]\big).$$

The right hand side satisfies the compatibility condition so it does indeed describe an element of $g^*(EG)$. The map ψ is G-equivariant and covers the identity map on BG so it is a principal G-bundle isomorphism. $\square$

Remark 2.2.8 It is not true that $B\phi$ must be homotopic to the indentity if ϕ is an outer automorphism of G. For example, if $G = \big(\mathbb{Z}/(2\mathbb{Z})\big) \times \big(\mathbb{Z}/(2\mathbb{Z})\big)$ and ϕ is the outer automorphism of G which interchanges the two factors then $B\phi : \mathbb{R}P^\infty \times \mathbb{R}P^\infty$ interchanges the two factors and this map is not homotopic to the identity, as seen, for example, by the fact that it does not induce the identity map on cohomology.

Chapter 3

Homotopy Theory of Bundles

In the previous chapter, we reduced the classification of numerable principal bundles to homotopy theory. In this chapter we present some basic homotopy theory, More detail can be found in [10].

For topological spaces X and Y, the set of all continuous functions from X to Y is denoted Y^X, or $\mathrm{Map}\,(X,Y)$. We make it into a topological space by means of the "compact-open" topology, described as follows. Given subsets $A \subset X$ and $B \subset Y$, let $\langle A, B\rangle = \{f \in Y^X \mid f(A) \subset B\}$. The *compact-open* topology on Y^X is the topology generated by the subsets $\{\langle K, U\rangle \mid K \subset X \text{ is compact and } U \subset Y \text{ is open}\}$. If X is a compact Hausdorff space and Y is a metric space, the compact-open topology is given by the corresponding metric.

Theorem 3.0.9 (Exponential Law) *If X is locally compact Hausdorff then $(Y^X)^Z \cong Y^{Z\times X}$ for all Y and Z.* □

For pointed topological spaces, X and Y, the subset of basepoint-preserving maps within $\mathrm{Map}\,(X,Y)$ is denoted $\mathrm{Map}_*\,(X,Y)$.

Theorem 3.0.10 (Reduced Exponential Law) *If X, Y, and Z are pointed topological spaces with X locally compact Hausdorff then*

$$\mathrm{Map}_*\big(Z, \mathrm{Map}_*\,(X,Y)\big) \cong \mathrm{Map}_*\big((Z \wedge X), Y\big).$$

□

The space $\mathrm{Map}_*\,(I,X)$ is called the (*based*) *path space* on X and denoted PX. It is contractible and can be regarded as a dual of the cone CX. The subspace $\mathrm{Map}_*\,(S^1,X)$ is called the (*based*) *loop space* on X, denoted ΩX. The space $\mathrm{Map}\,(S^1, X)$ is called the *free loop space* on X, often denoted ΛX or LX. The (reduced) exponential gives a natural equivalence $\mathrm{Map}_*\,(SX, Y) \cong \mathrm{Map}_*\,(X, \Omega Y)$, so in the language of homological algebra $S(\ )$ and $\Omega(\ )$ are "adjoint functors".

For pointed topological spaces, X and Y, the set of equivalence classes of pointed map from X to Y under the equivalence relation of basepoint-preserving homotopies is denoted $[X, Y]$. Any continuous function $f : Y \to Z$ induces a map denoted $f_\# : [X, Y] \to [X, Z]$, and similarly any $f : W \to X$ induces a map $f^\# : [X, Y] \to [W, Y]$. We write $\pi_n(Y)$ for $[S^n, Y]$. It follows from the definitions that for any pointed space Z, $\pi_0(Z)$ is the set of path components of Z, and $\pi_0\big(\mathrm{Map}_*(X, Y)\big) = [X, Y]$.

In general $[X, Y]$ has no natural group structure, but a natural group structure exists under suitable conditions on either X or on Y. We begin by considering appropriate conditions on Y.

Clearly if Y is a topological group then $[X, Y]$ clearly inherits an induced group structure. But from the homotopy point of view, the rigid structure of a topological group is overkill: all that is really needed from Y is a "group up to homotopy".

An *H-space* consists of a pointed space (H, e) together with a continuous map $m : H \times H \to H$ such that $mi_1 \simeq 1_H$ and $mi_2 \simeq 1_H$ where $i_1(x) = (x, e)$ and $i_2(x) = (e, x)$. An H-space H is called *homotopy associative* if $m \circ (1_H \times m) \simeq m \circ (m \times 1_H)$. If H is an H-space, a map $c : H \to H$ is called a *homotopy inverse* for H if $m \circ (1_H, c) \simeq *$ and $m \circ (c, 1_H) \simeq *$. A homotopy associative H-space together with a homotopy inverse c for H is called an *H-group*. An H-space is called *homotopy abelian* if $mT \simeq m$, where $T : H \times H \to H \times H$ is given by $T(x, y) = (y, x)$. A map $f : X \to Y$ between H-spaces is called an H-map if $m_Y \circ (f \times f) \simeq f \circ m_X$.

Proposition 3.0.11 *Let H be an H-group. Then for all W, the operation $([f], [g]) \to [m \circ (f, g)]$ gives a natural group structure on $[W, H]$. An H-map $f : H \to H'$ induces a group homomorphism $f_\# : [W, H] \to [W, H']$. If H is homotopy abelian then the group $[W, H]$ is abelian.* □

Examples

- Any topological group is an H-group.
- Concatenation of paths, $(\alpha, \beta) \mapsto \alpha \cdot \beta$ where
$$(\alpha \cdot \beta)(t) := \begin{cases} \alpha(2t) & \text{if } t \leq 1/2 \\ \beta(2t - 1) & \text{if } t \geq 1/2 \end{cases}$$
turns ΩY into an H-group for any space Y. The homotopy identity is the constant path at the basepoint, and the homotopy inverse is given by $\alpha^{-1}(t) := \alpha(1 - t)$.

- S^7 becomes an H-space under the multiplication inherited from the multiplication of "Cayley Numbers" (also called "Octonians") on $\mathbb{R}^8$. Since the multiplication is not associative, S^7 is not a topological group and it turns out that it is an example of an H-space which is not an H-group, although the fact that the multiplcation is not homotopy associative is non-trivial. S^7 is the only known example of a finite CW-complex which is an H-space but not a Lie group.

$\square$

A natural group structure on $[X, Y]$ can come from extra structure on X instead of coming from an H-space structure on Y. A *co-H-space* consists of a space X together with a continuous map $\psi : X \to X \vee X$ such that $(1_X \perp *) \circ \psi \simeq 1_X$ and $(* \perp 1_X) \circ \psi \simeq 1_X$ where $f \perp g : A \vee B \to Z$ denotes the map induced by $f : A \to Z$, $g : B \to Z$. Dualizing the other notions in the obvious way gives the concepts of *homotopy (co)associative*, *homotopy inverse*, *co-H-group*, and *homotopy (co)abelian* for a co-H-space, and of a *co-H-map* between co-H-spaces.

Example 3.0.12 For any space X, the map $\psi : SX \to SX \vee SX$ which pinches the equator to a point produces a co-H-space structure on SX.

It is easy to see that co-H-space structure on X yields an H-space structure on $\mathrm{Map}_*(X, Y)$.

Theorem 3.0.13 *Let X be a co-H-group and Y an H-group. Then the group operation on $[X, Y]$ induced by the co-H-structure on X equals the group operation induced by the H-structure on Y. The resulting group structure is abelian.*

$\square$

(See [10], Thm. 7.2.3.) In the special case where $X = SW$, it follows by adjointness from the fact that ΩY is homotopy abelian whenever Y is an H-space, (a statement whose proof is a question that has appeared several times on the Topology Qualifying Exams).

It follows from the preceding discussion that for any pointed space Y, the set $\pi_n(Y)$ has a natural group structure when $n \geq 1$ and that this group structure is abelian if $n \geq 2$. $\pi_n(Y)$ is called the nth homotopy group of Y.

3.1 Fibrations

As in the case of a topological group, the precise structure of a fibre bundle is too rigid from a homotopy viewpoint. For example, in a fibre bundle, the fibres over different points are homeomorphic, where from a homotopy viewpoint it would be more natural to consider a structure in which they were only homotopy equivalent. To this end, we generalize the notion of fiber bundle to that of fibration.

A map $p : E \to B$ is said to have the *homotopy lifting property* with respect to a space Y if, given a homotopy $H : Y \times I \to B$ and a lift $f' : Y \to E$ of H_0, there exists a lift $H' : Y \times I \to E$ of H such that $H'_0 = f'$. A surjection $p : E \to B$ is called a (*Hurewicz*) *fibration* if it has the homotopy lifting property with respect to Y for all Y. The fibre, F, of the fibration $p : E \to B$ is defined as $p^{-1}(*)$.

It is easy to check that

Proposition 3.1.1

a) The composition of fibrations is a fibration.

b) Let $p : E \to B$ be a fibration and let $f : A \to B$ be any map. Then the induced map $p' : P \to A$ from the pullback, P, of p and f is a fibration. □

The standard properties of covering spaces imply that a covering projection is a fibration, and it is also clear that a trivial bundle $\pi_2 : F \times B \to B$ is a fibration. But is it not so obvious that an arbitrary fibre bundle is a fibration.

Theorem 3.1.2 (Hurewicz) *A numerable fibre bundle is a fibration.*

Proof: Let $\xi := p : E \to B$ be a numerable bundle and choose a local trivialization of ξ over some numerable open cover $\mathcal{U} := \{U_j\}_{j \in J}$ of B. Let $\mathcal{S}$ be the set of finite subsets of $\mathcal{U}$. Let $\{\lambda_j : B \to [0,1]\}_{j \in J}$ be a partition of unity relative to $\mathcal{U}$. Given $S \in \mathcal{S}$ define $\lambda_S : B^I \to I$ by

$$\lambda_S(\beta) := \inf_{i=1,\ldots,n} \left\{ \lambda_i \left(\beta \left[\frac{i-1}{n}, \frac{i}{n} \right] \right) \right\}.$$

Set $\theta_n(\beta) := \sum_{|S|<n} \lambda_S(\beta)$ and define $\gamma_S(\beta) = \max(\{\lambda_S(\beta) - n\theta(\beta)\}_{|S|=n} \cup \{0\})$, where $|S|$ denotes the cardinality of S. Set $V_S := \{\beta \in B^I \mid \gamma_S(\beta) > 0\}$. For each $\beta \in B^I$, $S_\beta := \{S \in \mathcal{S} \mid \beta \in V_S\}$ is finite. Choose a total order on $\mathcal{S}$. For each β, this induces a total order on the subset S_β. For $\beta \in B^I$ and $[a,b] \subset [0,1]$, let $\beta_{a,b}$ be the reparameterization of the restriction of β to $[a,b]$ under the linear homeomorphism from $[a,b] \to [0,1]$. That is, $\beta_{a,b}(t) = \beta\big(ta + (1-t)b\big)$.

Let ΓE be the pullback

$$\begin{tikzcd}
\Gamma E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
B^I \arrow[r, "\mathrm{ev}_0"] & B
\end{tikzcd}$$

of p and ev_0, where ev_0 denotes evaluation of the path at 0. Define $\Phi : \Gamma E \to E$ as follows. Given $(\beta, e) \in \Gamma E$, write $S_\beta = \{S_1, \ldots, S_q\}$ where $S_1 < S_2 < \ldots < S_q$. For $j = 0, \ldots, q$, set $t_j := \left(\sum_{i=1}^{j} \gamma_{S_i}(\beta) / \sum_{i=1}^{q} \gamma_{S_i}(\beta)\right)$. Notice that $t_0 = 0$ and $t_1 = 1$ and that $\beta([t_{i-1}, t_i])$ lies in U_{S_i}. Define a sequence of points $e_0, \ldots, e_q \in E$ as follows. Set $e_0 := e$. Then $p(e_0) = \beta(0) = \beta(t_0) \in U_{S_1}$. Using the chosen trivialization $p^{-1}(U_{S_1}) \cong F \times U_{S_1}$, let e_1 be the point in $p^{-1}(U_{S_1})$ corresponding to $\big(f, \beta(t_1)\big)$, where $\big(f, \beta(t_0)\big)$ corresponds to e_0. Next use the chosen trivialization $p^{-1}(U_{S_2}) \cong F \times U_{S_2}$, and let e_2 be the point in $p^{-1}(U_{S_2})$ corresponding to $\big(f, \beta(t_2)\big)$, where $\big(f, \beta(t_1)\big)$ corresponds to e_1. Continuing in this fashion define the sequence $e_0, \ldots, e_q$ and define $\Phi : \Gamma E \to E$ by $\Phi\big((\beta, e)\big) := e_q$.

Now suppose given a diagram

$$\begin{tikzcd} Y \arrow[r, "f'"] \arrow[d, hook, "i_0"] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "H"] & E \end{tikzcd}$$

where f' is a lift of H_0. For $y \in Y$, let $H_y \in B^I$ be the path given by $H_y(s) := H(a, s)$, and let $H_{y,t} = (H_y)_{0,t}$, where, following the notation above, $(H_y)_{0,t}$ denotes the reparameterization of the restriction of H_y to $[0, t]$. Define $H' : Y \times I \to E$ by $H'(y, t) := \Phi\big(H_{y,t}, f'(a)\big)$. Then H' is a lift of H such that $H'_0 = f'$. Therefore ξ is a fibration. □

The analogue of Theorem 2.0.9 holds for fibrations.

Theorem 3.1.3 *Let $p : E \to B$ be a fibration and let f, $\hat{f} : X \to B$ be homotopic. Then the fibrations over X induced from p by taking pullbacks with f and $\hat{f}$ are homotopy equivalent by a homotopy covering the identity of X.*

Proof: As in the proof of Theorem 2.0.9 it suffices to consider the special case where the base has the form $X \times I$ and the maps f, $\hat{f}$ are the inclusions $f = i_0$, $\hat{f} = i_1$ of X at the two ends. Let $E_s = p^{-1}(X \times \{s\})$ denote the pullback of p with i_s. Define $h : X \times I \times I \times I \to X \times I$ by $h(x, r, s, t) = \big(x, (1-t)r + st\big)$. Applying the homotopy lifting property to the commutative diagram

$$\begin{tikzcd} E \times I \arrow[r, "\pi_1"] \arrow[d, "E \times i_o"] & E \arrow[d, "p"] \\ E \times I \times I \arrow[r, "h \circ (p \times 1_I \times 1_I)"] & X \times I \end{tikzcd}$$

gives a map $F : E \times I \times I \to E$ making both resulting triangles commute. Set $K = F_1 : E \times I \to E$ (where F_1 means $F|_{E\times I\times\{1\}}$, following our usual notation for homotopies).

$F(e, \pi_2 pe, t)$ for $0 \le t \le 1$ provides a homotopy from 1_E to the map $k(e) = K(e, \pi_2 pe)$ and satisfies $pF(e, \pi_2 pe, t) = pe$ for all t. Notice that $pK(e, s) = (\pi' pe, s)$ and so in particular $K(e, s)$ belongs to E_s. Define $\alpha : E_0 \to E_1$ by $\alpha(e) = K(e, 1)$ and $\beta : E_1 \to E_0$ by $\beta(e) = K(e, 0)$. Then $E_0 \times I \to E_0$ given by $(e, s) \mapsto K\big(K(e, 1-s), 0\big)$ gives a homotopy from $\beta \circ \alpha$ to $k \circ k|_{E_0}$ covering the constant homotopy $1_X \simeq 1_X$. Similarly $(e, s) \mapsto K\big(K(e, s), 1\big)$ gives a homotopy of $\alpha \circ \beta$ to $k \circ k|_{E_1}$ covering this constant homotopy. Therefore α and β are inverse homotopy equivalences over 1_X. □

Corollary 3.1.4 *Let $p : E \to B$ be a fibration. If b and b' are in the same path component of B, then $p^{-1}(b) \simeq p^{-1}(b')$.*

Proof: Apply Thm. 3.1.3 to the maps $f, f' : * \to B$ given by $f(*) = b$ and $f'(*) = b'$. □

Thm. 3.1.3 also gives

Corollary 3.1.5 *If $F \to E \to B$ is a fibration sequence in which B is contractible, then $E \simeq F \times B$ and in particular the inclusion $F \hookrightarrow E$ is a homotopy equivalence.*

□

Dualizing the concept of fibration gives the following. An inclusion $i : A \hookrightarrow X$ is said to have the *homotopy extension property* with respect to a space Y if given a homotopy $H : A \times I \to Y$ and an extension $f' : X \to Y$ of H_0, there exists an extension $H' : X \times I \to Y$ of H such that $H'_0 = f'$. An inclusion $i : A \hookrightarrow X$ is called a *cofibration* if it has the homotopy extension property with respect to Y for all Y.

Using the exponential law it is straightforward to check that

Proposition 3.1.6 *Let $A \to X$ be a cofibration with A and X locally compact. Then for any space Y the induced maps $Y^X \to Y^A$ and $\mathrm{Map}_*(X, Y) \to \mathrm{Map}_*(A, Y)$ have the homotopy lifting property with respect to Z for all Z and therefore are fibrations if they are surjective.*

□

The following proposition, which includes the dual of Prop. 3.1.1 is easy to check.

Proposition 3.1.7

a) An inclusion $A \hookrightarrow X$ is a cofibration if and only if the inclusion $(A \times I) \cup (X \times 0) \hookrightarrow X \times I$ has a retraction.

b) The composition of cofibrations is a cofibration.

c) Let $i : A \to X$ be a cofibration and let $f : A \to B$ be any map. Then the induced map $i' : B \to Q$ from B to the pushout, Q, of i and f, is a cofibration. □

This shows, in particular, that the inclusion of a subcomplex into a CW complex is a cofibration.
Note: The pushout of maps $f : A \to X$ and $g : A \to Y$ is defined as $(X \amalg Y)/\sim$, where $f(a) \sim g(a)$ for all $a \in A$.

The dual of Thm 3.1.3 also holds.

Theorem 3.1.8 *Let $j : A \to X$ be a cofibration with A closed in X and let $f,\, g : A \to Y$ be homotopic. Then the pushouts of j with f and g are homotopy equivalent.*

(See [10]; Thm. 7.1.8 for a proof.)

This yields the dual of Cor. 3.1.5

Corollary 3.1.9 *Let $A \hookrightarrow X$ be a cofibration where A is contractible and closed in X. Then the quotient map $X \to X/A$ is a homotopy equivalence.*

which was alluded to earlier in the discussion of reduced suspensions.

Of critical importance in homotopy theory is the fact that every map can be "turned into a fibration" according to the following theorem.

Theorem 3.1.10 *Let $f : X \to Y$ be a map of pointed spaces which induces a surjective map on path components. Then there exists a factorization $f = p\phi$ where $\phi : X \to X'$ is a homotopy equivalence and $p : X' \to Y$ is a fibration.*

Proof:

An explicit construction satisfying the conditions is as follows.

Applying Prop. 3.1.6 to the cofibration $\{0\} \cup \{1\} \hookrightarrow I$ gives a fibration

$$\mathrm{ev}_0 \times \mathrm{ev}_1 : Y^I \to Y^{\{0\}\cup\{1\}} = Y \times Y.$$

It follows that $1_X \times \mathrm{ev}_0 \times \mathrm{ev}_1 : X \times Y^I \to X \times Y \times Y$ is also a fibration. Define $q : X \times Y \to X \times Y \times Y$ by $q(x, y) := \big(x, f(x), y\big)$. Taking the pullback of $1_X \times \mathrm{ev}_0 \times \mathrm{ev}_1$ with q. gives a fibration $P^f \to X \times Y$. Explicitly

$$\begin{aligned} P^f &:= \big\{(x, y, x', \gamma) \in X \times Y \times X \times Y^I \mid x' = x, \gamma(0) = f(x), \gamma(1) = y\big\} \\ &\cong \{(x, \gamma) \mid \gamma(0) = f(x)\}\,, \end{aligned}$$

with projection map $\bar{p} : P^f \to X \times Y$ given by $\bar{p}(x, \gamma) = \gamma(1)$. Since the map $\pi_2 : X \times Y \to Y$ is also a fibration, the composition $p := \pi_2 \circ \bar{p} : P^f \to Y$ by composing $\bar{p}$ is a fibration, where $p(x, \gamma) = \gamma(1)$.

It now suffices to prove that there exists a homotopy equivalence $\phi : X \to P^f$. Define $\phi : X \to P^f$ by $\phi(x) := (x, c_{f(x)})$, where $c_{f(x)}$ denotes the constant path at $f(x) \in Y$. Define $\psi : P_f \to X$ by $\psi(x, \gamma) = x$. By definition, $\psi \circ \phi = 1_X$,. A homotopy $H : P^f \times I \to P^f$ from $\phi \circ \psi$ to 1_{P^f} is given by $H(x, \gamma, s) := (x, \gamma_s)$, where for any path γ the path γ_s is given by $\gamma_s(t) := \gamma(st)$. $\square$

The space P^f is called the *mapping path fibration* of f, and can be regarded as a dual of the mapping cylinder $M_f := \big((X \times I) \amalg Y\big)/\sim$, where $f(x, 0) \sim f(t)$.

Given $f : X \to Y$, if $p : X' \to Y$ is a fibration with $\phi : X \simeq X'$ and $f = p\phi$ the the fibre of p is called the "homotopy fibre" of f. The preceding results (particularly Thm. 3.1.3) imply that the homotopy type of F depends only on f and not on the particular manner in which f is converted to a fibration.

If Y is path connected, applying the preceding construction to the map $* \to Y$ gives a fibration $\mathrm{ev} : PY \to Y$ with fibre ΩY, which is thus the homotopy fibre of $* \to Y$.

Proposition 3.1.11 *Let G be a topological group. Then $\Omega BG \simeq G$.*

Proof: Since PBG and EG are contractible, ΩBG and G are both the homotopy fibre of $* \to BG$. $\square$

Lemma 3.1.12 *For any $f : X \to Y$, the homotopy fibre of f is homotopy equivalent to the pullback of f and* $\mathrm{ev} : PY \to Y$.

Proof: By Thm. 3.1.3, if $\psi : X' \to X$ is a homotopy equivalence than the homotopy type of the pullback is not affected by replacing $f : X \to Y$ by $f' := f \circ \psi : X' \to Y$. Thus, in the diagram

$$\begin{tikzcd}
 & \Omega Y \arrow[r, equal] \arrow[d] & \Omega Y \arrow[d] \\
F \arrow[r] \arrow[d, equal] & Q' \arrow[r] \arrow[d] & PY \arrow[d, "\mathrm{ev}_0"] \\
F \arrow[r] & X' \arrow[r, "f'"] & Y
\end{tikzcd}$$

where the bottom right square is a pullback and all the rows and columns are fibrations, the space F is, by definition, the homotopy fibre of f and the the pullback Q' is homotopy equivalent to the pullback of f and ev_0. However the map $F \to Q'$ is a homotopy equivalence by Cor. 3.1.5, since PY is contractible. $\square$

Corollary 3.1.13 *Let ξ be a principal G-bundle. Then the homotopy fibre of the classifying map $f : B(\xi) \to BG$ of ξ is $E(\xi)$.*

Proof: This follows from the Lemma since $EG \simeq *$. $\square$

From the diagram in the previous proof, we see that the fibre of the map $Q' \to X'$ is ΩY. Under the homotopy equivalences $Q' \simeq F$ and $X' \simeq X$, this corresponds to saying that the homotopy fibre of $F \to X$ is ΩY. In other words, from a homotopy fibration $F \xrightarrow{j} X \xrightarrow{f} Y$, we get an induced homotopy fibration $\Omega Y \xrightarrow{\partial} F \xrightarrow{j} X$. For reasons that will soon become apparent, the induced map $\partial : \Omega Y \to F$ is sometimes called the "connecting map" of the fibration.

The process above can be repeated, so from any map $f : X \to Y$ we get an infinite sequence

$$\begin{aligned} \ldots \longrightarrow \Omega^k F \longrightarrow \Omega^k X \longrightarrow \Omega^k Y \longrightarrow \Omega^{k-1} F \longrightarrow \ldots \\ \longrightarrow \Omega X \longrightarrow \Omega Y \xrightarrow{\partial} F \xrightarrow{j} X \xrightarrow{f} Y. \end{aligned}$$

in which every triple of consecutive maps forms a homotopy fibration. A more careful analysis, keeping track of the maps involved, (see [10]; Lemma 7.1.15) shows that the map $\Omega X \to \Omega Y$ is given by $(\Omega f)^{-1}$ and thus the map $\Omega F \to \Omega X$ is given by $(\Omega j)^{-1}$ and by induction the maps $\Omega^k X \to \Omega^k Y$ and $\Omega^k F \to \Omega^k X$ are given by $(-1)^k \Omega^k f$ and $(-1)^k \Omega^k j$ respectively. (Recall that $[W, \Omega Y]$ forms a group and that $[W, \Omega^k Y]$ is abelian for $k > 1$, so we switch to additive notation.)

An important property of fibrations (and thus of fiber bundles) is:

Proposition 3.1.14 *Let $F \xrightarrow{j} E \xrightarrow{p} B$ be a fibration. Then $[W, F] \xrightarrow{j_\#} [W, E] \xrightarrow{p_\#} [W, B]$ is an exact sequence of pointed sets for any space W. (That is, $\operatorname{Im} j_\# = \{x \in [W, F] \mid p_\#(x) = *\}$.)* $\square$

Proof: Since $p \circ f = *$, it is trivial that $p_\# \circ f_\# = *$. Conversely, suppose $p_\#([h]) = *$. Using the homotopy lifting property of p, we can replace h by a homotopic representative h' such that $p \circ h' = *$. Thus the image of h' lies in the subspace F of E and so $[h] \in \operatorname{Im} j_\#$. $\square$

Applying the preceding proposition with $W = S^0$ to the infinite sequence of fibrations above gives

Theorem 3.1.15 *Let $F \to E \to B$ be a homotopy fibration. Then there is a natural (long) exact homotopy sequence*

$$\begin{aligned} \ldots \to \pi_k(F) \to \pi_k(X) \to \pi_k Y \xrightarrow{\partial} \pi_{k-1} F \to \ldots \\ \to \pi_1(X) \to \pi_1(Y) \xrightarrow{\partial} \pi_0(F) \to \pi_0(X) \to \pi_0(Y). \end{aligned}$$

$\square$

If $A \to X$ is a cofibration with A and X locally compact, then for any space Y, such that $\mathrm{Map}_*(X,Y) \to \mathrm{Map}_*(A,Y)$ is surjective, applying the preceding to the fibration $Y^X \to Y^A$ of Prop 3.1.6 gives a long exact sequence

$$\begin{aligned} \ldots \to [S^n(X/A),Y] \to [S^nX,Y] \to [S^nA,Y] \xrightarrow{\partial} [S^{n-1}(X/A),Y] \to \ldots \\ \to [SX,Y] \to [SA,Y] \xrightarrow{\partial} [X/A,Y] \to [X,Y] \to [A,Y]. \end{aligned}$$

Exercise: Let $F \to E \to S^n$ be a fibre bundle. Then the composition $\partial \circ \alpha_{S^{n-1}} : S^{n-1} \to F$ is homotopic to the clutching function of the fibration, where $\alpha_X : X \to \Omega SX$ is the adjoint of the identity map 1_{SX}, given explicitly by $\alpha_X(t) := (x,t)$.

Remark 3.1.16 For an inclusion $A \subset X$ of pointed spaces one can define relative homotopy sets $\pi_n(X,A)$ as basepoint-preserving-homotopy classes of pairs $(D^n, S^{n-1}) \to (X,A)$. For $n \geq 2$, these sets have a natural group which is abelian if $n > 2$. It follows from the definitions that there is an associated long exact homotopy sequence

$$\ldots \to \pi_n(A) \to \pi_n(X) \to \pi_n(X,A) \to \pi_{n-1}(X,A) \to \ldots$$

and so if F is the homotopy fibre of F, the 5-Lemma implies that $\pi_n(X,A) \cong \pi_{n-1}(F)$.

Let $F \to X \xrightarrow{p} B$ and $F' \to Y \xrightarrow{q} B$ be fibration sequences and let $f : X \to Y$ satisfy $qf = p$. Naturality gives a map of the corresponding long exact homotopy sequences, and so it is clear from the 5-Lemma that $f : X \to Y$ induces an isomorphism on homotopy groups if and only if its restriction $f|_F : F \to F'$ induces an isomorphism on homotopy groups. The following stronger statement in proved in [6]; Thms. 1.5 and 2.6.

Theorem 3.1.17 *Let $F \to X \xrightarrow{p} B$ and $F' \to Y \xrightarrow{q} B$ be fibration sequences and let $f : X \to Y$ satisfy $qf = p$. Then $f : X \to Y$ is a homotopy equivalence if and only if $f_b : F_b \to F'_b$ is a homotopy equivalence for all $b \in B$.*

□

Note1: Since the fibres over points in a given path component are homotopy equivalent, it suffices to consider one point b in each path component.

Note 2: For the special case of simply connected CW-complexes, a map which is an isomorphism on homotopy groups is a homotopy equivalence (see [10]; Corollary 7.5.6) so the proof of the theorem in this special case follows from the preceding discussion.

Note 3: Existence of a homotopy cross-section $s : B \to E$ (with $ps \simeq 1_B$) is weaker than the existence of a homotopy retraction $r : E \to B$ and does **not** imply that $E \simeq F \times B$. The situation is analogous to that of a short exact sequence $1 \to A \to B \to C \to 1$ of non-abelian groups: existence of a retraction $r : B \to A$ implies that $B \cong A \times C$, but existence of a splitting $s : C \to B$ says only that B is some semidirect product of A and C.

If $F \xrightarrow{j} E \xrightarrow{p} B$ is a fibration sequence in which j is a homotopy equivalence then it is clear from the long exact homotopy sequence that the homotopy groups of B are trivial. Using Thm. 3.1.17 we get the following stronger converse of Cor. 3.1.5.

Corollary 3.1.18 *Let B be path connected. Let $F \xrightarrow{j} E \xrightarrow{p} B$ be a fibration sequence in which j is a homotopy equivalence. Then B is contractible.*

Proof:

Let $r : E \to F$ be a homotopy inverse to j. Applying Thm. 3.1.17 to the map of fibration sequences

$$\begin{tikzcd}
F \arrow[r, "j"] \arrow[d, "rj"] & E \arrow[r, "p"] \arrow[d, "{(r,p)}"] & B \arrow[d, equal] \\
F \arrow[r, "i_1"] & F \times B \arrow[r, "i_2"] & B
\end{tikzcd}$$

shows that $(r, p) : E \to F \times B$ is a homotopy equivalence. Thus the left square of the diagram shows that $i_1 : F \to F \times B$ is a homotopy equivalence. Therefore applying Thm. 3.1.17 to

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & F \arrow[r, equal] \arrow[d, "i_1"] & F \arrow[d, equal] \\
B \arrow[r] & F \times B \arrow[r] & F
\end{tikzcd}$$

gives that $* \to B$ is a homotopy equivalence. □

Corollary 3.1.19 *Let B and B' be path connected.*

$$\begin{tikzcd}
F' \arrow[r] \arrow[d, "\alpha"] & E' \arrow[r, "p'"] \arrow[d, "\beta"] & B' \arrow[d, "\gamma"] \\
F \arrow[r] & E \arrow[r, "p"] & B
\end{tikzcd}$$

be a map between fibration sequences in which α and β are homotopy equivalences. Then γ is a homotopy equivalence.

Of course, this corollary serves only to strengthen the fact that γ induces an isomorphism on homotopy groups which we can derive immediate from the 5-lemma.

Proof:

By Thm. 3.1.10 we can write γ as a composition $\gamma'' \circ \phi$ where $\phi : B' \simeq B''$ and $\gamma'' : B'' \to B$ is a fibration. Let Q be the pullback of γ'' and p. By Prop. 3.1.1, the induced maps $k_1 : Q \to E$ and $k_2 : Q \to B''$ are both fibrations. From p' and β, the universal property of pullback induces a map $\tau : E' \to Q$ whose composition with k_1 and k_2 are p' and β respectively. Applying Thm. 3.1.10 allows us to write $\tau = \tau'' \circ \psi$ where $\psi : E' \simeq E''$ and τ'' is a fibration. Let $\beta'' = \tau'' \circ k_1 : E'' \to E$ and $p'' := \tau'' \circ k_2 : E'' \to B''$. By Prop. 3.1.1, β'' and τ'' are fibrations. In this way we have replaced the right hand square by one in which β'' and γ'' are fibrations. Let F'', R, and Q be the fibres of the fibrations p'', $p \circ \beta''$, and γ'' respectively. In the diagram

$$\begin{tikzcd}
F'' \arrow[r, hook] \arrow[d, equal] & R \arrow[r] \arrow[d, hook] & Q \arrow[d, hook] \\
F'' \arrow[r, hook] & E'' \arrow[r, "p''"] \arrow[d, "p \circ \beta''"] & B'' \arrow[d, "\gamma''"] \\
 & B \arrow[r, equal] & B
\end{tikzcd}$$

$R = {p''}^{-1}(Q)$, so the top right square is a pullback and thus the induced map $R \to Q$ is also a fibration.

Composing a map with a homotopy equivalence does not change the homotopy type of its homotopy fibre. Thus $R \simeq F$ and the inclusion $F'' \to R$ is a homotopy equivalence corresponding to α. Therefore by Corollary 3.1.18, Q is contractible and so that γ'' is a homotopy equivalence by Thm. 3.1.17, which in turn implies that γ is a homotopy equivalence. □

Remark 3.1.20 The procedure used in the previous proof of replacing commutative diagrams by ones in which the maps are fibrations is a standard technique.

Let X be a G-space. To any principal G-bundle α_X there is an associated fibre bundle with structure group G and fibre X whose total space is given by $\big(X \times E(\alpha)\big)/G$. The case where α is the universal G-bundle has special significance.

Definition 3.1.21 Let X be a G-space. The space $(X \times EG)/G$ is called the *Borel construction* on X, sometimes written X_G. The cohomology groups $H^*(X_G)$ are called the *G-equivariant cohomology groups* of X, denoted $H^*_G(X)$.

In the special case when $X = *$ we get $H^*_G(*) = H^*(BG)$.

Lemma 3.1.22 *Let $X \to X/G$ be a principal G-bundle. Then $X_G \simeq X/G$. Thus if $X \to X/G$ is a principal G-bundle then $H_G(X) = H^*(X/G)$.*

Proof: Let f is the classifying map of the bundle $G \to X \to X/G$. Using Corollary 3.1.13 we have a commuting diagram of homotopy fibrations

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & X_G \arrow[r] \arrow[d] & BG \arrow[d, equal] \\ X \arrow[r] & X/G \arrow[r, "f"] & BG \end{tikzcd}$$

It follows from Theorem 3.1.17 that $X_G \to X/G$ is a homotopy equivalence. □

It is not true that $X_G \simeq X/G$ if the action of G is not free. For example, if $X = *$ then $X/G = *$. But in this case $H^*_G(*) = H^*(BG)$, and as we shall see, $H^*(BG)$ is rarely the same as $H^*(*)$.

Let K be a closed subgroup of a topological group G such that the map $G \to G/K$ is a principal fibration. For example, these hypotheses would hold if K is a compact subgroup of a Lie group G. Given a principal K bundle ξ let ξ_G be the fibre bundle with structure group K and fibre G associated to the principal ξ and the K-space G. That is, $E(\xi_G) = (G \times EK)/K$. There is an action of G on $E(\xi_G)$ given by $g' \cdot [g, x] = [g'g, x]$. Notice that $E(\xi_G)/G \cong E(\xi)/K = B(\xi)$. The local trivialization of the bundle ξ gives a cover of $B(\xi)$ with respect to which the projection map $q : E(\xi_G) \to B(\xi)$ is locally trivial. Thus $q : E(\xi_G) \to B(\xi)$ forms a principal G-bundle.

Lemma 3.1.23 *Let K be a closed subgroup of a topological group G such that $G \to G/K$ is a principal fibration. Let γ be Milnor's universal K-bundle $EK \to BK$ and let η be the principal G-bundle $q : E(\gamma_G) \to BK$. Then the total space of η is homotopy equivalent to the space G/K of cosets and the classifying map $f : BK \to BG$ of the bundle η is homotopic to the map induced by the group homomorphism $K \hookrightarrow G$.*

Proof: Lemma 3.1.22 gives immediately that the total space of η is homotopy equivalent to the space G/K of cosets. The proof is final statement is a matter of writing down the classifying map of a bundle as described in the proof of Theorem 2.2.1. Let $\tilde{f} : E(\gamma_G) \to EG$ be the map defined in proof of Theorem 2.2.1. The trivialization of η over U_i is given by $\phi_i : q^{-1}(U_i) \to G \times U_i$ where $\phi_i(g, y) = (gk_i, [y])$ for $y = (k_0, t_0, k_1, t_1, \ldots, k_n, t_n, \ldots) \in EK$. Therefore the definition of the map $\tilde{f}$ is $\tilde{f}(g, y) = (gk_0, t_0, gk_1, t_1, \ldots, gk_n, t_n, \ldots) \in EG$. The identification $BK \cong E(\gamma_G)/G$ is given by $[y] \mapsto [1, y]$. Thus the map on the base spaces induced by $\tilde{f}$ is just the standard inclusion of BK into BG. $\square$

Theorem 3.1.24 *Let K be a closed subgroup of a topological group G and let $j : K \hookrightarrow G$ denote the inclusion. Suppose that $G \to G/K$ is a principal fibration. Then the homotopy fibre of $Bj : BK \to BG$ is the space of cosets G/K.*

Proof: This is an immediate consequence of Lemma 3.1.23 and Corollary 3.1.13. $\square$

We conclude this section with the converse to Cor. 2.2.5.

Corollary 3.1.25 *Let $\xi = p' : G \to E'G \to B'G$ be a principal G-bundle such that $E'G$ is contractible. Then ξ is a universal G bundle.*

Proof: Let $\xi = p : EG \to BG$ be Milnor's universal bundle. By the universal property, there is a bundle map $\phi : \xi' \to \xi$. Since $\phi_{\text{fibre}} = 1_G$, and $\phi_{\text{tot}} : E'G \to EG$ is a homotopy equivalence, both sides being contractible, it follows from the previous corollary that $\phi_{\text{base}} : B'G \to BG$ is a homotopy equivalence. Thus for any space B, the induced map $\phi_{\#} : [B, B'G] \to [B, BG]$ is a natural bijection, so the bijection between $[B, BG]$ and isomorphism classes of G-bundles over B induced a similar bijection on $[B, B'G]$. $\square$

To summarize, we have shown:

- A principal G-bundle $G \to E \to B$ is a universal G-bundle if and only if E is contractible.
- The base spaces of any two universal G-bundles are homotopy equivalent.

Henceforth we shall use the notation BG to denote any space which is the base space of some universal G-bundle. Thus BG denotes a homotopy type. Any particular topological space which has the homotopy type BG, such as the base space of Milnor's construction, will be called a model for the classifying space.

Chapter 4

Characteristic Classes

Let G be a topological group. If x is a cohomology class in $H^*(BG)$, then for any G-bundle $\xi = p : E \to B$ there is a corresponding cohomology class $f^*(x) \in H^*(B)$, where $f : B \to BG$ is the classifying map of the bundle ξ. Cohomology classes formed in this way are called characteristic classes for the bundle ξ. Many such families of cohomology classes have been well studied and have particular names associated to them. For example, we shall show that $H^*\big(BO(n);\mathbb{F}_2\big) \cong \mathbb{F}_2[w_1,\ldots,w_n]$ with $|w_j| = j$ (where we write $|x| = j$ to mean $x \in H^j(\;)$, and $\mathbb{F}_2$ denotes the field with two elements). The element $w_j \in H^j\big(BO(n);\mathbb{F}_2\big)$ is called the universal jth Stiefel-Whitney class and for any real vector bundle ξ classified by $f : B \to BG$, the element $f^*(w_j) \in H^j\big(B;\mathbb{F}_2\big)$ is called the jth Stiefel-Whitney class of ξ.

For the purpose of calculating $H^*\big(BO(n)\big)$ and $H^*\big(U(n)\big)$ it is convenient to introduce new models for $BO(n)$ and $BU(n)$ different from Milnor's construction. The new models are called the Grassmannians. After describing the Grassmannian model itself, we will describe a cell decomposition for the base of the model which we can use to calculate $H^*\big(BO(n)\big)$ and $H^*\big(BU(n)\big)$. For simplicity, the notation used in the following is that of the case $O(n)$ where the field is $\mathbb{R}$, but one may replace $O(n)$ and $\mathbb{R}$ everywhere by $U(n)$ and $\mathbb{C}$ respectively.

Define the *Stiefel manifold* $V_k(\mathbb{R}^N)$ as the sets of k ordered orthonormal vectors in $\mathbb{R}^N$. That is,

$$V_k(\mathbb{R}^N) := \{(v_1,\ldots v_k) \in (\mathbb{R}^N)^k \mid \|v_i\| = 1 \text{ for all } i \text{ and } v_i \perp v_j \text{ for all } i \neq j\}.$$

If $A \in O(N)$ then columns of $O(N)$ are mutually orthogonal unit length vectors. Define $p : O(N) \to V_k(\mathbb{R}^N)$ to be the map which sends a matrix to its last k columns. For any $x \in V_k(\mathbb{R}^N)$, the elements of $p^{-1}(x)$ differ by the action of the subgroup $O(N-k) \subset O(N)$, so $V_k(\mathbb{R}^N)$ is homeomorphic to the space of cosets $O(N)/O(N-k)$. The topology on $V_k(\mathbb{R}^N)$ can be described either as the subspace topology from $V_k(\mathbb{R}^N) \subset (\mathbb{R}^N)^k$ or equivalently as the quotient topology coming from p. By Thm. 1.4.6, the quotient map $p : O(N) \to V_k(\mathbb{R}^N)$ is a principal $O(N-k)$ bundle, and the Stiefel manifold is indeed a manifold.

Define the *Grassmannian* $G_k(\mathbb{R}^N)$ as the set of k-dimensional subspaces of $\mathbb{R}^N$. There is a canonical map $q : V_k(\mathbb{R}^N) \to G_k(\mathbb{R}^N)$ which sends each set of vectors to its linear span. For a subspace $W \in G_k(\mathbb{R}^N)$, $q^{-1}(W)$ is the set of ordered orthonormal bases for W, any two differing by an element of $O(k)$. Thus $G_k(\mathbb{R}^N) \cong V_k(\mathbb{R}^N)/O(k)$, where $O(k)$ acts by matrix multiplication on the elements of V_k. We topologize $G_k(\mathbb{R}^N)$ by giving it the quotient topology coming from q. Thus $G_k(\mathbb{R}^N)$ is a manifold and $q : V_k(\mathbb{R}^N) \to G_k(\mathbb{R}^N)$ is a principal $O(k)$ bundle. Since $V_k(\mathbb{R}^N) \cong O(N)/O(N-k)$ we can write $G_k(\mathbb{R}^N) \cong O(N)/\big(O(k) \times O(N-k)\big)$ where the inclusion $O(k) \times O(N-k) \to O(N)$ is given by $(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

Taking the limit as $N \to \infty$, we let $V_k^{\mathbb{R}} := V_k(\mathbb{R}^\infty)$ be the sets of k ordered orthonormal vectors in $\mathbb{R}^\infty$ and let $G_k^{\mathbb{R}} := G_k(\mathbb{R}^\infty)$ be the set of k-dimensional subspaces of $\mathbb{R}^N$. Since we are discussing the real case at the moment, we shall simply write G_k and V_k for $G_k^{\mathbb{R}}$ and $V_k^{\mathbb{R}}$. Again there is an action of $O(k)$ on V_k and an induced quotient map $q : V_k \to G_k$ which we use to define the topology on G_k. The spaces V_k and G_k are infinite dimensional analogues of manifolds.

Example 4.0.26 If $k = 1$, then $V_1 = S^\infty$ and $G_1 = \mathbb{R}P^\infty$.

Theorem 4.0.27 *$\gamma := q : V_k \to G_k$ is a principal $O(k)$-bundle.*

Proof:

It is clear from Lemma 1.1.5 that the action of $O(k)$ in V_k is free with a continuous translation function. We must show that γ is locally trivial.

Since, by definition, each element of $\mathbb{R}^\infty$ has only finitely many nonzero components, the standard inner product on $\mathbb{R}^N$ induces an inner product on $\mathbb{R}^\infty$. Write pr_{W} for the orthogonal projection $\mathrm{pr}_W : \mathbb{R}^\infty \to W$.

Given $W \in G_k$, set

$$U_W := \{W' \in G_k \mid \mathrm{pr}_W : W' \to W \text{ is an isomorphism}\}.$$

Then

$$q^{-1}(U_W) = \big\{(v_1, \dots v_k) \in V_k \mid \{\mathrm{pr}_W(v_1), \dots, \mathrm{pr}_W(v_k)\} \text{ forms a linear independent set in } W\big\}.$$

For each N, the complement of $q^{-1}(U_W) \cap V_k(\mathbb{R}^N)$ in $V_k(\mathbb{R}^N)$, consisting of k-tuples whose projections to W are linearly dependent, forms a closed subset of $V_k(\mathbb{R}^N)$, so $q^{-1}(U_W)$ is open in V_k and thus U_W is open in G_k.

Pick a fixed ordered orthonormal basis $w \in q^{-1}(W)$ for W. Given $v = (v_1 \dots v_k) \in q^{-1}(U_W)$, applying Gram-Schmidt to the ordered linearly independent set $\big(\mathrm{pr}_W(v_1), \dots, \mathrm{pr}_W(v_k)\big)$, gives another ordered orthonormal basis b for W and so determines an element $r(v) \in O(k)$ such that $r(v) \cdot w = b$. The map $q^{-1}(U_W) \to O(k) \times U_W$ given by $v \mapsto \big(r(v), q(v)\big)$ defines a homeomorphism giving a local trivialization of q. $\square$

Lemma 4.0.28 *V_k is contractible for all k.*

Proof: The proof is by induction on k. To begin the proof, note that $V_1 = S^\infty$ which contractible since it equals $E\big(Z/(2Z)\big)$.

Let $O(\infty) = O$ be the infinite orthogonal group. Let $O(\infty - m)$ denote the subgroup

$$O(\infty - m) := \begin{pmatrix} I_m & 0 \\ 0 & O(\infty) \end{pmatrix} \subset O(\infty)$$

where I_m denotes the $m \times m$ identity matrix.

Define $q_k : O(\infty) \to V_k$ to be the map which picks out the first k columns of an (infinite) matrix $A \in O(\infty)$. Then as in the finite case, we get that V_k is homeomorphic to the space of cosets $O(\infty)/O(\infty - k)$. Since V_1 is contractible, the inclusion $O(\infty - 1) \hookrightarrow O(\infty)$ is a homotopy equivalence, and more generally, the inclusion $O(\infty - m - 1) \to O(\infty - m)$ is a homotopy equivalence. Therefore the inclusion $O(\infty - k) \hookrightarrow O(\infty)$ is the composition of the k homotopy equivalences $O(\infty - k) \hookrightarrow O(\infty - k + 1) \hookrightarrow O(\infty - k + 2) \ldots \hookrightarrow O(\infty)$ and so it is a homotopy equivalence and thus it follows that V_k is contractible. $\square$

From Theorem 4.0.27 we conclude

Corollary 4.0.29 *$G_k \simeq BO(k)$.*

$\square$

The elements of the total space of the associated universal vector bundle $\gamma^k := (V_k \times \mathbb{R}^k)/O(k) \to G_k$ can be described as pairs (W, v) where W is a k-dimensional subspace of $\mathbb{R}^\infty$ and $v \in W$.

4.1 Cell Structure for Grassmannians

Let W belong to $G_n(\mathbb{R}^N) \subset G_n$. In the sequence of integers $0 \le \dim(W \cap \mathbb{R}^1) \le \dim(W \cap \mathbb{R}^2) \le \ldots \le \dim(W \cap \mathbb{R}^N) = n$, each pair of consecutive integers differ by at most 1. Thus there are precisely n strict inequalities in the sequence. Let $\big(\sigma(W)\big)_i$ be the position where the above sequence changes from $i-1$ to i. Equivalently $\big(\sigma(W)\big)_i$ is the least j such that $\dim(W \cap \mathbb{R}^j) = i$. The sequence $\sigma(W) := \Big(\big(\sigma(W)\big)_1, \big(\sigma(W)\big)_2, \ldots, \big(\sigma(W)\big)_n\Big)$ is called the *Schubert symbol* of W. Given a sequence $\sigma = (\sigma_1, \ldots, \sigma_n)$ with $1 \le \sigma_1 < \ldots < \sigma_n \le N$, set $C_\sigma := \{W \in G_n \mid \sigma(W) = \sigma)$.

Let H^k be the half-plane $H^k := \{x \in R^N \mid \mathrm{Re}(x_k) > 0 \text{ and } x^j = 0 \text{ for } j > k\}$. If $W \in C_\sigma$, then by inductively extending from a basis for $W \cap \mathbb{R}^j$ we can form a basis $w_1, \ldots, w_n$ for W

in which w_i lies in $H^{\sigma(i)}$ for all i. Normalize by dividing w_i by its (nonzero) $\sigma(i)$th component. If we were to write our chosen basis as the rows of an $n \times N$ matrix, it would look like

$$\begin{pmatrix} * & \dots & * & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ * & \dots & * & * & * & \dots & * & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & \dots & * & * & * & \dots & * & * & * & \dots & * & 1 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

Subtracting the appropriate multiple of each basis vector from the ones after it, produces a new basis for W in which the $\sigma(j)$th component of w_i is 0 for $j < i$. This corresponds to doing row operations on our matrix, which now looks like

$$\begin{pmatrix} * & \dots & * & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ * & \dots & * & 0 & * & \dots & * & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & \dots & * & 0 & * & \dots & * & 0 & * & \dots & * & 1 & 0 & \dots & 0 & 0 & 0 \end{pmatrix} \tag{1}$$

Conversely, any W which has a basis of this form lies in C_σ.

Lemma 4.1.1 *Every element of C_σ has a unique basis of the preceding form.*

Proof: Let $w_1, \dots, w_n$ and $w'_1. \dots, w'_n$ be bases for W whose rows form a matrix of the desired form. The element w_1 is uniquely determined as the only element of the one dimensional space $W \cap \mathbb{R}^{\sigma_1}$ which has 1 as its $\sigma(1)$st coordinate, so $w_1 = w'_1$. Assuming by induction that $w_j = w'_j$ for $j < i$, write $w'_i = \sum c_j w_j$ in the basis $w_1, \dots w_i$ for $W \cap \mathbb{R}^{\sigma_i}$. Examining the jth component for each $j \le i$ shows that $c_j = 0$ for $j < i$ and that $c_i = 1$. Thus $w_i = w'_i$. □

Since the elements marked $*$ of the matrix can be filled in with any real number, we get

Corollary 4.1.2 *C_σ is homeomorphic to $\mathbb{R}^{d(\sigma)}$ where $d(\sigma) := (\sigma_1 - 1) + (\sigma_2 - 2) + \dots + (\sigma_n - n)$.*

Notice that $\overline{C_\sigma}$ is contained in $\{C'_\sigma \mid d(\sigma') < d(\sigma)\}$, since the matrix of a limit of elements in C_σ must have zeros in all the places required of an element of C_σ, although the limiting process may introduce more zeros which can might lower $d\big(\sigma(W)\big)$ but cannot raise it. Thus $\{C_\sigma\}$ gives a CW-structure on G_k.

The number of cells σ with $d(\sigma) = j$ equals the number of ways of creating a matrix of the form (1) with $d(\sigma)$ $*$'s, which equals the number of monomials of total degree $d(\sigma)$ in the graded polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$ in which $|x_j| = j$. In the case of the complex Grassmannian, the degree of the cell C_σ is $2d(\sigma)$, so since all the cells are in even degrees there can be no nonzero differentials in the cellular chain complex and conclude that as a group

$H^*\big(BU(n);\mathbb{Z}\big) \cong \mathbb{Z}[c_1,\dots,c_n]$. We will later show that this holds as rings. In the case of the real Grassmannian, the possibility of nonzero differentials exists and we will require a finer analysis to establish that $H^*\big(BO(n);\mathbb{F}_2\big) \cong \mathbb{F}_2[w_1,\dots,w_n]$ even as groups. However there can never be more homology classes than cells, so we do at least get an upper bound for the Betti numbers of $H^*\big(BO(n);\mathbb{F}_2\big)$, namely those of $\mathbb{F}_2[w_1,\dots,w_n]$.

Example 4.1.3 Using the isomorphism $\mathbb{R}\oplus\mathbb{R}^\infty \cong \mathbb{R}^\infty$, we get a map $j : G_{n-1} \to G_n$ given by $W \mapsto \mathbb{R}\oplus W$ corresponding to the inclusion $O(n-1) \hookrightarrow O(n)$ given by $A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$. In other words, j is the limit as $N \to \infty$ of the maps $j_N : G_n(\mathbb{R}^N) \hookrightarrow G_{n+1}(\mathbb{R}^{N+1})$ given by $W \mapsto \mathbb{R}\oplus W$. Under this map, the cell C_σ

$$C_\sigma = \begin{pmatrix} * & \dots & * & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ * & \dots & * & 0 & * & \dots & * & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & \dots & * & 0 & * & \dots & * & 0 & * & \dots & * & 1 & 0 & \dots & 0 & 0 & 0 \end{pmatrix} \subset G_{n-1}$$

is sent to the cell

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & * & \dots & * & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & * & \dots & * & 0 & * & \dots & * & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & * & \dots & * & 0 & * & \dots & * & 0 & * & \dots & * & 1 & 0 & \dots & 0 & 0 & 0 \end{pmatrix} \subset G_n.$$

Thus G_{n-1} is a subcomplex of G_n. The cell of G_n of smallest degree which is not a cell of G_{n-1} is

$$C_{(2,3,\dots,n+1)} = \begin{pmatrix} * & 1 & 0 & \dots & 0 \\ * & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & 0 & 0 & \dots & 1 \end{pmatrix}$$

of degree n.

Let $G_\infty^{\mathbb{R}} := \cup_n G_n^{\mathbb{R}}$ denote the direct limit (union) of the inclusions $G_{n-1} \to G_n$. It consists of all finite dimensional subspaces of $\mathbb{R}^\infty$. Our compatible CW-structures on $G_n^{\mathbb{R}}$ give an induced CW-structure on $G_\infty^{\mathbb{R}}$.

4.2 Thom spaces

In a vector bundle ξ with chosen Riemannian metric, let $D(\xi) = \{x \in E(\xi) \mid \|x\| \le 1\}$, called the *disk bundle* of ξ and let $S(\xi) = \{x \in E(\xi) \mid \|x\| = 1\}$, called the *sphere bundle* of ξ. If $\dim \xi = n$ then the action of $O(n)$ on each fibre of ξ restricts to an action on the fibres of $S(\xi)$, so $S(\xi)$ is also a locally fibre bundle with structure group $O(n)$, but its fibre is S^{n-1} (as compared to fibre $\mathbb{R}^n$ for ξ)).

Example 4.2.1 Let γ^n be the canonical (universal) n-bundle $(V_n \times \mathbb{R}^n)/O(n) \to G_n$ over G_n. An element of $S(\gamma^n)$ is a pair (W, v) where W is an n-dimensional subspace of $\mathbb{R}^\infty$ and $v \in W$ with $\|v\| = 1$. Suppose (W, v) lies in $S(\gamma^n)$. The Riemannian metric determines an complementary $n-1$ dimensional subspace $v^\perp$ to v within W. The association $(W, v) \mapsto v^\perp$ gives a map $q : S(\gamma^n) \to G_{n-1}$ which is locally trivial with respect to the standard covering of G_{n-1} (described in Thm. 4.0.27). The fibres of the bundle q over any subspace $X \in G_{n-1}$ are homeomorphic to the contractible space S^∞ (the fibre F_X consisting of the unit ball in the orthogonal complement of $X \subset \mathbb{R}^\infty$) and so $S(\gamma^n) \simeq G_{n-1}$. Another way to think of it is as follows. Using the isomorphism $\mathbb{R} \oplus \mathbb{R}^\infty \cong \mathbb{R}^\infty$, any $(X, x) \in D(\gamma^{n-1})$ uniquely determines an element $(\mathbb{R} \oplus X, \sqrt{1 - \|x\|^2} e_1 + x) \in S(\gamma^n)$, giving a homeomorphism $D(\gamma^{n-1}) \cong S(\gamma^n)$ and thus a homotopy equivalence $S(\gamma^n) \cong D(\gamma^{n-1}) \simeq G_{n-1}$. These are reflections of the fact that $O(n)/O(n-1) \cong S^{n-1}$ and so the homotopy fibre of the inclusion $BO(n-1) \subset BO(n)$ is S^{n-1}, the fibre of $S(\gamma^n)$, agreeing with the result of Thm. 3.1.24.

Definition 4.2.2 Let ξ be a vector bundle. The space $D(\xi)/S(\xi)$ is called the *Thom space* of ξ, sometimes written $\mathrm{Th}(\xi)$.

Note: In the ambiguous notation X/A which is used for both the space of orbits when A is a group acting on X and for the space obtained by collapsing a subspace A to a point, this is the latter notation.

An alternative description is that $\mathrm{Th}(\xi)$ is the one-point compactification of $E(\xi)$, which generalizes the fact that D^n/S^{n-1} is homeomorphic to the one-point compactification of $\mathbb{R}^n$.

Reflecting the fact that $(D^j \times D^k)/S^{j+k-1} \cong (D^j/S^{j-1}) \wedge (D^k/S^{k-1})$ we get

Proposition 4.2.3 *For vector bundles $\xi_1, \ldots, \xi_r$,*

$$\mathrm{Th}(\xi_1 \times \cdots \times \xi_r) \cong \mathrm{Th}(\xi_1) \wedge \cdots \wedge \mathrm{Th}(\xi_r)$$

□

Example 4.2.4 Let γ^1 be the canonical (universal) line bundle over $\mathbb{R}P^\infty$. Then $D(\gamma^1)$ is homotopy equivalent to its zero cross-section $\mathbb{R}P^\infty$ and $S(\gamma^1) \cong S^\infty$ is contractible, so $\mathrm{Th}(\gamma^1) \cong \mathbb{R}P^\infty$.

Theorem 4.2.5 *(Thom Isomorphism Theorem) Let $\xi := p : E \to B$ be an n-dimensional real vector bundle over a connected base B. Then there exists unique $u \in H^n\big(\mathrm{Th}(\xi); \mathbb{F}_2\big)$ (called the Thom class of ξ) which restricts nontrivially to $H^n\big(F_b/S(F_b); \mathbb{F}_2\big) \cong \mathbb{F}_2$ for each fibre F_b. The Thom class has the property that relative cup product $y \mapsto y \cup u$ is an isomorphism $H^*\big(E; \mathbb{F}_2\big) \to \tilde{H}^{*+n}\big(\mathrm{Th}(\xi); \mathbb{F}_2\big)$. In particular $\tilde{H}^i\big(\mathrm{Th}(\xi); \mathbb{F}_2\big) \cong \mathbb{F}_2$ and $\tilde{H}^i\big(\mathrm{Th}(\xi); \mathbb{F}_2\big) = 0$ for $i < n$.*

Note1: Since the fibres are contractible, the map $p : E \to B$ is a homotopy equivalence so induces an isomorphism $H^*(B) \to H^*(E)$.
Note2: Since $S(\xi)$ is a strong deformation retract of one of its open neighbourhoods in E, $\tilde{H}^*\big(\mathrm{Th}(\xi)\big) \cong H^*\big(E, S(\xi)\big)$.
Proof:
Case 1: ξ is a trivial bundle.

Let $E(\xi) = \mathbb{R}^n \times B$. Then $S(\xi) = S^{n-1} \times B$. In general.

$$\frac{B \times Z}{A \times Z} \cong \frac{(B/A) \times Z}{* \times Z}.$$

so $\mathrm{Th}(\xi) = (D^n \times B)/(S^{n-1} \times B) \cong \big((D^n/S^{n-1}) \times B\big)/(* \times B)$. Since $H^*(D^n/S^{n-1})$ is a free abelian group, from the Künneth Theorem we get $H^*\big((D^n/S^{n-1}) \times B\big) = H^*(D^n/S^{n-1}) \otimes H^*(B)$ and so from the long exact sequence,

$$\tilde{H}^*\big(\mathrm{Th}(\xi)\big) \cong \tilde{H}^*(D^n/S^{n-1}) \otimes H^*(B) \cong H^{*-n}(B) \cong H^{*-n}(E).$$

Let $u_{\mathbb{Z}} \in H^n\big(\mathrm{Th}(\xi)\big)$ correspond under this isomorphism to $u_n \otimes 1 \in \tilde{H}^n(D^n/S^{n-1}) \otimes H^0(B)$, where u_n is a generator of $H^n(D^n/S^{n-1}) \cong \mathbb{Z}$. Since the Künneth isomorphism holds as rings, it follows that $y \mapsto y \cup u_{\mathbb{Z}}$ is an isomorphism $H^*(E) \to \tilde{H}^{*+n}\big(\mathrm{Th}(\xi)\big)$. The image, u, of $u_{\mathbb{Z}}$ under $\tilde{H}^n\big(\mathrm{Th}(\xi)\big) \to \tilde{H}^n\big(\mathrm{Th}(\xi); \mathbb{F}_2\big) \cong H^0(E; \mathbb{F}_2) \cong H^0(B; \mathbb{F}_2) \cong \mathbb{F}_2$ is the unique nonzero element of that group and has the desired properties.

Note: In this case, there was an isomorphism on integral cohomology without need for reduction modulo 2. As we will discuss later, this is a consequence of the fact that the trivial bundle is "orientable". For the remainder of the proof, all cohomology will be with mod 2 coefficients and we will simply write $H^*(\ )$ to mean mod 2 cohomology.
Case 2: B is the union of open sets B' and B'' such that the theorem is known to hold for the pullbacks $\xi' := \xi|_{B'}$, $\xi'' := \xi|_{B''}$, and $\xi''' := \xi|_{B' \cap B''}$.

In the Mayer-Vietoris sequence in mod 2 cohomology

$$\longrightarrow H^{i-1}\big(\mathrm{Th}(\xi''')\big) \longrightarrow H^i\big(\mathrm{Th}(\xi)\big) \longrightarrow H^i\big(\mathrm{Th}(\xi')\big) \oplus H^i\big(\mathrm{Th}(\xi'')\big) \longrightarrow H^i\big(\mathrm{Th}(\xi''')\big) \longrightarrow$$

by uniqueness, the images in $H^n\big(\mathrm{Th}(\xi''')\big)$ of the Thom classes $u' \in H^n\big(\mathrm{Th}(\xi')\big)$ and $u'' \in H^n\big(\mathrm{Th}(\xi'')\big)$ are both u'''. Since the Thom isomorphism theorem for ξ''' shows that

$$H^{n-1}\big(\mathrm{Th}(\xi''')\big) = 0,$$

the sequence gives a unique $u \in H^n\big(\mathrm{Th}(\xi)\big)$ mapping to (u', u''). For $b \in B'$, the restriction of u to the fibre F_b will be nonzero since it is the same as the restriction of u' and similarly u restricts nontrivially to fibres over $b \in B''$. Mapping the Mayer-Vietoris sequence above to

$$\longrightarrow H^{i-1}\big(E(\xi''')\big) \longrightarrow H^i\big(E(\xi)\big) \longrightarrow H^i\big(E(\xi')\big) \oplus H^i\big(E(\xi'')\big) \longrightarrow H^i\big(E(\xi''')\big) \longrightarrow$$

via the corrspondence $y \mapsto y \cup u$ and applying the 5-Lemma, shows that $y \mapsto y \cup u$ is an isomorphism $H^*\big(E\big) \to \tilde{H}^{*+n}\big(\mathrm{Th}(\xi)\big)$.

Case 3: B is covered by finitely many open sets $B_1 \dots B_r$ such that $\xi|_{B_i}$ is a trivial bundle for each i.

Since a subbundle of a trivial bundle is trivial, by Case 1 the theorem holds for the restrictions of ξ to each B_i and to any of their intersections so by induction it holds for ξ.

Case 4: General Case

By case 3, the theorem holds for the restriction of ξ to any compact subset of B.

Note that by the compactness axiom for singular homology,

$$H_*(B) = \varinjlim_{\{\text{compact } K \subset B\}} H_*(K) \qquad \text{and} \qquad H_*\big(\mathrm{Th}(\xi)\big) = \varinjlim_{\{\text{compact } K \subset B\}} H_*\big(\mathrm{Th}(\xi|_K)\big).$$

Since we are working with field coefficients, $H^*(\ ) \cong \mathrm{Hom}\big(H_*(\ ; \mathbb{F}_2); \mathbb{F}_2\big)$, so

$$H^*(B) = \varprojlim_{\{\text{compact } K \subset B\}} H^*(K) \qquad \text{and} \qquad H^*\big(\mathrm{Th}(\xi)\big) = \varprojlim_{\{\text{compact } K \subset B\}} H^*\big(\mathrm{Th}(\xi|_K)\big).$$

The uniqueness condition guarantees that the collection u_K of Thom classes for $\xi|_K$ is compatible and so forms an element of the inverse limit yielding a unique Thom class $u \in H^*(B)$ with the desired properties. □

It follows from the uniqueness property that the Thom class is natural with respect to bundle maps. The Thom isomorphism $\Phi : H^*\big(B(\xi); \mathbb{F}_2\big) \to \tilde{H}^{*+n}\big(\mathrm{Th}(\xi); \mathbb{F}_2\big)$ is defined as the composition of the isomorphisms $p^* : H^*(B; \mathbb{F}_2) \cong H^*\big(E(\xi); \mathbb{F}_2\big)$ and the isomorphism $H^*\big(E(\xi); \mathbb{F}_2\big) \cong \tilde{H}^{*+n}\big(\mathrm{Th}(\xi); \mathbb{F}_2\big)$ given by $y \mapsto y \cup u$.

Making the replacements from in the isomorphisms in the long exact homology sequence of the pair $\big(E(\xi), S(\xi)\big)$ gives

Corollary 4.2.6 *(Gysin sequence) Let ξ be an n-dimensional vector bundle. Then there is a long exact cohomology sequence (with mod 2 coefficients)*

$$\dots \to H^{j-n}(B; \mathbb{F}_2) \xrightarrow{\cup u} H^j(B; \mathbb{F}_2) \to H^j\big(S(\xi); \mathbb{F}_2\big) \to H^{j-n+1}(B; \mathbb{F}_2) \to \dots$$

4.3 Cohomology of $BO(n)$ and $BU(n)$

4.3.1 $BO(n)$

Throughout this subsection, $H^*(\ )$ will denote cohomology with mod 2 coefficients unless specified otherwise.

Let D be the subgroup of diagonal matrices in $O(n)$. Then each diagonal entry in any element of D is ± 1 so $D \cong (Z/2)^n$. Let $P \cong S_n$ be the group of permutation matrices in $O(n)$ (those having precisely one 1 in each row an column with all other entries 0). Let N be the subgroup of $O(n)$ generated by D and P. If $d \in D$ and $\sigma \in P$, then $\sigma d \sigma^{-1}$ lies in D, $D \triangleleft N$ with $N/D \cong P$. and N is the semidirect product $N \cong D \ltimes P$.

Let γ^n be the canonical (universal) n-bundle over $G_n = BO(n)$. Let ξ be the n-fold Cartesian product $\xi := \gamma^1 \times \gamma^1 \times \cdots \gamma^1$. From the definition of the Cartesian product we see that the classifying map $(BO(1))^n \to BO(n)$ is induced by $(A_1, A_2 \ldots A_n) \mapsto \begin{pmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_n \end{pmatrix}$, i.e. the inclusion $j : D = O(1) \times O(1) \times \cdots O(1) \hookrightarrow O(n)$. We will also write j for the bundle map $\xi \to \gamma^n$.

Recall that $H^*\big(BO(1)\big) = H^*\big(\mathbb{R}P^\infty\big) \cong \mathbb{F}_2[t]$ and so $H^*(BD) \cong \mathbb{F}_2[t_1, \ldots, t_n]$. The action $D \mapsto \sigma D \sigma^{-1}$ of each $\sigma \in P$ gives an automorphism of $\sigma : D \to D$ and thus an induced action of the symmetric group on $H^*(BD)$ given by $\sigma \cdot x := (B\sigma^{-1})^*(x)$ for $x \in H^*(BD)$. (The introduction of the inverse makes it a left action.) Evidently the action corresponds to permutation of the polynomial variables in $H^*(BD) \cong \mathbb{F}_2[t_1, \ldots, t_n]$.

Proposition 4.3.1 *The image of* $\mathrm{Im}(Bj)^* : H^*\big(BO(n)\big) \to H^*(BD)$ *lies in the subgroup* $H^*(BD)^{S_n}$ *of elements invariant under the action of the symmetric group.*

Proof: Let σ belong to P. The conjugation action of σ on $O(n)$ is an inner automorphism of $O(n)$ which restricts to our given action on D. Thus there is a commutative diagram

$$\begin{tikzcd} H^*\big(BO(n)\big) \arrow[r, "(B\sigma)^*"] \arrow[d, "(Bj)^*"] & H^*\big(BO(n)\big) \arrow[d, "(Bj)^*"] \\ H^*(BD) \arrow[r, "(B\sigma)^*"] & H^*(BD). \end{tikzcd}$$

According to Prop. 2.2.7, $B\sigma \simeq 1_{BO(n)} : BO(n) \to BO(n)$. Thus the top map in the diagram is the identity and so the commutativity of the diagram forces the image of the bottom map to lie in the invariants of S_n. □

Note: The argument does not imply that the bottom map in the diagram above is the identity since $\sigma|_D$ is an outer automorphism, not an inner automorphism.

Let w_k be the kth elementary symmetric polynomial in $\{t_1, \dots t_n\}$. From MAT1100, we know that $F[t_1, \dots, t_n]^{S_n} = F[w_1, \dots w_n]$ for any field F.

Theorem 4.3.2 *The map* $(Bj)^* : H^*\big(BO(n)\big) \to \mathbb{F}_2[t_1 \dots, t_n]^{S_n} \cong \mathbb{F}_2[w_1 \dots, w_n]$ *is an isomorphism.*

Proof: As in Example 4.2.4, $\mathrm{Th}(\gamma^1) \simeq \mathbb{R}P^\infty$, so by Prop. 4.2.3, $\mathrm{Th}(\xi) \simeq (\mathbb{R}P^\infty)^{(n)}$, where $(\ )^{(n)}$ denotes the n-fold smash product.

$$\begin{tikzcd}
S(\xi) \arrow[r] \arrow[d, "S(j)"] & D(\xi) \simeq (\mathbb{R}P^\infty)^n \arrow[r] \arrow[d, "D(j)"] & \mathrm{Th}(\xi) \arrow[d, "\mathrm{Th}(j)"] \\
S(\gamma^n) \arrow[r] & D(\gamma^n) \simeq BO(n) \arrow[r] & \mathrm{Th}(\gamma^n)
\end{tikzcd}$$

The top right map corresponds to the canonical map $(\mathbb{R}P^\infty)^n \twoheadrightarrow (\mathbb{R}P^\infty)^{(n)}$, which is injective on cohomology.

Let u_ξ and u_{γ^n} denote the respective Thom clases. Then $\mathrm{Th}(j)^*(u_{\gamma^n}) = u_\xi$. We know that $t_1 t_2 \cdots t_n$ is the only nonzero element of $H^n(\mathrm{Th}\,\xi) \cong H^n\big((\mathbb{R}P^\infty)^{(n)}\big)$ and so $u_\xi = t_1 t_2 \cdots t_n$. Thus diagram chasing shows that the nth symmetric polynomial, $w_n = t_1 t_2 \cdots t_n$ lies in $\mathrm{Im}\big(D(j)\big)^* = \mathrm{Im}(Bj)^*$. In particular, the Thom isomorphism gives $\tilde{H}^n(\mathrm{Th}\,\xi) \cong H^2\big(BO(n)\big) \cong \mathbb{F}_2$, and diagram chasing says that the bottom right induces an injection on $H^n(\ )$.

As we noted earlier in Example 4.2.1, $S(\gamma^n) \simeq BO(n-1)$ and the inclusion

$$BO(n-1) \simeq S(\gamma^n) \subset D(\gamma^n) \simeq BO(n)$$

corresponds to the map induced by the inclusion $Bi : BO(n-1) \hookrightarrow BO(n)$. For $k < n$, since the Thom isomorphism theorem tells us that $\tilde{H}^k\big(\mathrm{Th}(\gamma^n)\big) = 0$, we conclude from the long exact cohomology sequence of the bottom row that $H^k(BO(n)) \xrightarrow{(Bi)^*} H^k(BO(n-1))$ is an isomorphism. (Another way to deduce this is to use the long exact homotopy sequence of the bundle $O(n-1) \to O(n) \to S^{n-1}$, or equivalently the fibration $S^{n-1} \to BO(n-1) \to BO(n)$ to determine the connectivity of the pair $\big(BO(n), BO(n-1)\big)$.) Thus for each $k < n$, there exists a unique pre-image $x_k \in H^*\big(BO(n)\big)$ of the Thom class $u_{\gamma^k} \in H^k(BO(k)$. By the preceding, $D(j)^*(x_k)$ projects to $t_1 t_2 \cdots t_k$ under the map $\mathbb{F}_2[t_1 \ldots, t_n]^{S_n} \longrightarrow \mathbb{F}_2[t_1 \ldots, t_k]^{S_k}$ induced by the inclusion $\big((\mathbb{R}P)^\infty\big)^k \to \big((\mathbb{R}P)^\infty\big)^n$. Considering the inclusions $O(1)^{|S|} \hookrightarrow O(n)$ induced by other subsets $S \subset \{1 \ldots n\}$, we see that $D(j)^*(x_k) = w_k$. Since $D(j)^*$ is a ring homomorphism and its image contains the ring generators of $\mathbb{F}_2[w_1 \ldots, w_n]$, we deduce that $D(j)^*$ is surjective. However we already know from our cell decomposition that for each q the dimension of $H^*\big(BO(n)\big)$ is no larger than that of the elements of total degree q in $\mathbb{F}_2[w_1 \ldots, w_n]$, it follows that $D(j)^*$ is an isomorphism. □

For the infinite orthogonal group, using the definition $O := \varinjlim_n \cup_n O(n)$ gives

Corollary 4.3.3 $H^*\big(BO\big) \cong \mathbb{F}_2[w_1 \ldots, w_n, \ldots]$

Remark 4.3.4 There are methods of obtaining the the cohomology of $BO(n)$ which do not rely on the Thom isomorphism theorem. An alternate approach to the construction of the element $w_n \in H^n\big(BO(n)\big)$ is as follows. From the homotopy fibration sequence

$$S^{n-1} \to BO(n-1) \to BO(n)$$

we see that the least nonvanishing relative homotopy group of the pair $\big(BO(n), BO(n-1)\big)$ is

$$\pi_n\big(BO(n), BO(n-1)\big) \cong \pi_{n-1}\big(S^{n-1}\big) \cong \mathbb{Z}.$$

The Relative Hurewicz Theorem gives that

$$\pi_n\big(BO(n), BO(n-1)\big) \cong \pi_{n-1}\big(S^{n-1}\big) \to H_n\big(BO(n), BO(n-1)\big)$$

is onto with kernel generated by elements which differ by the action of $\pi_1\big(BO(n)\big) \cong \mathbb{Z}/(2\mathbb{Z})$ on $\pi_n\big(BO(n), BO(n-1)\big) \cong \mathbb{Z}$. In this case, the action is the antipodal action $m \mapsto -m$, so we get that the least nonvanishing homology group of $\mathrm{Th}(\gamma^n)$ is

$$H_n\big(\mathrm{Th}(\gamma^n)\big); \mathbb{Z}) \cong H_n\big(BO(n), BO(n-1)\big); \mathbb{Z}) \cong \mathbb{Z}/(2\mathbb{Z})$$

The universal coefficient theorem then gives $H^n\big(\mathrm{Th}(\gamma^n)\big) \cong \mathbb{Z}/(2\mathbb{Z})$. Some diagram chasing is required to check that the nonzero element of $H^n\big(\mathrm{Th}(\gamma^n)\big)$ maps nontrivially under $(Bj)^*$.

The element $w_k \in H^*\big(BO(n)\big)$ is called the *kth universal Stiefel-Whitney* class. Given an n dimensional vector bundle ξ classified by a map $f : B(\xi) \to BO(n)$, define the kth Stiefel-Whitney class $w_k(\xi)$ of the bundle by $w_k(\xi) := f^*(\xi) \in H^k\big(B(\xi)\big)$. It follows from the preceding discussion that the top Stiefel-Whitney class $w_n(\xi)$ is the Thom class of ξ. We extend the definition to a definition of w_k for all $k > 0$ by setting $w_k(\xi) = 0$ for $k > n$.

Example 4.3.5 Consider the inclusion $Bi : BO(n-1) \to BO(n)$. According to the preceding discussion, $(Bi)^* : \mathbb{F}_2[w_1 \ldots, w_n] \to \mathbb{F}_2[w_1 \ldots, w_{n-1}]$ is ring homomorphism which sends w_n to 0 while sending w_k to itself for $k < n$. We see from the long exact sequence that $\tilde{H}^*(\mathrm{Th}(\gamma^n))$ is the kernel of $(Bi)^*$, consisting of the ideal (w_n). The Thom isomorphism for γ^n becomes the statement that multiplication by w_n is an isomorphism from $\mathbb{F}_2[w_1 \ldots, w_n]$ to the ideal (w_n).

Proposition 4.3.6 *Suppose $p + q = m$. Let $\alpha : O(p) \times O(q)$ be the homomorphism $(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Abusing notation by writing $(B\alpha)^*$ for its corresponding map after applying the isomorphisms given by $H^*\big(BO(m)\big) \cong \mathbb{F}_2[w_1 \ldots, w_m]$, and the Künneth Theorem, and using the convention that $w_r = 0 \in H^*\big(BO(m)\big)$ if $r > m$, we have $(B\alpha)^*(w_k) = \sum_{i+j=k} w_i \otimes w_j$*

Proof:

$$\begin{tikzcd}
H^*\big(O(m)\big) \arrow[r, "(B\alpha)^*"] \arrow[d, hook] & H^*\big(O(p) \times O(q)\big) \arrow[r, "\cong"] \arrow[d, hook] & \mathbb{F}_2[w_1, \ldots, w_p] \otimes \mathbb{F}_2[w_1, \ldots, w_q] \arrow[d, hook] \\
H^*\big(O(1)^m\big) \arrow[r, equal] & H^*\big(O(1)^p \times O(1)^q\big) \arrow[r, "\cong"] & \mathbb{F}_2[x_1, \ldots, x_p] \otimes \mathbb{F}_2[y_1, \ldots, y_q]
\end{tikzcd}$$

After composing with the injection into $\mathbb{F}_2[x_1, \ldots, x_p] \otimes \mathbb{F}_2[y_1, \ldots, y_q] = \mathbb{F}_2[x_1, \ldots, x_p, y_1, \ldots, y_q]$ the statement becomes equivalent to showing the equality

$$s_k(x_1, \ldots, x_p, y_1, \ldots, y_q) = \sum_{i+j=k} s_i(x_1, \ldots, x_p) s_j(y_1, \ldots, y_q)$$

in $\mathbb{F}_2[x_1, \ldots, x_p, y_1, \ldots, y_q]$, where $s_k(z_1, \ldots, z_m)$ denotes the kth symmetric polynomial in $z_1, \ldots, z_m$. Set $p(t) = \prod_{i=1}^{p}(t + x_i)$ and $q(t) = \prod_{j=1}^{q}(t + y_i)$. Then

$$\begin{aligned}
s_k(x_1, \ldots, x_p, y_1, \ldots, y_q) &= \text{coefficient of } t^k \text{ in } p(t)q(t) \\
&= \sum_{i+j=k} \big(\text{coefficient of } t^i \text{ in } p(t)\big)\big(\text{coefficient of } t^j \text{ in } q(t)\big) \\
&= \sum_{i+j=k} s_i(x_1, \ldots, x_p) s_j(y_1, \ldots, y_q)
\end{aligned}$$

Note: The fact that p and q do not appear in the formula is masked by our convention that $w_r = t_r = 0$ for $r > m$ and also the fact that setting $t_r = 0$ for $r > m$ in $s_i(t_1, \ldots t_n)$ yields $s_i(t_1, \ldots t_m)$ □

4.3.2 $BU(n)$

In this subsection we return to $\mathbb{Z}$ coefficients.

The methods of the preceding subsection can be duplicated to give

Theorem 4.3.7 *$H^*\big(BU(n)\big) \cong \mathbb{Z}[c_1 \ldots, c_n]$ and $H^*\big(BU\big) \cong \mathbb{Z}[c_1 \ldots, c_n, \ldots]$, where $|c_n| = 2n$.*

Proposition 4.3.8 *Suppose $p+q=n$. Let $\alpha : O(p) \times O(q)$ be the homomorphism $(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Abusing notation by writing $(B\alpha)^*$ for its corresponding map after applying the isomorphisms given by $H^*\big(BU(m)\big) \cong \mathbb{Z}[c_1 \ldots, c_m]$, and the Künneth Theorem, and using the convention that $c_r = 0 \in H^*\big(BU(m)\big)$ if $r > m$. we have $(B\alpha)^*(w_k) = \sum_{i+j} c_i \otimes c_j$.*

In comparison with the real case, the complex case is, if anything, easier in some ways. To begin, all the cells of G_n are in even degrees and so we start with an exact count on the size of the cohomology, not just a bound. Also, the spaces $BU(n)$ are all simply connected as compared to $\pi_1\big(BO(n)\big) \cong \mathbb{Z}/(2\mathbb{Z})$. This makes alternate approaches to the calculation, such as a spectral sequence approach, easier. In the real case, the spectral sequence is complicated by the fact that the non-simply-connected base results in dealing with a nontrivial local coefficient system. Computation of the differentials in the spectral sequence is also trivial in the complex case since all the terms are in even degrees.

In the complex case, the group D of diagonal matrices is a maximal abelian subgroup of $U(n)$, known in Lie theory as a maximal torus. The group $N(D)/D \cong S_n$ is the Weyl group W of $U(n)$ and the final answer is often stated in the form $H^*\big(BU(n)\big) \cong \big(H^*(T)\big)^W$ where T is a maximal torus and W is the Weyl group.

The cohomology class c_n is called the nth universal *Chern class* and can be used to define the nth Chern class $c_n(\xi) \in H^{2n}\big(B(\xi)\big)$ of any complex vector bundle ξ by setting $c(\xi) := f^*(c_n)$. where f is the classifying map of ξ.

4.4 Applications of Stiefel-Whitney Classes

Theorem 4.4.1

1. *For any bundle ξ, $w_0(\xi) = 1$.*
2. *Naturality: For any bundle map $f : \xi \to \eta$, $w_j(\xi) = (f_{\text{base}})^*\big(w_j(\eta)\big)$*
3. *Whitney Product Fromula: For any bundles ξ, η over the same base B, $w_k(\xi \oplus \eta) = \sum_{i+j=k} w_i(\xi) w_j(\eta)$*

4. $w_1(\gamma_1) \neq 0$ *for the universal bundle* γ_1 *over* $\mathbb{R}P^1 \cong S^1$.

Proof: Items (1), (2), and (4) are immediate from the definitions. Item (3) follows Prop. 4.3.6 and the fact that if $f : B \to BO(p)$ is the classifying map of ξ and $f' : B \to BO(q)$ then by the definition of Whitney sum, the classifying map of $\xi \oplus \eta$ is the composition

$$B \xrightarrow{\Delta} B \times B \xrightarrow{f \times f'} BO(p) \times BO(q) \xrightarrow{B\alpha} BO(p+q),$$

where $\alpha : O(p) \times O(q) \to O(p+q)$ as in Prop. 4.3.6. □

Theorem 4.4.2 *Stiefel-Whitney classes are the only assignment of cohomology classes* $\tilde{w}_j \in H^j\bigl(B(\xi); \mathbb{F}_2\bigr)$ *to vector bundles* ξ*, satisfying the properties in the preceding theorem.*

Proof: Details of the following outline are left to the reader.

Let $\tilde{w}_j$ be a collection of classes satisfying the properties. Let $\xi_n = \gamma^1 \times \cdots \gamma^1$ over $(\mathbb{R}P^\infty)^n$. Use the Whitney Product Formula and naturality to establish a formula for the (external) Cartesian product of bundles and apply it to compute the classes $\tilde{w}_j(\xi_n)$ from those given by (4) for γ_1. Then apply naturality to the bundle map $\xi_n \to \gamma_n$ to compute the classes $\tilde{w}_j(\gamma_n)$ of the universal bundle from those for ξ. Once it is known that $\tilde{w}_j(\gamma_n) = w_j(\gamma_n)$ for all j, this equality follows for all vector bundles by natural. □

Theorem 4.4.3

1. *If* ϵ *is a trivial vector bundle, then* $w_k(\epsilon) = 0$ *for all* $k > 0$.
2. *If* $\xi = \eta \oplus \epsilon$*, where* ϵ *is a trivial bundle, then* $w_k(\xi) = w_k(\eta)$ *for all* k.

Proof:

(1) follows from the fact that the classifying map of trivial bundle is null homotopic, and (2) follows from (1) via the Whitney Product formula. □

By definition $H^*(X) = \oplus_n H^n(X)$. It is convenient to introduce the notation $H^{\Pi}(X) = \prod_n H^n(X)$. We define the "total Stiefel-Whitney class", $w(\xi) \in H^{\Pi}\bigl(B(\xi)\bigr)$ to be the element whose nth component is $w_n(\xi)$. In this notation, the Whitney product formula can be expressed as saying $w(\xi \oplus \eta) = w(\xi)w(\eta)$. We will write elements of $\prod_n H^n(X)$ is series form: $a_0 + a_1 + \ldots + a_n + \ldots$ will denote the element whose nth factor is a_n.

Lemma 4.4.4 *The group of units of the ring* $\prod_n H^n(X; \mathbb{F}_2)$ *consists of those series with nonzero leading coefficient.*

Proof: Since the leading term of ab is the product of the leading terms of a and b, it is clear that all units have invertible leading coefficients. Conversely, if $a = 1 + a_1 + \ldots + a_n + \ldots$, we can inductively solve for the coefficients of b such that $ab = 1$. □

Since $w_0 = 1$ for any bundle, for any bundle ξ, $w(\xi)$ is always invertible in $H^{\Pi}\bigl(B(\xi)\bigr)$

4.4.1 Immersions

Let $M \subset \mathbb{R}^N$ be an immersion of a manifold M. The immersion determines a normal bundle ν such that $T(M) \oplus \nu \cong T(\mathbb{R}^N) \cong \epsilon^N$. Therefore Whitney's formula gives $w\big(T(M)\big)w(\nu) = 1$ and so $w(\nu) = w\big(T(M)\big)^{-1}$. Notice that $w(\nu)$ is independent of N, provided only that N is large enough so that the bundle ν exists, and can be calculated completely from information about the tangent bundle to M. We will write $w\big(\nu(M)\big)$ to reflect the fact that is it determined entirely by M and does not depend on the choice of immersion. Since $w_k(\xi) = 0$ when $k > \dim \xi$, we conclude that a necessary condition for the existance of an immersion of M in $\mathbb{R}^N$ is that the components $\Big(w\big(\nu(M)\big)\Big)_k$ be zero for $k > N - \dim M$. Equivalently $N \geq \dim M + K$ where K is the largest integer for which $\Big(w\big(\nu(M)\big)\Big)_K \neq 0$. This can be used to give lower bounds least N such that M has an immersion in R^N.

Remark 4.4.5 Whitney gave upper bounds for the "immersion number" of a manifold M by showing that any n-dimensional manifold can be immersed in $\mathbb{R}^{2n-1}$, and this bound was refined by the Immersion Conjecture (Ralph Cohen, 1983) which says that any n-dimensional manifold can be immersed in $\mathbb{R}^{2n-\alpha(x)}$ where $\alpha(n)$ is the number of 1's in the diadic expansion of n. Of course any specific manifold might be immersable in a lower number than the generic number for manifolds of its dimension.

Example 4.4.6 Write $H^*(\mathbb{R}P^n; \mathbb{F}_2) = \mathbb{F}_2[t]/(t^{(n+1)})$ which, because of the truncation, is the same as $H^{\prod}(\mathbb{R}P^n; \mathbb{F}_2)$. For the canonical line bundle γ_n^1 we have $w(\gamma_n^1) = 1+t$. Using Thm. 1.3.5 we calculate $\Big(w\big(T(\mathbb{R}P^n)\big)\Big) = (1+t)^{(n+1)}$ and so $\Big(w\big(\nu(\mathbb{R}P^n)\big)\Big) = (1+t)^{-(n+1)}$.

For example, if $n = 9$, calculation shows that $(1+t)^{-10} = 1 + t^2 + t^4 + t^6$ so the least possible degree for the dimension of a normal bundle to $\mathbb{R}P^9$ is 6 and thus it is not possible to immerse $\mathbb{R}P^9$ in R^N for any $N < 15$. Whitney's theorem guarantees that it **is** possible to immerse $\mathbb{R}P^9$ in $\mathbb{R}^{17}$ and the immersion conjecture improves on this, saying that it should be immersible in $\mathbb{R}^{16}$, but it is more difficult to determine whether or not it has an immersion in $\mathbb{R}^{15}$.

The problem of determining the best possible immersion number for $\mathbb{R}P^n$ for all n is still open.

4.4.2 Parallizability

If M is stably parallelizable then $w\big(T(M)\big) = 1$.

Theorem 4.4.7 *If $n \neq 2^s - 1$ for some s, then $\mathbb{R}P^n$ is not stably parallelizable.*

Proof: $w\big(T(\mathbb{R}P^n)\big) = (1+t)^{n+1}$, so $w\big(T(\mathbb{R}P^n)\big) = (1+t)^{n+1} = 1$ if and only if $\binom{n+1}{j} \equiv 0 \pmod 2$ for $j = 1, \dots n$ and this happens if and only if $n + 1 = 2^s$ for some s. □

A fortiori, $\mathbb{R}P^n$ cannot be parallizable unless $n = 2^s - 1$ for some s. In the converse direction we have,

Theorem 4.4.8 *If there exists a continuous bilinear operation (not necssarily associative nor necessarily having an identity) $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ without zero divisors, then S^{n-1} and $\mathbb{R}P^{n-1}$ are parallelizable.*

Proof: Suppose such a multiplication exists. Write $\mathbb{R} = \langle e_1, \dots e_n \rangle$. Since the multiplication has no zero divisors, $y \mapsto ye_j$ is a self-homeomorphism of $\mathbb{R}^n$ for all j. Thus for each fixed j, every element of $\mathbb{R}^n$ has a unique expression in the form ye_j. For $i = 2, \dots n$, define $v_i : \mathbb{R}^n \to \mathbb{R}^n$ by $v_i(ye_1) = ye_i$ which is therefore a well defined continuous map. Given $x \in S^{n-1}$, let $L_x \subset \mathbb{R}^n$ be the line joining x to $-x$. Similarly given $x \in \mathbb{R}P^{n-1}$, let $L_x \subset \mathbb{R}^n$ be the line joining the two representatives of x. Let $\sigma_i(x)$ be the orthogonal projection of $v_i(x)$ onto $L_x^\perp$. The pair $\big(x, \sigma_i(x)\big) \in T_x(\mathbb{R}P^n)$ is independent of choice of representative x, since $\big(x, \sigma_i(x)\big) \sim \big(-x, \sigma_i(-x)\big)$. The cross-sections $\{x \mapsto \big(x, \sigma_i(x)\big)\}_{i=2,\dots n}$ are linearly independent $T_x(S^{n-1})$ and $T_x(\mathbb{R}P^{n-1})$ respectively and so give a trivialization of the tangent bundle $T(S^{n-1})$ and $T(\mathbb{R}P^n)$ respectively. □

It follows from the multiplication of complexes, quaternions, and Cayley numbers (octonians), which are norm-preserving and thus have no zero divisors, that S^n and $\mathbb{R}P^n$ are parallelizable if $n = 1$, 3, or 7. It turns out that these are the only cases in which $\mathbb{R}P^n$ is parallelizable.

4.5 Cobordism

Definition 4.5.1 Two compact n-dimensional manifolds M and N are called *cobordant* if there exists an $n+1$ dimensional manifold-with-boundary W, such that $\partial W = M \amalg N$. W is called a *cobordism* between M and N.

It is easy to see that cobordism is an equivalence relation among the diffeomorphism classes of n-dimensional manifolds. (For transitivity, glue the cobordisms along their common boundary component.)

For an n-manifold M, let $[M] \in H^n(M; \mathbb{F}_2)$ denote its orientation class. (Recall that every manifold is mod 2 orientable so we do not need to add orientability as a hypothesis.)

Theorem 4.5.2 *Suppose M and N are cobordant n-manifolds. Then $\langle Y, [M] \rangle = \langle Y, [N] \rangle$ for any degree n product of Stiefel-Whitney classes $Y = w_1^{r_1} w_2^{r_2} \cdots w_n^{r_n}$ (where $\sum_{j=1}^n j r_j = n$).*

Proof: Let W be a cobordism between M and N. We can regard W as a cobordism between $M \amalg N$ and $\emptyset$. Since $[M \amalg N] = [M] - [N] \in H^n(M \amalg N; \mathbb{F}_2) = H^n(M; \mathbb{F}_2) \oplus H^n(N; \mathbb{F}_2)$, it suffices to consider the case where $N = \emptyset$ so that $M = \partial W$. Write $[W, M] \in H^{n+1}(W, M; \mathbb{F}_2)$ for the orientation class of the manifold-with-boundary W. Let $i : M \hookrightarrow W$ denote the inclusion. At every $x \in M$, the Riemannian metric on W uniquely determines a tangent direction within TW which is perpendicular to TM and points outward. This gives a decomposition $TW|_M \cong TM \oplus \epsilon$ so $w(M) = w\big((TW|_M\big)$. In particular, $w_j \in \operatorname{Im} i^* : H^j(W; \mathbb{F}_2) \to H^j(M; \mathbb{F}_2)$ for all j and so $Y \in \operatorname{Im} i^* : H^n(W; \mathbb{F}_2) \to H^n(M; \mathbb{F}_2)$. Thus the long exact sequence

$$\ldots \to H^n(W; \mathbb{F}_2) \xrightarrow{i^*} H^n(M; \mathbb{F}_2) \xrightarrow{\delta} H^{n+1}(W, M; \mathbb{F}_2) \to \ldots$$

implies that $\delta(Y) = 0 \in H^{n+1}(W, M; \mathbb{F}_2)$. Therefore letting $\partial : H_*(W, M) \to H_{*-1}(M)$ denote the connecting homomorphism we have

$$\langle Y, [M] \rangle = \langle Y, \partial[W, M] \rangle = (-1)^{n+2} \langle \delta Y, [W, M] \rangle = 0.$$

$\square$

Example 4.5.3

$w\big(T(\mathbb{R}P^2)\big) = (1+t)^3 = 1 + t + t^2$ so $w_1\big(T(\mathbb{R}P^2)\big) = t$ and $w_2\big(T(\mathbb{R}P^2)\big) = t^2$. Therefore using either $r_1 = 2, r_2 = 0$ or $r_1 = 0, r_2 = 1$ shows that $Y = t^2$ is a product of Stiefel-Whitney classes. Since $\langle t^2, [\mathbb{R}P^2] \rangle = 1 \neq 0$, this shows that $\mathbb{R}P^2$ is not the boundary of any 3-manifold.

Chapter 5

Orientable Bundles

A linear transformation $A \in \mathrm{GL}_n(\mathbb{R})$ is orientation-preserving or orientation-reversing according to the sign of its determinant. An orientation on a real vector space V is a choice of equivalence class of basis for V where two bases are equivalent if the linear transformation which takes one to the other is orientation-preserving.

A *orientation for a real vector bundle* $\xi = p : E \to B$ consists of an assignment of an orientation to each fibre F_b which is compatible in the sense that there is a local trivialization of ξ in the neighbourhood of U_b such that for each $x \in U_b$ the orientation on F_x induced by the isomorphism $F_b \cong F_b \times \{x\} \cong F_x$ coming from the restriction of the homeomorphism $F_b \times U_b \cong p^{-1}(U_b)$ agrees with the chosen orientation on F_x. Equivalently, an orientation is an assigment of a generator $u_F \in H^n\bigl(F_b, S(F_b)\bigr) \cong \mathbb{Z}$ for each fibre F_b which is compatible in the sense that for each for each b there is a neighbourhood U_b and a cohomology class $u \in H^n\bigl(\xi|_{U_b}, S(\xi|_{U_b})\bigr)$ such that $u|_{\bigl(F_x, S(F_x)\bigr)} = u_{F_x}$ for all $x \in U_b$.

Generalizing the latter formulation, for any ring R, we can define an orientation with coefficients in R to be an an assignment of a generator $u_F \in H^n\bigl(F_b, S(F_b); R\bigr) \cong R$ for each fibre F_b which is compatible in sense above. Any ring homomorphism $R \to S$, takes an orientation with coefficients in R to one with coefficients in S. Thus if a bundle is orientable (over $\mathbb{Z}$) then it is orientable over R for all R.

Note: The universal bundle γ_n is **not** orientable.

Theorem 5.0.4 *(Thom Isomorphism Theorem for orientable bundles) Let $\xi := p : E \to B$ be an oriented n-dimensional real vector bundle over a connected base B and let $u_{F_b} \in H^n\bigl(F_b, S(F_b)\bigr)$ be the chosen orientation. Then there exists unique $u \in H^n\bigl(\mathrm{Th}(\xi); R\bigr)$ which restricts to u_{F_b} for each fibre F_b. The class u has the property that relative cup product $y \mapsto y \cup u$ is an isomorphism $H^*\bigl(E; R\bigr) \to \tilde{H}^{*+n}\bigl(\mathrm{Th}(\xi); R\bigr)$. In particular $\tilde{H}^i\bigl(\mathrm{Th}(\xi); R\bigr) \cong R$*

and $\tilde{H}^i\big(\mathrm{Th}(\xi)\big) = 0$ *for* $i < n$.

Proof: In the case where R is a field or the spaces are compact, the proof of the mod 2 Thom isomorphism carries over intact, given our hypothesis on the capatiblity of the u_F's. If R is not a field, there is a problem with Case 4. We can repeat the proof there as far as the statement

$$H_*\big(\mathrm{Th}(\xi)\big) = \varinjlim_{\{\text{compact } K \subset B\}} H_*(\mathrm{Th}(\xi|_K), \tag{5.0.1}$$

but cannot proceed to the next line since without field coeffients it is not true in general that $H^*(\ ;R) \cong \mathrm{Hom}\big(H_*(\ ;R);R\big)$, so we will use the Universal Coefficient Theorems to deduce the general case from that for fields and compact spaces.

Since the theorem holds for compact spaces, it follows that $H_k(\mathrm{Th}(\xi|_K) = 0$ for $k < n$ whenever K is a compact subset of B. Therefore Eqn. 5.0.1 tells us that $H_{n-1}\big(\mathrm{Th}(\xi)\big) = 0$. However one version of the Universal Coefficient Theorem says that for any X there is a short exact sequence

$$0 \to \mathrm{Ext}^R\big(H_{n-1}(X), R\big) \to H^n\big(\mathrm{Hom}_R(X, R)\big) \to \mathrm{Hom}_R\big(H_n(X), R\big) \to 0$$

and in particular $H^n(X;R) \cong \mathrm{Hom}\big(H_n\big(X;R);R\big)$ whenever $H_{n-1}(X;R) = 0$. Therefore $H_{n-1}\big(\mathrm{Th}(\xi);R\big) = 0$ and thus the argument in Case 4 goes through to produce the desired class u. The proof that $y \mapsto y \cup u$ is an isomorphism can be reduced to the case of a field by the following fact which follows via the 5-Lemma from the Universal Coefficient Theorems.

Fact Let $f : C \to C'$ be a chain map between free chain complexes such that $f^* : H^*(C';K) \to H^*(C;K)$ is an isomorphism for every field K. Then f_* and f^* are isomorphisms with any coefficients. □

Remark 5.0.5 The hypothesis that the chain complexes are free is required by the Universal Coeffient Theorem but is automatic in the case of chain complexes arising in the (co)homology of topological spaces. In the finitely generated case, the hypothesis that the isomorphism is known for every field can be weakened to require only that it be known for $K = \mathbb{Q}$ and for $K = \mathbb{F}_p$ for every prime p.

As before, the composition of the isomorphisms $p^* : H^*\big(B;R\big) \cong \tilde{H}^*(E(\xi);R)\big)$ and the isomorphism $H^*\big(E(\xi);R\big) \cong \tilde{H}^{*+n}\big(\mathrm{Th}(\xi);R\big)$ is called the Thom isomorphism $\Phi : H^*\big(B(\xi);R\big) \to \tilde{H}^{*+n}\big(\mathrm{Th}(\xi);R\big)$.

In the case $R = \mathbb{Z}$, we refer to u as the "oriented Thom class". The image of u under the composite map $H^n\big(\mathrm{Th}(\xi)\big) \to H^n\big(E(\xi)\big) \cong H^n\big(B(\xi)\big)$ is called the *Euler class of* ξ, often written $e(\xi)$. It is clear from uniquess/naturality that the reduction mod 2 of the Euler class is the top Stiefel-Whitney class w_n.

Corollary 5.0.6 *(Gysin sequence) Let ξ be an orientable n-dimensional vector bundle. Then there is a long exact cohomology sequence*

$$\ldots \to H^{j-n}(B) \xrightarrow{\cup u} H^j(B) \to H^j\big(S(\xi)\big) \to H^{j-n+1}(B) \to \ldots$$

Remark 5.0.7 The existence of a Gysin sequence for any orientable fibration whose homotopy fibre is a sphere, whether or not it comes from a vector bundle, can be derived from the Serre spectral sequence.

Proposition 5.0.8 *If n is odd then $2e(\xi) = 0$.*

Proof 1:
Any bundle map $\alpha \to \ \eta$ takes $e(\alpha)$ to $e(\eta)$. If $\dim V$ is odd the automorphism $v \mapsto -v$ of V is orientation-reversing. Therefore the selfmap of the bundle given on each fibre by $(b, v) \mapsto (b, -v)$ is orientation reversing so we conclude $e(\xi) = -e(\xi)$. □

Proof 2:
Let $e' = (p^*)^{-1}\big(e(\xi)\big) \in H^n\big(E(\xi)\big)$ and let $q : E(\xi) \to \mathrm{Th}(\xi)$ be the quotient map. Then by definition, $e' = q^*(u)$. Let $\Phi : H^*\big(B(\xi)\big) \to \tilde{H}^{*+n}\big(\mathrm{Th}(\xi)\big)$ be the Thom isomorphism. Then $\Phi\big(e(\xi)\big) = e' \cup u = q^*(u) \cup u$. But for classes in $\operatorname{Im} q^*$, the relative cup product $q^*(x) \cup u$ equals the absolute cup product $x \cup u$ in $H^*\big(\mathrm{Th}(\xi)\big)$ so $2\Phi\big(e(\xi)\big) = 2u \cup u = 0$ since $|u| = n$ is odd. □

Theorem 5.0.9 *Let ξ and η be oriented bundles and give $\xi \times \eta$ the orientation coming from the direct sum orientation of the fibres. Then $e(\xi \times \eta) = e(\xi) \times e(\eta)$. If ξ and η are bundles over the same base then $e(\xi \oplus \eta) = e(\xi)e(\eta)$.*

Proof:
For a bundle α, wirte $u(\alpha)$ for the Thom class u_α. Suppose $\dim \xi = m$ and $\dim \eta = n$ and set $p := p(\xi)$ and $p' := p(\eta)$. Let Φ denote the Thom isomorphism. Given $x \in H^j\big(B(\xi)\big)$ and $y \in H^k\big(B(\xi)\big)$, we have

$$\begin{aligned}
(p \times p')^*(x \times y) \cup \big(u(\xi) \times u(\eta)\big) &= (-1)^{km}\big(p^*(x) \cup u(\xi)\big) \times \big((p')^*(y) \cup u(\eta)\big) \\
&= (-1)^{km}\Phi_\xi(x) \times \Phi_\eta(y) = (-1)^{km}(\Phi_\xi \times \Phi_\eta)(x \times y) \\
&= (-1)^{km}(p \times p')^*(x \times y) \cup u(\xi \times \eta)
\end{aligned}$$

Setting $x = y = 1$ gives $u(\xi \times \eta) = u(\xi) \times u(\eta)$. Now setting $x = e(\xi)$ and $y = e(\eta)$ gives $\Phi_{\xi\times\eta}\big(e(\xi \times \eta)\big) = (-1)^{km}\Phi_{\xi\times\eta}\big(e(\xi) \times e(\eta)\big)$. Since $\Phi_{\xi\times\eta}$ is an isomorphism, it follows that

$e(\xi \times \eta) = (-1)^{km} e(\xi) \times e(\eta)$. The sign is negative only when k and m are both odd. But if both (or indeed either) k, m are odd then the right hand side has order 2 by Prop. 5.0.8 in which case the sign is irrelevant.

If ξ and η are bundles over the same base B, then the final statement follows by taking the pullback under the diagonal map $\Delta : B \to B \times B$. $\square$

Proposition 5.0.10 *Let ξ be an orientable bundle with a nowhere zero cross-section. Then the Euler class $e(\xi) = 0$.*

Proof: Let s be a nowhere zero cross-section of ξ. Normalizing, we may assume $s : B \to S(\xi)$. Let $j : S(\xi) \hookrightarrow E(\xi)$ and $q : E(\xi) \twoheadrightarrow \mathrm{Th}(\xi)$ be the canonical maps. Then $p \circ j \circ s = 1_{B\xi}$. Therefore

$$e(\xi) = s^* j^* p^* \big(e(\xi)\big) = s^* j^* q^* (u_\xi) = 0$$

since $q \circ j$ is trivial. $\square$

Theorem 5.0.11 *(Tubular Neighourhood Theorem)*

Let $M \hookrightarrow W$ be an embedding of a differentiable manifold M into a differentiable manifold W. Then there exists an open neighbourhood N of M in W such that M is a strong deformation retract of N and N is diffeomorphic to the total space of the normal bundle to M in W under a diffeomorphism whose restriction to M is the zero cross-section of the normal bundle.

Proof: (Sketch)

We present the proof in the case where M is compact.

Choose a Riemannian metric on W. The Riemannian metric determines geodesics in W. Let ν be the normal bundle. For $\epsilon > 0$, set $E(\epsilon) := \{(x, v) \in E(\nu) \mid \|v\| < \epsilon\}$. Observe that the correspondence $(x, v) \mapsto \left(x, v/\sqrt{1 - \frac{\|v\|^2}{\epsilon^2}}\right)$ is a diffeomorphism from $E(\epsilon)$ to E.

Define an "exponential map" $\exp : E(\epsilon) \to W$ by $\exp(x, v) :=$ endpoint of γ where $\gamma : [0, 1] \to W$ is the (unique) parameterized geodesic of length $\|v\|$ beginning at x and satisfying $\gamma'(0) = v$. By uniqueness theorems for initial value problems, exp is a smooth map defined in some neighbourhood of the zero cross-section $M \times 0 \subset E(\mu)$. By the Inverse Function Theorem, every point $(x, 0)$ in the zero cross-section has an open neighbourhood which exp maps diffeomorphically to an open subset of W.

Claim: If ϵ is sufficiently small then exp maps $E(\epsilon)$ diffeomorphically onto an open subset of W.

Proof of Claim: The exponential map is a local diffeomorphism so it suffices to show that it is $1-1$ on $E(\epsilon)$ if ϵ is sufficiently small. Suppose not. Then for each integer $k>0$, taking $\epsilon=1/k$ gives two distinct points $(x_k,v_k)\neq(x_k,v'_k)$ in $E(1/k)$ for which $\exp(x_k,v_k)=\exp(x_k,v'_k)$. Since M is compact, thus sequentially compact, there exists a choice of indices $k_{j\,j=1}^{\infty}$ such that $\lim_{j\to\infty}(x_{k_j},v_{i_j}=(x,0)$ and $\lim_{j\to\infty}(x'_{k_j},v'_{i_j}=(x',0)$ for some $x,x'\in M$. Therefore $x=\exp(x,0)=\lim_{j\to\infty}\exp(x_{k_j},v_{k_j})$ and similarly for x', so $x=x'$. But then $\exp(x_{k_j})=\exp(x'_{k_j})$ contradicting the fact that $(x,0)$ has a neighbourhood througout which exp is injective. This contradiction completes the proof of the claim.

It follows from the claim that $E(\epsilon)$ is diffeomorphic to its image N_ϵ for small ϵ and it is clear that the inclusion $M\subset N_\epsilon$ is a strong deformation retract. □

Let M be a differentiable manifold and let $\Delta: M\to M\times M$ be the diagonal map. Then $\Delta(M)$ is diffeomorphic to M and the inclusion $\Delta(M)\hookrightarrow M\times M$ is an embedding.

Lemma 5.0.12 *The normal bundle ν of the embedding $\Delta(M)\hookrightarrow M\times M$ is isomorphic to the tangent bundle TM.*

Proof: For $x\in M$, a vector $(u,v)\in T_{(x,x)}(M\times M)$ lies in $T_{(x,x)}\Delta(M)$ if and only if $u=v$ and lies in $E(\nu)$ if and only if $u+v=0$. The correspondence $(x,v)\mapsto\big((x,x),(v,-v)\big)$ describes a bundle isomorphism from TM to ν. □

Suppose that M is a connected compact oriented n-dimensional differentiable manifold. Let E denote the total space of the normal bundle ν of $\Delta(M)\hookrightarrow M\times M$ and let E_0 denote the complement of the zero cross-section. Let N be a tubular neighbourhood of $\Delta(M)$ in $M\times M$. Then using excision we have

$$\tilde{H}^*\big(\mathrm{Th}(\nu)\big)\cong H^*(E,E_0)\cong H^*\big(N,N-\Delta(M)\big)\cong H^*\big(M\times M,M\times M-\Delta(M)\big).$$

Since M is oriented, so is $\nu\cong TM$. Let $u'\in H^n\big(M\times M,M\times M-\Delta(M)\big)$ correspond to the oriented Thom class of ν under the above isomorphism and let $u''\in H^n(M\times M)$ be the image of u' under the canonical map.

$$\begin{array}{ccc} H^n\big(M\times M,M\times M-\Delta(M)\big) & \xrightarrow{\cong} & H^n\big(\mathrm{Th}(\nu)\big)\\ \downarrow & & \downarrow\\ H^n(M\times M) & \xrightarrow{\Delta^*} & H^n(M)\end{array}$$

Then $\Delta^*(u'')=e(\nu)=e(TM)$.

Lemma 5.0.13 $(a \times 1) \cup u' = (1 \times a) \cup u'$ *for any* $a \in H^*(M)$.

Proof: Let $\pi_1, \pi_2 : M \times M \to M$ be the projections. By definition $a \times 1 = \pi_1^*(a) \cup 1 = \pi_1^*(a)$ and $1 \times a = 1 \cup \pi_2^*(a) = \pi_2^*(a)$.

$$\Delta(M) \xhookrightarrow{\simeq} N \xhookrightarrow{j} M \times M$$

Since the restrictions of π_1 and π_2 to $\Delta(M)$ are equal and the inclusion $\Delta(M) \hookrightarrow N$ is a homotopy equivalence, $\pi_1 \circ j \simeq \pi_2 \circ j$ and thus $j^*(a \times 1) = j^*(1 \times a)$.

$$\begin{tikzcd}
H^*(M \times M) \arrow[r, "j"] \arrow[d, "\cup u'"] & H^*(N) \arrow[d, "\cup u'\big|_{(N, N-\Delta(M))}"] \\
H^{*+n}\big(M \times M, M \times M - \Delta(M)\big) \arrow[r, "\cong"] & H^{*+n}\big((N, N - \Delta(M)\big)
\end{tikzcd}$$

Therefore the commutative diagram shows that $(a \times 1) \cup u' = (1 \times a) \cup u'$. □

Composing with the canonical map gives

$$(a \times 1) \cup u' = (1 \times a) \cup u' \tag{5.0.2}$$

where the relative cup product of the lemma has become an absolute cup product.

Let F be a field. We will use coefficients in F and simply write $H_*(\,)$ and $H^*(\,)$ for homology and cohomology with coeffients in F. Since we are using coefficients in a field, the Künneth theorem gives $H^*(M \times M) \cong H^*(M) \times H^*(M)$.

Let $[M] \in H_n(M)$ denote the orientation class of M. Let $1^* \in H^n(M) \cong F$ be the unique element having the property that $\big\langle 1^*, [M] \big\rangle = 1$ Let $\{b_i\}_{i \in I}$ be a homogeneous basis for $H^*(M)$ including $1^* \in H^n(M)$. By Poincaré Duality there is a "dual basis" $\{b_i^*\}_{i \in I}$ with $b_i^* \in H^{n-|b_i|}(M)$ having the property that $\big\langle b_i \cup b_j^*, [M] \big\rangle = \delta_{ij}$.

Theorem 5.0.14 $u'' = \sum_{i \in I} (-1)^{|b_i|} b_i \otimes b_i^*$

Proof: Write $u'' = \sum_{i \in I} b_i \otimes c_i$ for some $c_i \in H^{n-|b_i|}(M)$. Applying Eqn. 5.0.2 with $a := b_i$ gives

$$\sum_{j \in I} b_i b_j \otimes c_j = \sum_{j \in I} (-1)^{|b_i||b_j|} b_j \otimes b_i c_j$$

Equating terms from the component $H^{|b_i|}(M) \otimes H^n(M)$ gives

$$b_i \otimes 1^* = \sum_{j \in I'} (-1)^{|b_i||b_j|} b_j \otimes b_i c_j$$

where I' is the subset of I indexing those basis elements in $H^{n-|b_i|}(M)$. Comparing coefficients of $b_j \otimes (\_)$ gives $1^* = (-1)^{|b_i||b_i|} b_i c_i$ and $0 = (-1)^{|b_i||b_j|} b_i c_j$ for $j \neq i$. It follows from the definition of the dual basis that $c_i = (-1)^{|b_i|^2} b_i^* = (-1)^{|b_i|} b_i^*$, using $(-1)^{k^2} = (-1)^k$. $\square$

Recall that the Euler characteristic of a finite complex is defined by

$$\chi(X) := \sum_{i=0}^{\dim X} (-1) \dim H^i(X).$$

Corollary 5.0.15 $\langle e(TM), [M] \rangle = \chi(M)$

Proof:

$$\begin{aligned} \langle e(TM), [M] \rangle &= \langle \Delta^*(u''), [M] \rangle = \left\langle \Delta^* \left(\sum_{i \in I} (-1)^{|b_i|} b_i \otimes b_i^* \right), [M] \right\rangle = \left\langle \sum_{i \in I} (-1)^{|b_i|} b_i b_i^*, [M] \right\rangle \\ &= \sum_{i \in I} (-1)^{|b_i|} 1 = \chi(M) \end{aligned}$$

$\square$

For connected compact nonorientable n-manifolds, the analogous computation with mod 2 coefficients (and the mod 2 orientation class $[M]$) yields

Theorem 5.0.16 $\langle w_n(TM), [M] \rangle \equiv \chi(M) \pmod 2$

Chapter 6

K-Theory

Let M be an abelian monoid. A monoid homomorphism $i : M \to A$ from M to an abelian group A is called the *group completion* of M if for any monoid homomorphism $j : M \to B$ to an abelian group B, there exists a unique group homomorphism $k : A \to B$ such that

$$\begin{tikzcd} M \arrow[rr, "i"] \arrow[dr, "j"'] & & A \arrow[dl, dotted, "k"] \\ & B & \end{tikzcd}$$

A construction of the group completion A (which by its universal property is unique up to isomorphism) is given by $A = (M \times M)/\sim$, where $(x, y) \sim (x', y')$ iff there exists $z \in M$ such that $x+y'+z = x'+y+z$ holds in M and $i : M \to A$ is given by $i(x) := (x, 0)$. If M is a semi-ring (a monoid having an associative multiplication with identity such that the multiplication distributes over the monoid addition), we can similarly define its "ring completion". In our explicit construction above, the semi-ring multiplication would be extended to A by setting $(x, y)(x', y') := (xx' + yy', xy' + yx')$.

For a topological space X, let $\mathrm{Vect}^{\mathbb{C}}(X)$ (respectively $\mathrm{Vect}^{\mathbb{R}}(X)$) denote the set of isomorphisms classes of (finite dimensional) complex (respectively real) vector bundles over X. Whitney sum of vector bundles gives $\mathrm{Vect}(X)$ a monoid structure with the 0-dimensional bundle as the 0-element. Tensor product of bundles then turns $\mathrm{Vect}(X)$ into a semi-ring, with the trivial one-dimensional bundles as the multiplicative identity.

Let X be compact. We define $KU(X)$ (respectively $KO(X)$), the KU-theory (respectively KO-theory) of X, to be the ring completion of the semi-ring $\mathrm{Vect}^{\mathbb{C}}(X)$ (respectively $\mathrm{Vect}^{\mathbb{R}}(X)$). It is customary to write $K(X)$ to mean $KU(X)$. For simplicity, for the remainder of this section we shall use the notation for the complex case, although the theory for the real case is identical.

Elements of $K(X) \setminus \mathrm{Vect}(X)$ are referred to as "virtual bundles". A continuous function $r : X \to Y$ induces, via pullback, a semi-ring homomorphism $\mathrm{Vect}(Y) \to \mathrm{Vect}(X)$ and thus a ring homomorphism $K(Y) \to K(X)$. Thus $K(\ )$ is a contravariant functor from topological spaces to rings.

It is clear from the definition that $\mathrm{Vect}(*) \cong \mathbb{Z}^+ = \{n \in \mathbb{Z} \mid n \geq 0\}$ and $K(*) \cong \mathbb{Z}$, corresponding to the dimension of the virtual bundle. The unique map $X \to *$ induces a homomorphism $\mathbb{Z} \cong K(*) \to K(X)$ whose cokernel $K(X)/\mathbb{Z}$ is called the "reduced K-theory of X", written $\tilde{K}(X)$. Notice that in the group $\tilde{K}$ all trivial bundles become 0, and so $\tilde{K}$ can be regarded as isomorphism classes of bundles module the relation of stable equivalence.

If X is a pointed (compact) space with basepoint $*$, the inclusion $* \hookrightarrow X$ induces a ring homomorphism $\epsilon : K(X) \to K(*) \cong \mathbb{Z}$, called the "augmentation", given by the dimension of the virtual bundle. Since $* \hookrightarrow X \twoheadrightarrow *$ is the identity map, as abelian groups $K(X) \cong \tilde{K}(X) \oplus \mathbb{Z}$ whenever $X \neq \emptyset$. In this case the kernel of the augmentation map is an ideal in $K(X)$ which is isomorphic to $\tilde{K}(X)$.

Let $G_\infty(\mathbb{C})$ be the infinite Grassmannian $G_\infty(\mathbb{C}) := \cup_{n=0}^\infty G_n(\mathbb{C})$, which forms a model for the classifying space BO of the infinite unitary group $U := \cup_{n=0}^\infty U(n)$, where $G_n(\mathbb{C}) \hookrightarrow G_{n+1}(\mathbb{C})$ is given by $W \mapsto W \oplus \mathbb{C}$ and correspondingly $U(n) \hookrightarrow U(n+1)$ is given by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. Any vector bundle ξ over a compact connected space X gives rise to a classifying map $f_\xi : X \to G_n(\mathbb{C}) \hookrightarrow G_\infty(\mathbb{C})$, where $n = \dim \xi$. Conversely, if $f : X \to G_\infty(\mathbb{C})$, then since X is compact the image of f lies in some compact subset of $G_\infty(\mathbb{C})$, and so lies in $G_n(\mathbb{C})$ for some sufficiently large n. Thus every homotopy class f in $[X, G_\infty(\mathbb{C})]$ produces a vector bundle over X. Because of the way $G_n(\mathbb{C})$ includes into $G_{n+1}(\mathbb{C})$, the composition $f_\xi : X \to G_n(\mathbb{C}) \hookrightarrow G_{n+1}(\mathbb{C})$ is $f_{\xi \oplus \epsilon}$, where ϵ is a trivial bundle. Thus ξ and $\xi \oplus \epsilon$ produce the same element of $[X, G_\infty(\mathbb{C})]$. Therefore there is a bijection between $\tilde{K}(X)$ and $[X, G_\infty(\mathbb{C})]$.

What corresponds to the group operation on $\tilde{K}(X)$? The maps $G_n(\mathbb{C}) \times G_k(\mathbb{C}) \to G_{n+k}(\mathbb{C})$ given by $(W, W') \mapsto W \oplus W'$ pass to the limit to give a map $G_\infty(\mathbb{C}) \times G_\infty(\mathbb{C}) \to G_\infty(\mathbb{C})$, giving $G_\infty(\mathbb{C})$ the structure of a topological monoid. (One has to use an isomorphism $\mathbb{C}^\infty \times \mathbb{C}^\infty \cong \mathbb{C}^\infty$ such as odd and even components to set up the compatibility to take the limit.) This produces a natural operation on $[X, G_\infty(\mathbb{C})]$ which agrees with the group operation on $\tilde{K}(X)$.

Remark 6.0.17 Since the operation makes $\tilde{K}(X)$ into a group, whereas superficially the operation on $[X, G_\infty(\mathbb{C})]$ appears to produce only a monoid structure, there appears to be something missing in the correspondence. The missing link is provided by the fact that any CW-complex Y which is a connected associative H-space always has a homotopy inverse and is thus an H-group. This is proved by observing that the shearing map $Y \times Y \to Y \times Y$ given by $(a, b) \mapsto (a, ab)$ induces the isomorphism $(f, g) \mapsto (f, fg)$ of homotopy groups and is thus a homotopy equivalence. In this case, we could rephrase the result as saying that the inclusion of the topological monoid $G_\infty(\mathbb{C})$ into its topological group completion is a homotopy equivalence. Later we will discover that a different way of looking at the reason $G_\infty(\mathbb{C})$ is an H-group is that it is a loop space according to Bott periodicity.

There is also an operation $G_\infty(\mathbb{C}) \times G_\infty(\mathbb{C}) \to G_\infty(\mathbb{C})$ induced from the map $G_n(\mathbb{C}) \times G_k(\mathbb{C}) \to G_{nk}(\mathbb{C})$ given by $(W, W') \mapsto W \otimes W'$. This corresponds to the multiplication on $\tilde{K}(X)$ induced from that on $K(X)$.

According to the preceding discussion, Thm. 2.2.2 can be restated as

Theorem 6.0.18 *Let X be compact and connected. Then $\tilde{K}(X) \cong [X, BU]$ preserving group and multiplication structures.*

□

To obtain the corresponding unreduced version we must keep track of the dimension of the bundle, which was discarded in passing to $\tilde{K}(X)$.

Theorem 6.0.19 *Let X be compact connected. Then there is a natural ring isomorphism $K(X) \cong [X, BU] \times \mathbb{Z}$ whose components are $K(X) \twoheadrightarrow \tilde{K}(X) \to BU$ and $K(X) \xrightarrow{\epsilon} \mathbb{Z}$, where ϵ is the augmentation.*

□

If $j : A \to X$ is a cofibration, mapping into BU gives an induced sequence

$$\tilde{K}(X/A) \xrightarrow{q^*} \tilde{K}(X) \xrightarrow{j^*} \tilde{K}(A)$$

which is exact at $\tilde{K}(X)$, as in the dual of Prop. 3.1.14 discussed after Thm. 3.1.15. The exactness shows that if two bundles, ξ, η over X have isomorphic restrictions to A then $[\xi]$ and $[\eta]$ differ by $q^*([\tau])$ for some bundle τ over X/A. There may be more than one such τ but any particular choice of isomorphism determines a particular τ. More generally, let $x = [(\alpha, \beta)]$ and $x' = [(\alpha', \beta')]$ be elements of $\tilde{K}(X)$, where α, β, α', β' are bundles over X. If $(\alpha \oplus \beta')|_A \cong (\alpha' \oplus \beta)|_A$, then $j^*(x - x') = 0$ and so x and x' differ by an element of $\tilde{K}(X/A)$. Any particular choice of isomorphism $\phi : (\alpha \oplus \beta')|_A \cong (\alpha' \oplus \beta)_A$ uniquely determines $\tau_\phi \in \tilde{K}(X/A)$ such that $q^*(\tau_\phi) = x - x'$.

Example 6.0.20 Let ξ be an n-dimensional complex vector bundle over X. Let $s : X \to E$ be a nowhere 0 cross-section of ξ. Then s determines a decomposition $\xi \cong \xi' \oplus \operatorname{Im} s$, where $\operatorname{Im} s$ is a trivial 1-dimensional bundle. Let λ_i denote the ith component of the exterior algebra functor Λ : {Vector Spaces} $\to$ {Vector Spaces}. Then λ_i induces an operation {Vector Bundles over X} $\to$ {Vector Bundles over X} where if F_x is the fibre ξ over x then $\lambda_i(F_x)$ is the fibre of $\lambda_i(\xi)$ over x. In particular, $\lambda_i(\xi)$ is an $\binom{n}{i}$ dimensional bundle.

Suppose $w \in \mathbb{C}$ is nonzero. Then the map $\lambda_{i+1}(V) \oplus \lambda_i(V) \to \lambda_{i+1}(V \oplus \mathbb{C})$ given by $(\alpha, \beta) \mapsto \alpha + (\beta \wedge w)$ is an isomorphism of vector spaces. Since $s(x) \neq 0$ for each x, the section s determines a corresponding bundle isomorphism

$$\lambda_{i+1}(\xi') \oplus \lambda_i(\xi') \xrightarrow{\cong} \lambda_{i+1}(\xi) \tag{6.0.1}$$

Repeated application of this isomorphism gives an isomorphism

$$\lambda_0(\xi) \oplus \lambda_2(\xi) \oplus \lambda_4(\xi) \oplus \ldots \cong \lambda_0(\xi') \oplus \lambda_1(\xi') \oplus \lambda_2(\xi') \oplus \lambda_3(\xi') \oplus \lambda_4(\xi') \oplus \ldots.$$

Note that these are finite sums since for dimension reasons $\lambda_k(\xi) = 0$ for $k > n$ and $\lambda_k(\xi') = 0$ for $k > n-1$. Similarly equation 6.0.1 also gives

$$\lambda_1(\xi) \oplus \lambda_3(\xi) \oplus \lambda_5(\xi) \oplus \ldots \cong \lambda_0(\xi') \oplus \lambda_1(\xi') \oplus \lambda_2(\xi') \oplus \lambda_3(\xi') \oplus \lambda_4(\xi') \oplus \ldots.$$

Thus the cross-section s determines an isomorphism

$$\lambda_0(\xi) \oplus \lambda_2(\xi) \oplus \lambda_4(\xi) \oplus \ldots \cong \lambda_1(\xi) \oplus \lambda_3(\xi) \oplus \lambda_5(\xi) \oplus \ldots.$$

In particular, the existence of such a cross section implies that $\sum_{i=0}^{n} [\lambda_i(\xi)] = 0$ in $\tilde{K}(X)$. This can be considered analogous to the fact that given a vector space W and any nonzero $w \in W$, the map $\lambda_i(W) \to \lambda_{i+1}(W)$ given by $\beta \mapsto \beta \wedge w$ is a differential d_w, and the cohomology of the chain complex $\big(\Lambda(W), d_w\big)$ is always 0.

$\square$

Notation: For a complex vector bundle ξ, set $\lambda(\xi) := \sum_{i=0}^{n} (-1)^i [\lambda_i(\xi)] \in \tilde{K}\big(B(\xi)\big)$.

As shown above:

Proposition 6.0.21 *Let ξ be a complex vector bundle. If ξ has a nowhere zero cross-section then $\lambda(\xi) = 0 \in \tilde{K}\big(B(\xi)\big)$.*

$\square$

From the preceding example and the discussion preceding it we get:

Proposition 6.0.22 *Let $j : A \to X$ be a cofibration. Let ξ be a complex vector bundle over X. Let s be a cross-section of ξ and suppose that the restriction $s|_A$ is never 0. Then s determines an element $\tau \in \tilde{K}(X/A)$ such that $q^*(\tau) = \lambda(\xi)$ in $\tilde{K}(X)$.*

Proof: As in the preceding example, the restriction of the cross-section s to A determines an isomorphism

$$\lambda_0(\xi|_A) \oplus \lambda_2(\xi|_A) \oplus \lambda_4(\xi|_A) \oplus \ldots \cong \lambda_1(\xi|_A) \oplus \lambda_3(\xi|_A) \oplus \lambda_5(\xi|_A) \oplus \ldots$$

and according to the discussion preceding the example, this isomorphism uniquely determines an element $\tau \in \tilde{K}(X/A)$ such that $q^*(\tau) = \sum_{i=0}^{n} (-1)^i [\lambda_i(\xi)]$ in $\tilde{K}(X)$. $\square$

Chapter 7

Bott Periodicity

The purpose of this chapter is to prove Bott Periodicity in the complex case which states

Theorem 7.0.23 (Bott Periodicity) *$\Omega U \simeq BU \times \mathbb{Z}$.*

Notice that $\Omega\mathbb{Z} = *$ since the only basepoint preserving loop in $\mathbb{Z}$ is $\omega(t) = 0$ for all t. Therefore Bott Periodicity can be stated in the equivalent form $\Omega^2 U \simeq U$ or $\Omega^2(BU \times \mathbb{Z}) \simeq BU \times \mathbb{Z}$. There is also a real version of Bott periodicity which states that $\Omega^8 O \simeq O$, but we will give the details of the proof only in the complex case.

If $\mathcal{K}$ is a Hilbert space and $B : \mathcal{K} \to \mathcal{K}$ is a bounded linear operator, we set

$$\mathrm{Gr}^B(\mathcal{K}) := \{\text{closed subspaces } W \text{ of } \mathcal{K} \mid B(W) \subset W\}.$$

Let $\mathcal{H}^{(n)}$ be the Hilbert space $\mathcal{H}^{(n)} := L^2(S^1;\mathbb{C}^n) \cong L^2(S^1;\mathbb{C}) \otimes \mathbb{C}^n$. We write simply $\mathcal{H}$ for $\mathcal{H}^{(1)}$. For $f \in \mathcal{H}$, let M_f denote the multiplication-by-f-operator given by $\big(M_f(g)\big)(z) := f(z)g(z)$. We will sometimes write $\mathrm{Gr}^f(\mathcal{H}^{(n)})$ for $\mathrm{Gr}^{M_f}(\mathcal{H}^{(n)})$.

Let $\{e_m\}_{m=1}^{\infty}$ be the standard basis for $\mathbb{C}^\infty$ and let $\{\zeta_m\}_{m=-\infty}^{\infty}$ be the standard Hilbert space basis for $\mathcal{H}$. That is, $e_m := (0, \ldots, 0, 1, 0 \ldots)$ with 1 in the mth position and $\zeta_m(z) := z^m$. For $S \subset \mathcal{H}$, let $\langle S \rangle$ denote the smallest closed subspace of $\mathcal{H}$ containing S, i.e. the closure of the linear span of S. Given $a \le b \in \mathbb{Z}$, set $\mathcal{H}_a^b := \langle \{\zeta^k\}_{a \le k \le b} \rangle$. Set $\mathcal{H}_+ := \mathcal{H}_0^\infty$.

An element $f \in \mathcal{H}$ is called a *Fourier polynomial* if it is a finite linear combination of the basis elements $\{\zeta_m\}$.

Define the *polynomial loops* $\Omega_{\text{poly}} U(n)$ by

$$\Omega_{\text{poly}} U(n) = \{f \in \Omega U(n) \mid f = \sum_{j=p}^{q} a_j z^j \text{ for some } p \le q \in \mathbb{Z}\}.$$

That is, each entry of the matrix f is a Fourier polynomial.

Given $a \le b \in \mathbb{Z}$, set

$$\mathrm{Gr}_{\mathrm{bdd},a,b}(\mathcal{H}^{(n)}) := \{\text{closed subspaces } W \text{ of } \mathcal{H}^{(n)} = \mathcal{H} \otimes \mathbb{C}^n \mid (\mathcal{H}_b^\infty \otimes \mathbb{C}^n) \subset W \subset (\mathcal{H}_a^\infty \otimes \mathbb{C}^n)\}$$

and write $\mathrm{Gr}_{\mathrm{bdd},k}(\mathcal{H}^{(n)}) := \mathrm{Gr}_{\mathrm{bdd},-k,k}(\mathcal{H}^{(n)})$. Set

$$\mathrm{Gr}_{\mathrm{bdd}}(\mathcal{H}^{(n)}) := \cup_{\{a \le b\}} \mathrm{Gr}_{\mathrm{bdd},a,b}(\mathcal{H}^{(n)}) = \cup_k \mathrm{Gr}_{\mathrm{bdd},k}(\mathcal{H}^{(n)}).$$

For a bounded linear operator B on $\mathcal{H}^{(n)}$ we write $\mathrm{Gr}^B_{\mathrm{bdd},a,b}(\mathcal{H}^{(n)})$, $\mathrm{Gr}^B_{\mathrm{bdd},k}(\mathcal{H}^{(n)})$ and $\mathrm{Gr}^B_{\mathrm{bdd}}(\mathcal{H}^{(n)})$ for the intersections with $\mathrm{Gr}^B(\mathcal{H}^{(n)})$.

Since $\dim\big(\mathcal{H}^\infty_{-k}/\mathcal{H}^\infty_k\big) = 2k$, using the isomorphism of vector spaces $\mathcal{H}^\infty_{-k}/\mathcal{H}^\infty_k \cong \mathbb{C}^{2k}$ given by $\zeta^j \mapsto e_{k-j}$ for $j = -k, \ldots, k-1$, the association $\phi_k(W) := W/\mathcal{H}_k \subset \mathcal{H}^\infty_{-k}/\mathcal{H}^\infty_k$ becomes a map from $\mathrm{Gr}_{\mathrm{bdd},k}(\mathcal{H})$ to the Grassmannian $\mathrm{Gr}^{\mathbb{C}}_\infty$ taking W into $G_{\dim(W/\mathcal{H}_k)}(\mathbb{C}^{2k}) \subset \mathrm{Gr}^{\mathbb{C}}_\infty$. Recall that we defined the inclusion $G_j(\mathbb{C}^m) \to G_{j+1}(\mathbb{C}^{m+1})$ by $V \mapsto \mathbb{C} \oplus V$ so that it would be compatible with our CW-decompositions. Notice that if $W \in \mathrm{Gr}_k(\mathcal{H})$ is regarded as an element of $\mathrm{Gr}_{k+1}(\mathcal{H})$, we have $\phi_{k+1}(W) = \mathbb{C} \oplus \phi_k(W)$ because of the way we chose to identify $\mathcal{H}^\infty_{-k}/\mathcal{H}^\infty_k$ with $\mathbb{C}^{2k}$. Thus with these conventions, the maps $\phi_k : \mathrm{Gr}_{\mathrm{bdd},k}(\mathcal{H}) \to G^{\mathbb{C}}_\infty$ are compatible as k increases and yield a map

$$\phi : \mathrm{Gr}_{\mathrm{bdd}}(\mathcal{H}) \to G^{\mathbb{C}}_\infty = BU.$$

However ϕ is not injective: if $W \in \mathrm{Gr}_{\mathrm{bdd},k}(\mathcal{H})$ then $zW \in \mathrm{Gr}_{\mathrm{bdd},k+1}(\mathcal{H})$ and it satisfies $zW/\mathcal{H}_{k+1} \cong W/\mathcal{H}_k$ so $\phi(W) = \phi(zW)$. Indeed, while $\mathrm{Gr}^{\mathbb{C}}_\infty \simeq BU$ is connected, $\mathrm{Gr}_\infty(\mathcal{H})$ is disconnected with components in correspondence with the integers. More precisely, given $W \in \mathrm{Gr}_{\mathrm{bdd},k}(\mathcal{H})$ define its index by $\mathrm{Ind}(W) := \big(\dim(W/\mathcal{H}^\infty_k) - \dim(\mathcal{H}^\infty_{-k}/W)\big)/2$. Regarding $W \in \mathrm{Gr}_{\mathrm{bdd},k}(\mathcal{H})$ as an element of $\mathrm{Gr}_{\mathrm{bdd},k+1}(\mathcal{H})$ increases both terms in the difference by one, so $\mathrm{Ind}(W) : \mathrm{Gr}_{\mathrm{bdd}}(\mathcal{H}) \to \mathbb{Z}$ is well defined. (The term "index" is used because is possible to interpret W as the graph of a Fredholm operator T and $\mathrm{Ind}(W)$ becomes the negative of the index of T in the sense of Fredholm operators.) Notice that $\mathrm{Ind}(zW) = \mathrm{Ind}(W) + 1$, so the index distinguishes between elements having the same image under ϕ. From the preceeding discussion we conclude

Theorem 7.0.24 *The map (ϕ, Ind) is a homeomorphism from $\mathrm{Gr}_{\mathrm{bdd}}(\mathcal{H})$ to $\mathrm{Gr}^{\mathbb{C}}_\infty \times \mathbb{Z} = BU \times \mathbb{Z}$.*

□

Lemma 7.0.25 $\mathrm{Gr}_{\mathrm{bdd}}(\mathcal{H}) = \cup_n \mathrm{Gr}^{z^n}_{\mathrm{bdd}}(\mathcal{H})$

Proof: Notice that if $W \in \mathrm{Gr}_{\mathrm{bdd},k}(\mathcal{H})$ and n is sufficiently large compared to k, then condition $z^n W \subset W$ is automatically satisfied. Specifically, any $W \in \mathrm{Gr}_{\mathrm{bdd},k}(\mathcal{H})$ satisfies $W \subset \mathcal{H}^\infty_{-k}$, so if $w \in W$ and $n \ge 2k$ then necessarily $z^n w$ belongs to $\mathcal{H}^\infty_k \subset W$. Therefore, given $W \in \mathrm{Gr}_{\mathrm{bdd},k}(\mathcal{H}) \subset \mathrm{Gr}_\infty(\mathcal{H})$, if we choose n large enough so that $n > 2k$ then W lies in $\mathrm{Gr}^{z^n}_{\mathrm{bdd},k}$.

□

Choosing a bijection $\{1,\ldots,n\}\times\mathbb{Z}\to\mathbb{Z}$ induces a Hilbert space isomorphism $\Psi_n : \mathcal{H}^{(n)} \cong \mathcal{H}$ by means of the corresponding bijection between the Hilbert space bases. To be explicit, we will settle upon the bijection $(m,k)\mapsto nk+(m-1)$. Then the automorphism M_{z^k} of $\mathcal{H}^{(n)}$ corresponds under Ψ_n to the operator which shifts each basis element by kn positions and so $\mathrm{Gr}^z(\mathcal{H}^{(n)})$ corresponds to $\mathrm{Gr}^{z^n}(\mathcal{H})$.

$$\begin{tikzcd}
\mathrm{Gr}^z(\mathcal{H}^{(n)}) \arrow[r, "\Psi_n", "\cong"'] \arrow[d, hook] & \mathrm{Gr}^{z^n}(\mathcal{H}) \arrow[d, hook] \\
\mathrm{Gr}^z(\mathcal{H}^{(n+1)}) \arrow[r, "\Psi_{n+1}", "\cong"'] & \mathrm{Gr}^{z^{n+1}}(\mathcal{H})
\end{tikzcd}$$

Taking the union and applying Lemma 7.0.25 gives

$$\cup_n \mathrm{Gr}^z_{\mathrm{bdd}}(\mathcal{H}^{(n)}) \cong \mathrm{Gr}_{\mathrm{bdd}}(\mathcal{H}).$$

So far we have shown

$$\cup_n \mathrm{Gr}^z_{\mathrm{bdd}}(\mathcal{H}^{(n)}) \cong \mathrm{Gr}_{\mathrm{bdd}}(\mathcal{H}) \cong \mathrm{Gr}^{\mathbb{C}}_{\infty} \times\mathbb{Z} = BU\times\mathbb{Z}.$$

To finish the proof of Bott Periodicity, we will show that $\Omega U(n)\simeq \mathrm{Gr}^z_{\mathrm{bdd}}(\mathcal{H}^{(n)})$ for each n, so that passing to the limit gives $\Omega U\simeq \cup_n \mathrm{Gr}^z_{\mathrm{bdd}}(\mathcal{H}^{(n)})$.

If $W\in\mathrm{Gr}^z(\mathcal{H}^{(n)})$ then by definition $zW\subset W$, so W becomes a module over the ring $\mathbb{C}[z]$.

Lemma 7.0.26 *Suppose $W\in\mathrm{Gr}^z_{\mathrm{bdd}}(\mathcal{H}^{(n)})$. Then any minimal generating set for W as a $\mathbb{C}[z]$-module contains n elements.*

Proof: By definition, there exists k such that $\mathcal{H}^{\infty}_{k}\otimes\mathbb{C}^n\subset W\subset\mathcal{H}^{\infty}_{-k}\otimes\mathbb{C}^n$. Since W is a finitely generated torsion-free module over the principal ideal domain $\mathbb{C}[z]$ it is a free $\mathbb{C}[z]$-module and its rank equals the number of elements in a minimal set of $\mathbb{C}[z]$-module generators. Notice that $\mathcal{H}^{\infty}_{k}\otimes\mathbb{C}^n$ and $\mathcal{H}^{\infty}_{-k}\otimes\mathbb{C}^n$ are each free $\mathbb{C}[z]$-modules of rank n (with basis $\{z^k\otimes e_j\}$ and $\{z^{-k}\otimes e_j\}$ respectively). In general if A, B, C are modules over a principal ideal domain with $A\subset B\subset C$, then if A and C free modules of rank n, so is B. Therefore the rank of W is n. $\square$

Corollary 7.0.27 *Suppose $W\in\mathrm{Gr}^z_{\mathrm{bdd}}(\mathcal{H}^{(n)})$. Then $\dim(W/zW)=n$. Equivalently $\dim(W\ominus zW)=n$.*

$\square$

Note: For $A \subset B$ recall the definition $B \ominus A \equiv \{b \in B \mid b \perp a \ \forall \ a \in A\}$.

Recall $\Omega_{\text{poly}} U(n) = \{f \in \Omega U(n) \mid f$ is a Fourier polynomial$\}$. I.e.$f(z) = \sum_{j=-m}^{m} a_j z^j$ for some m, where $a_j \in M_{n\times n}(\mathbb{C})$ such that $f(z) \in \Omega(n)$ for all $z \in S^1$.

Given $f \in \Omega U(n)$ set $W_f = M_f(\mathcal{H}_+ \otimes \mathbb{C}^n) \subset \mathcal{H}^{(n)}$ where $f(z)$ acts by matrix multiplication for each z. Since $f(z)$ is invertible for all z, the operator M_f is invertible so $W_f \cong \mathcal{H}_+ \otimes \mathbb{C}^n$ and, in particular, is a closed subspace of $\mathcal{H}^{(n)}$. Also, since H_+ is closed under multiplication by z, so is W_f, i.e. $W_f \in \mathrm{Gr}^z(\mathcal{H}^{(n)})$.

Theorem 7.0.28 *The map $f \mapsto W_f$ is a homeomorphism $\alpha : \Omega_{\text{poly}} U(n) \cong \mathrm{Gr}^z_{\text{bdd}}(\mathcal{H}^{(n)})$.*

Proof:

If $f(z) = \sum_{j=-m}^{m} a_j z^j \in \Omega_{\text{poly}} U(n)$, then it is clear from the definition that

$$\mathcal{H}^\infty_m \otimes \mathbb{C}^n \subset W \subset \mathcal{H}^\infty_{-m} \otimes \mathbb{C}^n,$$

so $W_f \in \mathrm{Gr}^z_{\text{bdd}}(\mathcal{H}^{(n)})$.

Define $\beta : \mathrm{Gr}^z_{\text{bdd}}(\mathcal{H}^{(n)}) \to \Omega_{\text{poly}} U(n)$ as follows. Let W belong to $\mathrm{Gr}^z_{\text{bdd},m}(\mathcal{H}^{(n)})$. Choose an ordered orthonormal basis $B = \{w_1, w_2, \ldots w_n\}$ for $W \ominus zW$. Write out $w_1, \ldots, w_n$ in the basis $e_1, \ldots e_n$ getting an $n \times n$ matrix $N_B(z)$ with entries in $\mathcal{H}$. Since $\mathcal{H}^\infty_m \otimes \mathbb{C}^n \subset W$, we have $\mathcal{H}^\infty_{m+1} \otimes \mathbb{C}^n \subset zW$ and thus since the elements w_j are perpendicular to zW, they have no component of ζ^k for $k > m$. On the other hand, $W \subset \mathcal{H}^\infty_{-m}$ says that elements of W have no component of ζ^k for $k < -m$. Thus each entry of $N_B(z)$ is a polynomial function in z and z^{-1} having degree at most m.

We wish to show that $N_B(z) \in U(n)$ for all z. Write $w_k(z) = \sum_{j=-m}^{m} w_{kj} \zeta^j$, where $w_{kj} \in \mathbb{C}^n$. For $x, y \in \mathbb{C}^n$, let $x \cdot y \in \mathbb{C}$ denote their inner product in $\mathbb{C}^n$ and for $f, g \in \mathcal{H}$, let $\langle f, g \rangle$ denote their Hilbert space inner product. Note that if $a, b \in \mathbb{C}$ and $x, y \in \mathbb{C}^n$ we have $(ax) \cdot (by) = (x \cdot y) a\bar{b}$ and that if $x, y \in \mathbb{C}^n$ are regarded as the constant functions $x\zeta^0$, $y\zeta^0$ in $\mathcal{H} \otimes \mathbb{C}^n$ then $x \cdot y = \langle x, y \rangle$ and $x \cdot y = \langle x, y\zeta^p \rangle = 0$ for $p \neq 0$. In addition, $\langle w_k, w_l \rangle = \delta_{kl}$ by construction.

Also, since $\langle \zeta^s, \zeta^t \rangle = 0$ for $s \neq t$, we have that for any p,

$$\langle w_k, z^p w_l \rangle = \sum_{j,r} \langle w_{kj} \zeta^r, z^p w_{lr} \zeta^r \rangle = \sum_j \langle w_{kj}, w_{l,j-p} \rangle.$$

Using these facts, for each z we have

$$\begin{aligned}
w_k(z) \cdot w_l(z) &= \sum_{j,r} (w_{kj}\zeta^j) \cdot (w_{lr}\zeta^r) \\
&= \sum_{j,r} (w_{kj} \cdot w_{lr})\zeta^{j-r} \\
&= \sum_{j,r} \langle w_{kj}, w_{lr}\rangle \zeta^{j-r} \\
&= \sum_{j,p} \langle w_{kj}, w_{l,l-p}\rangle \zeta^{p} \\
&= \sum_{p} \langle w_k, z^p w_l\rangle \zeta^p \\
&= \langle w_k, w_l\rangle \\
&= \delta_{kl}
\end{aligned}$$

Thus $N_B(z)$ lies in $U(n)$ for all z. Set $\beta(W) := N_B(z)N_B(1)^{-1}$ to get a based loop in $\Omega_{\text{poly}}U(n)$. If we were to pick a different ordered basis $B' = \{w'_1, w'_2, \dots w'_n\}$ for $W \ominus zW$, the resulting matrix would be $N_{B'}(z) = N_B(z)A$ where $A \in U(n)$ is the linear transformation taking the ordered basis B' to B, Therefore $N'_B(z)N_{B'}(1)^{-1} = N_B(z)AA^{-1}N_B(1)^{-1} = N_B(z)N_B(1)^{-1}$ and so $\beta(W)$ is well defined.

Suppose $f \in \Omega_{\text{poly}}U(n)$. Then $\{f(e_1), f(e_2), \dots f(e_n)\}$ forms an orthonormal basis for W_f showing that $\beta \circ \alpha = 1_{\Omega_{\text{poly}}U(n)}$. Conversely, if $W \in \text{Gr}^z_{\text{bdd}}(\mathcal{H}^{(n)})$ and $\{w_1, w_2, \dots w_n\}$ is an orthonormal basis for $W \ominus zW$, then $(\alpha \circ \beta)(W) = W_{\beta(W)}$ is the $\mathbb{C}[z]$-module generated by $\{w_1, w_2, \dots w_n\}$, which is W.

Thus β and α are inverse homeomorphisms. $\square$

To finish the proof we show that $\Omega_{\text{poly}}U(n) \simeq \Omega U(n)$ after which passing to the limit using $U = \varinjlim_n U(n)$ gives the desired homeomorphism $\Omega U \cong \cup_n \text{Gr}^z_{\text{bdd}}(\mathcal{H}^{(n)})$. We will need the properties of the infinite mapping telescope.

7.1 Infinite Mapping Telescopes

Suppose $X_0 \subset X_1 \subset \dots \subset X_n \subset \dots$ and let $X = \bigcup_{i=0}^{\infty} X_i$. The *infinite mapping telescope* of X is the space $T_X := X_0 \times [0,1] \cup X_1 \times [1,2] \cup \dots \cup X_i \times [i, i+1] \cup \dots \subset X \times \mathbb{R}$.

Note: Although we have written simply T_X, the space depends not just on X but also on the subsets X_i.

Proposition 7.1.1 *If X is paracompact for all $x \in X$ there exists i such that x lies in the interior of X_i then the projection map $\pi_1 : T_X \to X$ is a homotopy equivalence.*

Proof: Use a partition of unity to construct a map $f : X \to [0, \infty)$ such that $f(x) \geq i + 1$ for $x \notin X_i$. Then $g(x) := \big(x, f(x)\big)$ is a homeomorphism from X to $g(X) \subset T_X$ and the inclusion $j : g(X) \hookrightarrow T_X$ is a deformation retract and satisfies $\pi_1 \circ j \circ g = 1_X$. Therefore π_1 is a homotopy equivalence. □

Theorem 7.1.2 *$X = \bigcup_{i=0}^{\infty} X_i$ and let $Y = \bigcup_{i=0}^{\infty} Y_i$. Let $f : X \to Y$ be a continuous map such that for each n, $f(X_i) \subset Y_i$ and the restriction $f_i := f\big|_{X_i} : X_i \to Y_i$ is a homotopy equivalence. Then f is a homotopy equivalence.*

Proof:
Case 1: $X_i = Y_i$ and $f_i \simeq 1_{X_i}$.

For each n let $h^n : f_n \simeq 1_{X_n}$ be a homotopy. Define $h : T_X \times I \times T_X$ by

$$h(x, n+t, s) := \begin{cases} \big(h^n(x, s), n + 2t\big) & \text{for } 0 \leq t \leq 1/2; \\ \Big(h^n\big(x, (3 - 4t)s\big), n + 1\Big) & \text{for } 1/2 \leq t \leq 3/4; \\ \Big(h^{n+1}\big(x, (4 - 3t)s\big), n + 1\Big) & \text{for } 3/4 \leq t \leq 1. \end{cases}$$

Then (using the usual notation $h_s(x) := h(x, s)$), $h_0 = h^0 \simeq f$ while h_1 has the properties

$$\begin{array}{rcll} h_1(x, n+t) & = & (x, n + 2t) & \text{for } 0 \leq t \leq 1/2 \\ h_1(x, n+t) & \in & X_{n+1} \times \{n+1\} & \text{for } 1/2 \leq t \leq 1 \end{array}$$

Define $g : T_X \to T_X$ by

$$g(x, n+t) := \begin{cases} (x, n + 2t) & \text{for } 0 \leq t \leq 1/2; \\ h_1(x, n + \frac{3}{2} - t) & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

This is well defined since $h_1(x, n + \frac{1}{2}) = h_1(x, n+1) = (x, n+1)$.

$$h_1 g(x, n+t) := \begin{cases} (x, n + 4t) & \text{for } 0 \leq t \leq 1/4; \\ h_1(x, n + 2t) & \text{for } 1/4 \leq t \leq 1/2; \\ h_1(x, n + \frac{3}{2} - t) & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

Define $H : T_X \times I \to T_X$ by

$$H(x, n+t, s) := \begin{cases} h_1 g(x, n+t) & \text{for } 0 \leq t \leq (1-s)/2 \text{ and } \frac{1}{2} + s \leq t \leq 1; \\ h_1 g(x, n + 1 - s) & \text{for } (1-s)/2 \leq t \leq \frac{1}{2} + s. \end{cases}$$

This is well defined since $h_1 g\big(x, n+(1-s)/2\big) = h_1 g(x, n_{\frac{1}{2}} - s) = h_1(x, n+1-s)$. Then $H_0 = h_1$ and

$$H_{\frac{1}{2}} = \begin{cases} (x, n+4t) & \text{for } 0 \leq t \leq \frac{1}{4}; \\ (x, n+1) & \text{for } \frac{1}{3} \leq t \leq 1. \end{cases}$$

which is homotopic to 1_{T_X}. Thus $h_1 g \simeq 1_{T_X}$ and similarly $g h_1 \simeq 1_{T_X}$. Therefore f is a homotopy equivalence.

Case 2: General Case

For each n let $g_n : Y_n \to X_n$ be a homotopy inverse to f_n. Then

$$\begin{tikzcd} Y_n \arrow[r, "g_n"] \arrow[d, "j_n"] & X_n \arrow[d, "i_n"] \\ Y_{n+1} \arrow[r, "g_{n+1}"] & X_{n+1} \end{tikzcd}$$

is homotopy commutative since $i_n g_n \simeq g_{n+1} f_{n+1} i_n g_n = g_{n+1} j_n f_n g_n \simeq g_{n+1} j_n$. For each n choose a homotopy $h^n : i_n g_n \simeq g_{n+1} j_n$. Define $G : T_Y \to T_X$ by

$$G(y, n+t) := \begin{cases} \big(g_n(y), n+2t\big) & \text{for } 0 \leq t \leq 1/2; \\ \big(h^n_{2t-1}(y), n+1\big) & \text{for } 1/2 \leq t \leq 1/2. \end{cases}$$

Let $T^n_X = \{(x,t) \in T_X \mid t \leq n\}$. Then $Gf(T^n_X) \subset T^n X$. The inclusion $X_n = X_n \times \{n\} \subset T^n_X$ is a deformation retract. and $Gf\Big|_{T^n_X} \simeq 1_{X_n \times \{n\}}$ is $g_n f_n$ which is homotopic to $1_{X \times \{n\}}$. Therefore $Gf\Big|_{T^n_X} \simeq 1_{T^n_X}$. Applying Case 1 to $T^1_X \subset \ldots T^n_X \ldots$ shows that Gf is a homotopy equivalence. Similarly fG is a homotopy equivalence. Since f has both a left and a right homotopy inverse, it follows that f is a homotopy equivalence. □

7.2 Various forms of loop spaces

Set $\mathcal{K} := \mathcal{H}^{(n)} \cong \mathcal{H} \otimes \mathbb{C}^n$ and set $\mathcal{K}_+ := \mathcal{H}_+ \otimes \mathbb{C}^n \subset \mathcal{K}$.

Let $\Omega_{\text{psm}} U(n) = \{f \in \Omega U(n) \mid f \text{ is piecewise smooth}\}$. We will show that each of the inclusions $\Omega_{\text{poly}} U(n) \subset \Omega_{\text{psm}} U(n) \subset \Omega U(n)$ is a homotopy equivalence.

One approach is to use homology and show that the map induces an isomorphism on homology groups and then apply the Whitehead theorem. The homology of $\Omega_{\text{poly}} U(n)$ can be found by filtering it by subspaces consisting of Fourier polynomials having specific degrees, while the homology of $\Omega U(n)$ can be found by techniques such as the Serre or Eilenberg-Moore spectral sequence. Instead we describe the much more complicated proof of Pressley-Segal which has the advantage that it provides ideas that generalize more readily to other contexts, such as G-homotopy theory, where there is no Whitehead Theorem.

As in Theorem 7.0.28, the function $\alpha : \Omega_{\text{psm}}U(n) \to \text{Gr}^z(\mathcal{K})$ given by $f \mapsto W_f$ is a homeomorphism to its image with inverse given by the function β defined in the proof of that theorem.

Given a bounded linear operator $B \in B(\mathcal{K})$, the inclusions $\mathcal{K}_\pm \to \mathcal{K}$ and projections $\mathcal{K} \to \mathcal{K}_\pm$ give a description of B as a matrix of operators

$$B = \begin{pmatrix} B_{++} & B_{+-} \\ B_{-+} & B_{--} \end{pmatrix}.$$

In particular $(M_f)_{++}$ is the composite $\mathcal{K}_+ \xrightarrow{(M_f)|_{\mathcal{K}_+}} W_f \xrightarrow{P_+^{W_f}} \mathcal{K}_+$ and similarly $(M_f)_{+-}$ is the composite $\mathcal{K}_+ \xrightarrow{(M_f)|_{\mathcal{K}_+}} W_f \xrightarrow{P_-^{W_f}} \mathcal{K}_-$.

Lemma 7.2.1 *Suppose $f \in \Omega_{\text{psm}}M_{n\times n}(\mathbb{C})$. Then $(M_f)_{+-}$ and $(M_f)_{-+}$ are Hilbert-Schmidt operators and thus compact.*

Proof: let $f(z)$ be piecewise smooth. Then $f'(z)$ is piecewise smooth and thus, in particular, piecewise continuous and so square integrable on S^1. Let $\sum_{k=-\infty}^{\infty} c_k z^k$ be the Fourier expansion of $f(z)$ and let $\sum_{k=-\infty}^{\infty} d_k z^k$ be the Fourier expansion of $f'(z)$ Then the Fourier expansion of $(M_f)_{-+}(z^{-k})$ is $\sum_{j=k}^{\infty} c_k z^{k-j}$. Therefore the Hilbert=Schmidt norm of $(M_f)_{-+}$ is given by $\|(M_f)_{-+}\|_{\text{HS}}^2 = \sum_{k=1}^{\infty} \|(M_f)_{-+}(z^{-k})\|^2 = \sum_{k=1}^{\infty}\sum_{j=k}^{\infty} |c_j|^2 = \sum_{k=1}^{\infty} k|c_k|^2$.

Let $\gamma(t) = e^{2\pi i t}$ be the standard parameterization of S^1. Let $0 = b_0 < b_1 \ldots < b_i < \ldots < b_r = 1$ be a subdivision of $[0,1]$ such that $f \circ \gamma$ is smooth on (b_{i-1}, b_i) for $i = 1, \ldots r$. Let μ be the standard measure on S^1 normalized so that $\mu(S^1) = 1$. Letting $h(z) = \frac{z^{k+1}f'(z)}{2\pi i(k+1)}$, integration by parts gives

$$
\begin{aligned}
c_k = \langle f(z), z^{-k}\rangle &= \int_{S^1} z^k f(z)\, d\mu = \sum_{i=1}^{r} \int_{b_{i-1}}^{b_i} e^{2\pi i n t} f(e^{2\pi i t}) 2\pi i t \frac{dt}{2\pi} \\
&= \sum_{i=1}^{r} ih\big(\gamma(b_i)\big) - h\big(\gamma(b_i)\big) - \sum_{i=1}^{r} i \int_{b_{i-1}}^{b_i} \frac{e^{2\pi i(k+1)t} f'(e^{2\pi i t})}{k+1}\, dt \\
&= h(b_r) - h(b_0) - \int_{S^1} \frac{z^{k+1} f'(z)}{k+1}\, dz \\
&= 0 - \frac{2\pi}{k+1}\langle f'(z), z^{-(k+1)}\rangle = -\frac{2\pi}{k+1} d_{k+1}.
\end{aligned}
$$

Since $\sum_{k=-\infty}^{\infty} |d_k|^2$ converges, we get $\sum_{k=1}^{\infty} \frac{(k+1)^2}{4\pi^2} |c_k|^2 = \sum_{k=1}^{\infty} |d_{k+1}|^2$ converges. For large k, $k < (k+1)^2/(4\pi^2)$ and so it follows that $\|(M_f)_{-+}\|_{\mathrm{HS}}^2 = \sum_{k=1}^{\infty} k|c_k|^2$ converges and thus $(M_f)_{-+}$ is Hilbert-Schmidt.

Similary $(M_f)_{+-}$ is Hilbert-Schmidt. $\square$

Corollary 7.2.2 *Suppose $f \in \Omega_{\mathrm{psm}} \operatorname{GL}(n)$. Then $(M_f)_{++}$ is a Fredholm operator and its index $\operatorname{Ind}\big((M_f)_{++}\big)$ equals $-n$ times the degree of the homotopy class of the function $z \mapsto \det\big(f(z)\big)$ in $\pi_1(\mathbb{C} \smallsetminus \{0\}) \cong \mathbb{Z}$.*

Proof: Suppose

$$
B = \begin{pmatrix} B_{++} & B_{+-} \\ B_{-+} & B_{--} \end{pmatrix}
$$

is invertible with inverse

$$
\begin{pmatrix} W & X \\ Y & Z \end{pmatrix}.
$$

Then $B_{++}W + B_{+-}Y = I_{\mathcal{K}_+}$ and $B_{-+}X + B_{--}Z = I_{\mathcal{K}_-}$. Therefore if B_{+-} is compact then B_{++} is invertible modulo the ideal of compact operators and thus Fredholm by Atkinson's Theorem.

Since the integers are discrete, it follows that $\operatorname{Ind}((M_f)_{++})$ depends only on the homotopy class of $\det f(z)$. Therefore to verify the formula for the index it suffices to consider the special case where $f(z) = \begin{pmatrix} z^k & 0 \\ 0 & 1 \end{pmatrix}$ for some integer k. If $k \geq 0$ we get $\dim \operatorname{Ker}(M_f)_{++} = 0$ and $\dim \operatorname{CoKer}(M_f)_{++} = nk$ and if $k \leq 0$ we get $\dim \operatorname{Ker}(M_f)_{++} = nk$ and $\dim \operatorname{CoKer}(M_f)_{++} = 0$ and so the formula holds in both cases. $\square$

For $S \subset \mathbb{Z}$, set $\mathcal{K}_S$ be the closure of the subspace of $\mathcal{K}$ generated by $\{\zeta_n \mid n \in S\}$. Let $\pi^S : \mathcal{K} \to \mathcal{K}_S$ denote the orthogonal projection. For $W \in \mathrm{Gr}(\mathcal{K})$, set $W^S := \pi^S(W)$. In particular, for a singleton set $S = \{m\}$ we have $W^{\{m\}} \subset \langle \zeta^n \rangle \cong \mathbb{C}^n$. Set $d^m(W) := \dim W^{\{m\}}$. If $W \in \mathrm{Gr}^z(\mathcal{K})$ then d_m is an increasing function of m.

Let f belong to $\Omega_{\mathrm{psm}} U(n)$. Since $(M_f)_{++}$ is Fredholm, its kernel and cokernel are finite dimensional which says that there exists N such that $d_m(W_f) = 0$ for $m < -N$ and $d_m(W_f) = n$ for $m > N$.

Let $r(f) = \min\{m \mid d_m > 0$ and $R(f) = \min\{n \mid d_m = n\}$. Each element $w \in W_f$ has an associated integer $d(w) \in [r(f), R(f)]$ given by the least m such that $w \in \pi^{\{m\}}(W_f)$.

Beginning with a basis for $\pi^{\{r\}}(W_f)$, add to the basis whenever d_m increases eventually creating an ordered basis $B = \{u_1, \ldots u_n\}$ for $\mathbb{C}^n$ in which for each m an initial segment of B forms a basis for $\pi^{\{m\}}(W_f)$. By applying Gram-Schmidt, we may assume that B is an orthonormal basis for $\mathbb{C}^n$. Define a homomorphism $\lambda(f) : S^1 \to U(n)$ by $\lambda_f(z)(u_j) := z^{d(u_j)} u_j$. Thus λ_f is a diagonal matrix when written in the basis B.
Note: Although there may be some choice in the basis B since the dimension d_m may be more than 1 greater than d_{m-1} for some values of m, the homomorphism $\lambda(f)$ is uniquely determined by W_f. (The restriction of $\lambda(f)$ to the subspace generated by a collection of basis elements added at the same stage is a multiple of the identity.)

Let $\mathcal{O}$ denote the ring of infinite series $a(z) = \sum_{n=0}^{\infty} a_n z^n$ in non-negative powers of z which converge on the closed unit disk D^2 in $\mathbb{C}$. (In particular, by assumption such $a(z)$ are holomorphic on the interior of the unit disk.) Slightly abusing the notation, for a matrix valued function $A(z)$ we will write $A(z)\mathcal{O}$ if all its entries lie in $\mathcal{O}$.

Set

$$
\begin{aligned}
N^- &:= \{A(z) \in \mathcal{O} \mid (z) \text{ is invertible for all } z \text{ with } \|z\| \geq 1 \text{ and } A(\infty) \text{ is upper triangular}\} \\
L_1^- &:= \{A(z) \in \mathcal{O} \mid (z) \text{ is invertible for all } z \text{ with } \|z\| \geq 1 \text{ and } A(\infty) = I\} \\
\Sigma_\lambda &:= N^{-1} \lambda \mathcal{K}_+ \subset \mathrm{Gr}^z(\mathcal{K}) \\
U_\lambda &:= \lambda L_1^{-1} \mathcal{K}_+ \subset \mathrm{Gr}^z(\mathcal{K}) \\
\Sigma_{\mathrm{bdd},\lambda} &:= \Sigma_\lambda \cap \mathrm{Gr}^z_{\mathrm{bdd}}(\mathcal{K}) \\
U_{\mathrm{bdd},\lambda} &:= U_\lambda \cap \mathrm{Gr}^z_{\mathrm{bdd}}(\mathcal{K})
\end{aligned}
$$

The representation of an element of Σ_λ in the form $N^{-1}\lambda\mathcal{K}_+$ is not unique: different elements of N^- might yield the same subspace. By subtracting multiples of one row from another we get

Proposition 7.2.3 $\Sigma_\lambda = (N^- \cap \lambda L_1^- \lambda^{-1}) \lambda \mathcal{K}_+$.

The next statement follows from the contraction $H_t(A)(z) := A(t^{-1}z)$ of L_1^-

Proposition 7.2.4 *Σ_λ is contractible.*

Proposition 7.2.5 *$U_\lambda \cong \Sigma_\lambda \times \mathbb{C}^{d(\lambda)}$ for some integer $d(\lambda)$.*

Proof: Suppose $A(z) \in L_1^-$. Set $A_\lambda(z) := \lambda(z)A(z)\lambda^{-1}(z)$. Then $A_\lambda(z)$ can be written uniquely in the form $A_\lambda(z) = T(z)P(z)$ where $T(z) \in N^{-1} \cap \lambda L_1^- \lambda^{-1}$ and $P(z)$ is an upper triangular matrix with 1's on the diagonal and polynomial entries which are divisibly by z. (Birkhoff decomposition.) To see this multiply matrices $T(z)$, $P(z)$ of the given form and equate coefficients with $A_\lambda(z)$. The entries of $T(z)$ which are on or below the diagonal are seen to be indentical with those in $A_\lambda(z)$. Before solving for the entries of $T(z)$ above the diagonal it is necessary to find the entries of $P(z)$. This can be done by examining the coefficients of the positive powers of z in the matrix $T(z)P(z)$ and comparing to the corresponding entries in $A_\lambda(z)$. With $P(z)$ now determined, these equations now determine the entries of $T(z)$ above the diagonal.

It follows that $U_\lambda \cong \Sigma_\lambda \times \mathbb{C}^{d(\lambda)}$ where $d(\lambda)$ is the sum of the degrees of the polynomials p_{ij} where $zp_{ij}(z)$ is an off-diagonal entry of $P(z)$. $\square$

Remark 7.2.6 There is a formula for $d(\lambda)$ in terms of the entries of λ. (Pressley-Segal, page 133.) It is analogous to Corollary 4.1.2

Corollary 7.2.7 *U_λ is contractible.*

The space $\Omega_{\text{psm}}U(n)$ is the union of the subspaces U_λ. If we think if $\Omega_{\text{psm}}U(n)$ as an infinite dimensional manifold, the sets U_λ are its charts. To do the Mayer-Vietoris style arguments (for example to use these methods to compute the cohomology of $\Omega_{\text{psm}}U(n)$ we must understand the intersections of our charts. For this purpose we now give alternate discriptions of U_λ and Σ_λ. Until we have shown the equivalence, we shall denote these as $\tilde{U}_\lambda$ and $\tilde{\Sigma}_\lambda$.

Define

$$\tilde{U}_\lambda := \{W_f \mid \text{the orthogonal projection from } W \text{ to } \lambda\mathcal{K}_+ \text{ is an isomorphism}\},$$
$$\tilde{\Sigma}_\lambda := \{W_f \in \tilde{U}_\lambda \mid= \lambda(f) = \lambda\}$$

Lemma 7.2.8 *If $W_f \in \Sigma_\lambda$ the $\lambda(f) = \lambda$. $U_\lambda \subset \tilde{U}_\lambda$ and $\Sigma_\lambda \subset \tilde{\Sigma}_\lambda$.*

Proof: If $g(z) \in \mathcal{H}$ has the form $a_r z^r +$ lower terms, orthogonal projection of g onto $\langle \zeta^r \rangle$ is the composition of the multiplication by z^{-r} followed by evaluation at ∞ followed by multiplication by z^r. Thus the Lemma follows from the definitions of N^- and L_1^-, and the construction of λ. $\square$

Lemma 7.2.9 *Let $h(z) : S^1 \to \mathbb{C}$ be piecewise smooth. Suppose that the coefficient of z^k in the Fourier expansion of h is zero for $k < 0$. Then $h \in \mathcal{O}$.*

Proof: Let $\sum_{k=0}^{\infty} c_k z^k$ be the Fourier expansion of $h(z)$, where $c_k \in \mathbb{C}$. Since h is piecewise smooth, the Fourier expansion of $h(z)$ converges to $h(z)$. Since the series $\sum_{k=0}^{\infty} c_k z^k$ converges for all z with $|z| = 1$, its radius of convergence is greater than 1 so it defines a holomorphic function on the unit disk whose boundary value is $h(z)$. $\square$

Lemma 7.2.10 *Let $h(z)$ be holomorphic on a domain containing D^2. Suppose that the restriction of $h(z)$ to S^1 is never 0 and that $h|_{S^1} : S^1 \to \mathbb{C} \smallsetminus \{0\}$ is null homotopic. Then $h(z)$ has no zeros in D^2.*

Proof: Consider the curve $\gamma(z) := h(S^1) \subset \mathbb{C}$. By hypothesis, γ is null homotopic. According to the Argument Principle

$$\#\text{ of zeros of } h(z) \text{ on } D^2 = \int_{S^1} \frac{h'(z)}{h(z)}\, dz = \int_\gamma \frac{1}{w}\, dw = \text{winding } \# \text{ of } \gamma \text{ about the origin} = 0.$$

Hence the origin is not in $h(D^2)$. $\square$

Proposition 7.2.11 $\tilde{\Sigma}_\lambda = \Sigma_\lambda$ *and* $\tilde{U}_\lambda = U_\lambda$

Proof: We showed the containments $\Sigma_\lambda \subset \tilde{\Sigma}_\lambda$ and $U_\lambda \subset \tilde{U}_\lambda$ in Lemma 7.2.8 and as in the proof of that lemma the other containment follows provided the holomorphicity and invertibility conditions on the resulting $A(z)$ are satisfied. Applying Lemma 7.2.9 to the entries of $\lambda^{-1}A(z)$ shows that they are boundary values of holomorphic functions on $|z| > 1$.

To see invertibility, let $d(z) = \det A(z)$. According to Lemma 7.2.2, the homotopy class of the function $z \to d(z)$ is divisibly by n, so there exists r such that the homotopy class of $e(z) := \det z^r A(z)$ is zero. Applying Lemma 7.2.10 to $e(z^{-1})$ shows that $e(z^{-1})$ is never zero on $|z| \geq 1$ so $A(z)$ is invertible for $|z| \geq 1$. $\square$

A homomorphism λ determines a subspace $\lambda\mathcal{K}_+$ whose orthogonal projection under our identification of $\mathcal{K}$ with $\mathcal{H}$ determines a subset $S(\lambda) \subset \mathbb{Z}$ containing the values of s such that ζ^s is in the image or the orthogonal projection. The subset $S(\lambda)$ determines and is uniquely determined by λ, and invariants such as the dimension $d(\lambda)$ in the decomposition $U_\lambda \cong \mathbb{C}^{d(\lambda)}$ and the Fredholm index of $\lambda\mathcal{K}_+$ can be computed directly from $S(\lambda)$. Since $\lambda\mathcal{K}_+ \in \mathrm{Gr}_{\mathrm{bdd}}(\mathcal{K})$ the subset $S(\lambda)$ must have the property that there exists m such that $j \notin S(\lambda)$ for $j < m$ and there exists M such that $j \in S(\lambda)$ for all $j \geq M$.

We define a partial order on $\{\lambda\}$ as follows. Write $S(\lambda) = \{s_1, s_2, \ldots, s_j, \ldots\}$ and $S(\lambda') = \{s'_1, s'_2, \ldots, s'_j, \ldots\}$ Set $\lambda \leq \lambda'$ if $s'_j \leq s_j$ for all j. Choose a total order on $\{\lambda\}$ extending this partial order.

By definition $U_\lambda \cup \mathrm{Gr}^z_{<\lambda}(\mathcal{K}) = \mathrm{Gr}^z_{\leq\lambda}(\mathcal{K})$. To do Mayer-Vietoris style arguments, we must also understand the intersection.

Proposition 7.2.12 *Under the identification $U_\lambda \cong \Sigma_\lambda \times \mathbb{C}^{d(\lambda)}$, the intersection $U_\lambda \cap \mathrm{Gr}^z_{<\lambda}(\mathcal{K})$ corresponds to $\Sigma_\lambda \times (\mathbb{C} - \{0\})^{d(\lambda)}$ (the complement of the zero cross-section in this trivial bundle).*

Proof: Note that the inclusion $\Sigma_\lambda \subset U_\lambda$ corresponds, under the identification with the inclusion of the zero cross-section into the total space. Suppose $W_f \in U_\lambda \cap \mathrm{Gr}^z_{<\lambda}(\mathcal{K})$. Since $\lambda(f) < \lambda$, Lemma 7.2.8 implies that $W \notin \Sigma_\lambda$. That is, W lies in the complement of the zero cross-section. Conversely, if $W_f \in \mathrm{Gr}^z_{\leq\lambda}(\mathcal{K})$ is not in Σ_λ, by Lemma 7.2.8 $\lambda(f) \neq \lambda$ and therefore it lies in $W_f \in \mathrm{Gr}^z_{\leq\lambda}(\mathcal{K})$. □

There are "bounded" versions of the preceding discussion which we now record. The proofs are obtained by the same arguments restricted to the bounded subsets.

Proposition 7.2.13 $U_{\mathrm{bdd}\lambda} \cong \Sigma_{\mathrm{bdd},\lambda} \times \mathbb{C}^{d(\lambda)}$. □

Set $\mathrm{Gr}^z_{\mathrm{bdd},\leq\lambda}(\mathcal{K}) = \cup_{\lambda'\leq\lambda} U_{\mathrm{bdd},\lambda'}$ and $\mathrm{Gr}^z_{\mathrm{bdd},<\lambda}(\mathcal{K}) = \cup_{\lambda'<\lambda} U_{\mathrm{bdd},\lambda'}$.

Proposition 7.2.14 *Under the identification $U_{\mathrm{bdd},\lambda} \cong \Sigma_{\mathrm{bdd},\lambda} \times \mathbb{C}^{d(\lambda)}$, the intersection $U_{\mathrm{bdd},\lambda} \cap \mathrm{Gr}^z_{\mathrm{bdd},<\lambda}(\mathcal{K})$ corresponds to $\Sigma_{\mathrm{bdd},\lambda} \times (\mathbb{C} - \{0\})^{d(\lambda)}$ (the complement of the zero cross-section in this trivial bundle).* □

Theorem 7.2.15 *The inclusion $\mathrm{Gr}^z_{\mathrm{bdd},\leq\lambda}(\mathcal{K}) \subset \mathrm{Gr}^z_{\leq\lambda}(\mathcal{K})$ is a homotopy equivalence for all λ.*

Proof: In general, if a topological space X is a union $U \cup V$ and another space X' is also a union $U' \cup V'$, and $f : X \to X'$ is a map, then assuming all of the inclusion maps are cofibrations, we have that f is a homotopy-equivalence if it induces G-homotopy-equivalences $U \to U'$, $V \to V'$ and $U \cap V \to U' \cap V'$. (See e.g. Selick, Thm.7.1.8) Thus our assertion follows by induction from the comparison of Proposition 7.2.12 with Proposition 7.2.14. □

Applying Theorem 7.1.2 and the homeomorphism α gives

Corollary 7.2.16 *The inclusion of $\mathrm{Gr}^z_{\mathrm{bdd}}(\mathcal{K}) = \cup_\lambda U_{\mathrm{bdd},\lambda}$ into $\mathrm{Gr}^z(\mathcal{K}) = \cup_\lambda U_\lambda$ is a homotopy equivalence. Equivalently, the inclusion $\Omega_{\mathrm{poly}} U(n) \subset \Omega_{\mathrm{psm}} U(n)$ is a homotopy equivalence.*

Theorem 7.2.17 *The inclusion $k : \Omega_{\mathrm{psm}} U(n) \hookrightarrow \Omega U(n)$ is a homotopy equivalence.*

Proof: Let $d(x, y)$ denote the distance between points x and y in $U(n)$ under the Riemannian metric coming from the embedding of $U(n)$ in Euclidean space. There is a number c such that if $x, y \in U(n)$ such that $d(x, y) < c$ implies that there is a unique shortest geodesic joining x and y. (It turns out that c is the distance from the identity matrix to $\begin{pmatrix} e^{2\pi i}/n & 0 & \dots & 0 \\ 0 & e^{2\pi i}/n & \dots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \dots & e^{2\pi i}/n \end{pmatrix}$.) A subset $A \subset U(n)$ is called "geodesically convex" if there is a unique shortest geodesic between any two points in A. Thus for A to be geodesically convex it suffices for the diameter of A to be less than c. Let $\{V_j\}_{j \in J}$ be an open cover of $U(n)$ by geodesically convex sets. For each integer m, let

$$Y_m := \{f \in \Omega U(n) \mid \text{for each } i = 1, \dots, 2^m \text{ there exists } j \in J \text{ such that } f([\tfrac{i-1}{2^m}, \tfrac{i}{2^m}]) \subset V_j\}.$$

Then Y_m is open in $U(n)$ and $\cup_m Y_m = U(n)$.

Set $X_m := Y_m \cap \Omega_{\text{psm}} U(n)$. Given $f \in Y_m$, define $r(f) \in X_m$ to be the path which agrees with f at $\frac{i}{2^m}$ for each $i = 1, \dots, 2^m$ and whose restriction to the subinterval $[\frac{i-1}{2^m}, \frac{i}{2^m}]$ is the shortest geodesic joining $f\left(\frac{i-1}{2^m}\right)$ to $f\left(\frac{i}{2^m}\right)$. Define $H : Y_m \times I \to Y_m$ by letting $H_s(t)$ be the shortest geodesic from $f(t)$ to $r(t)$. Then H is a homotopy from 1_{Y_m} to $r \circ k$ showing that the inclusion $k\big|_{X_m} : X_m \hookrightarrow Y_m$ is a homotopy equivalence. Therefore applying Thm. 7.1.2 gives that $k : \Omega_{\text{psm}} U(n) \hookrightarrow \Omega U(n)$ is a homotopy equivalence. $\square$

This completes the proof of Bott periodicity.

Chapter 8

Symmetric Polynomials

In this section we review some properties of symmetric polynomials and the various bases for expressing them.

Let K be a field, and let V be a finite dimensional vector space over K with basis $\langle x_1, \ldots, x_n \rangle$. The *tensor algebra* $T(V)$ on V is defined as $T(V) := \oplus_{n=0}^{\infty} V^{\otimes n}$ with multiplication by juxtiposition. The symmetric algebra $S(V)$ is defined as $S(V) := T(V)/\!\sim$ where $\sim$ is the equivalence relation generated by $x \otimes y \sim y \otimes x$ and the exterior algebra $\Lambda(V)$ is defined as $\Lambda(V) := T(V)/\!\sim$ where $\sim$ is the equivalence relation generated by $x \otimes y \sim -y \otimes x$. Thus $S(V) \cong K[x_1, \ldots, x_n]$. We write $T_n(V)$, $S_n(V)$, and $\Lambda_n(V)$ for the nth component (tensors of length n) of $T(V)$, $S(V)$, and $\Lambda(V)$ respectively.

The symmetric group S_n acts on $T(V)$ by permuting the variables, and this induces an action on $S(V)$ and $\Lambda(V)$. We write $\Sigma(V)$ for the subalgebra of symmetric polynomials in $S(V)$ consisting, by definition, of those polynomials which are invariant under the action of the symmetric group.

The jth elementary symmetric polynomial $s_j(x_1, \ldots {}_n)$ can be defined as the coefficient of t^j in $(t + x_1)(t + x_2) \cdots (t + x_n)$. From MAT1100 we know that $\Sigma(V) = K[s_1, \ldots s_n]$. If n is sufficiently large (compared to m), the dimension of $\big(\Sigma(V)\big)_m$ is independent of n and equals the number of partitions of m into a sum of positive integers.

There are several bases of $\big(\Sigma(V)\big)_m$ in common use. We shall discuss three of them: basis of elementary symmetric polynomials; basis of homogeneous symmetric polynomials; basis of power functions. The elementary symmetric polynomials s_k were defined above. The kth homogeneous symmetric polynomial, h_k, is defined as the sum of all symmetric monomials of degree k. The kth power function, ψ_k is defined as the sum of all kth powers. In each case the degree of s_k, h_k, and ψ_k is k, and the set of polynomials in these functions having of total degree m forms a basis for $\big(\Sigma(V)\big)_m$.

Example 8.0.18 Suppose $m = 2$. There are 2 partitions of 2 into positive integers: (2); and $(1, 1)$. Using three variables, $(n = 3)$, the various bases of $\big(\Sigma(V)\big)_m$ would be:

	(2)	(1,1)
Elementary Symmetric Polynomials	$s_2 = xy + xz + yz$	$s_1 s_1 = (x + y + z)^2$
Homogenious functions	$h_2 = x^2 + y^2 + z^2 + xy + xz + yz$	$h_1 h_1 = (x + y + z)^2$
Power Functions	$\psi_2 = x^2 + y^2 + z^2$	$\psi_1\psi_1 = (x + y + z)^2$

Example 8.0.19 $x_1^2 + \ldots x_n^2 = (x_1 + \ldots x_n)^2 - 2x_1 \cdots x_n = s_1(x_1, \ldots, x_n)^2 - 2s_2(x_1, \ldots, x_n)$. Therefore $\psi^2(\xi) = \Lambda_1(\xi)^{\otimes 2} - 2\Lambda_2(\xi)$.

The polynomial N_k is called the *kth Newton polynomial.* A recursive method for computing it is given below.

Although for computations for any particular m we need only finitely many variables, to consider all at once we shall let $\hat{V} = \langle x_1, \ldots, x_n, \ldots\rangle$. In $\big(\Sigma(\hat{V})\big)[[t]]$, set $S(t) := \sum_{n=0}^{\infty} s_n t^n$, $H(t) := \sum_{n=0}^{\infty} h_n t^n$, and $\psi(t) := \sum_{n=1}^{\infty} \psi_n t^{n-1}$. Then $S(t) = \prod_{j=1}^{\infty}(1 + x_j t)$ and $H(t) = \prod_{j=1}^{\infty}(1 + x_j t + x_j^2 t^2 + x_j^3 t^3 + \ldots) = \prod_{j=1}^{\infty} \frac{1}{(1-x_j t)}$. Immediately from these formulas we get $S(t)H(-t) = 1$. Also

$$\psi(t) = \sum_{n=1}^{\infty} \psi_n^n t^{n-1} = \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} x_j^n t^{n-1} = \sum_{j=1}^{\infty} \frac{x_j}{1 - x_j t} = \sum_{j=1}^{\infty} -\frac{d}{dt}\log(1 - x_j t)$$
$$= \frac{d}{dt}\log\left(\prod_{j=1}^{\infty} \frac{1}{1 - x_j t}\right) = \frac{d}{dt}\log\big(H(t)\big) = \frac{H'(t)}{H(t)}.$$

To write the power functions in terms of elementary symmetric polynomials we use

$$\psi(-t) = \frac{H'(-t)}{H(-t)} = \frac{S'(t)}{S(t)}.$$

Writing this as $S'(t) = \psi(-t)S(t)$ and expanding both sides gives

$$\sum_{n=1}^{\infty} n s_n t^{n-1} = \sum_{n=1}^{\infty}\sum_{j=0}^{n}(-1)^{j-1}\psi_j s_{n-j} t^n.$$

Therefore equating coefficients of t^{n-1} gives

$$n s_n = \sum_{j=1}^{n}(-1)^{j-1}\psi_j s_{n-j}$$

which recursively gives the Newton polynomial $N_n(s_1, \dots s_n)$ for writing the power functions in terms of the elementary symmetric functions.

Various other expressions for the Newton polynomials appear in representation theory and combinatorics. For example, let N be the square matrix with rows and columns indexed by the partition of n in which $N_{\lambda\mu} :=$ coefficient of S_μ in $N_{\lambda_1}, \dots N_{\lambda_r}$ for partitions $\lambda = (\lambda_1, \dots \lambda_r)$ and $\mu = (\mu_1, \dots, \mu_s)$. Then $N = Z(K^t J)^{-1}$ where Z is the normalized character matrix of the symmetric group S_n and K^t is the transpose of the matrix of "Kostka numbers" and J is given by $J_{\lambda\mu} := \begin{cases} 1 & \text{if } \lambda \text{ and } \mu \text{ are dual partitions;} \\ 0 & \text{otherwise.} \end{cases}$

Chapter 9

K-Theory as a Generalized Cohomology Theory

Definition 9.0.20 A *reduced cohomology theory* consists of a sequence of functors

$$\tilde{Y}^n : \text{pointed topological spaces} \;\to\; \text{abelian groups}$$

together with natural transformations $\sigma : \tilde{Y}^n \to \tilde{Y}^{n+1} \circ S$ such that:

1) Homotopy: $f \simeq g \Rightarrow f_* = g_*$;

2) Suspension: $\sigma_X : \tilde{Y}^n(X) \to \tilde{Y}^{n+1}(SX)$ is an isomorphism for all X;

3) Exactness: For every (basepoint-preserving) inclusion $A \hookrightarrow X$, the induced sequence $\tilde{Y}^n(X/A) \to \tilde{Y}^n(X) \to \tilde{Y}^n(A)$ is exact for all n.

From these axioms, the infinite sequence of homotopy cofibrations

$$A \xrightarrow{j} X \xrightarrow{q} C \xrightarrow{\delta} SA \xrightarrow{(Sj)^{-1}} SX \to \ldots \to S^{k-1}C$$
$$\xrightarrow{\delta} S^kA \xrightarrow{(S^kj)^{(-1)^k}} S^kX \xrightarrow{(S^kq)^{(-1)^k}} S^kC \to \ldots$$

associated to a cofibration $A \to X$ allows the construction of a long exact sequence and Mayer-Vietoris follows.

A corresponding unreduced theory is obtained by setting $Y^n(X) := \tilde{Y}^n(X \amalg *)$.

Definition 9.0.21 An Ω-*spectrum* is a collection of pointed spaces $\{Y_n\}_{n\in\mathbb{Z}}$ together with homotopy equivalences $e_n : Y_n \to \Omega Y_{n+1}$.

An Ω-spectrum $\{Y_n\}$ gives rise to a cohomology theory $\tilde{Y}^*(\ )$ given by $\tilde{Y}^n(X) := [X, Y_n]$, where the suspension isomorphism is induced from ϵ_n.

Remark 9.0.22 According to the Brown Representability Theorem, all cohomology theories satisfing the "Milnor Wedge Axiom" which are defined on CW-complexes arise in this way from some Ω-spectrum. (See [10].)

Example 9.0.23 Let G be an abelian group and consider ordinary cohomology theory $\tilde{H}^*(\ ;G)$ with coefficients in G. By Brown Representability, there exists an Ω-spectrum $\tilde{Y}_n$ such that $\tilde{H}(X;G) = [X, \tilde{Y}_n]$ for all CW-complexes X. Letting $X = S^q$, be see that

$$\pi_q(\tilde{Y}_n) = \begin{cases} G & \text{if } q = n; \\ 0 & \text{if } q \neq n. \end{cases}$$

The space $\tilde{Y}_n$ is usually written as $K(G, n)$ and called an "Eilenberg - Mac Lane" space (a space with one nonzero homotopy group). Any two models for $K(G, n)$ must be homotopy equivalent. One can construct Eilenberg - Mac Lane spaces directly, without appealing to the Brown Representability Theorem, by creating an n-skeleton with the correct $\pi_n(\ )$ using wedges of spheres in degrees n and $n+1$ and then inductively attaching higher dimensional spheres to kill the higher homotopy groups. To consider this example in the light of bundle theory, make G into a topological group by giving it the discrete topology. Then $G = K(G, 0)$ and $BG = K(G, 1)$. If G were not abelian it would not be possible to construct $K(G, 2)$ since $\pi_2(\ )$ is always abelian. However since G is abelian the multiplication map $G \times G \to G$ is a group homomorphism and so it induces a map $BG \times BG \to BG$. Usually we think of BG only as a homotopy type, but to proceed one must chose a model for BG under which the preceding map becomes strictly associative and turns BG into a topological group. Then $B^2G := B(BG) = K(G, 2)$. With the right machinery, one can arrange that B^2G is also a topological group and repeat the procedure indefinitely. For example, $S^1 = K(\mathbb{Z}, 1)$ so its classifying space $\mathbb{C}P^\infty$ is $K(\mathbb{Z}, 2)$. The model $\mathbb{C}P^\infty$ for $K(\mathbb{Z}, 2)$ is only an H-space, not a topological group, but it is possible to construct a space homotopy equivalent to $\mathbb{C}P^\infty$ which is a topological group.

Example 9.0.24 Using Bott periodicity we can construct the "BU-spectrum" in which

$$Y_n = \begin{cases} BU \times \mathbb{Z} & \text{if } n \text{ is even}; \\ U & \text{if } n \text{ is odd}, \end{cases}$$

and the structure maps are given by the homotopy equivalence $\Omega(BU \times \mathbb{Z}) \cong U$ of Prop. 3.1.11 and the Bott periodiciy homotopy equivalence $\Omega U \cong BU \times \mathbb{Z}$.

We noted earlier (6.0.18), that for that for a compact connected space X, there is an isomorphism $\tilde{K}(X) \cong [X, BU]$. More generally, if the compact space X is a union of path components, $\tilde{K}(X) \cong [X, BU \times \mathbb{Z}]$. (The map $X \to BU$ describes, for each component of X, a stable equivlence class bundles over that component, and the map $X \to \mathbb{Z}$ describes, for each component, the difference between its dimension and that of the bundle over the component of the basepoint.) Of course, if X is connected than $[X, BU] = [X, BU \times \mathbb{Z}]$. For an arbitrary space X, we **define** $\tilde{K}(X) := [X, BU \times \mathbb{Z}]$ and we set $K(X) := \tilde{K}(X \amalg *)$, or equivalently $K(X) = [X, BU \times \mathbb{Z}] \times \mathbb{Z}$, where the second component keeps track of the dimension of the restriction to the component of the basepoint, as in 6.0.19.

Remark 9.0.25 This is sometimes called "representable K-theory". People sometimes look at another version of K-theory formed by taking the inverse limit of the K-theory of compact subsets of X. If X is the union of nested compact subspaces $\{X_n\}$ then the two are related by the Milnor exact sequence

$$0 \to \varprojlim_n{}^1 \tilde{K}(SX_n) \to \tilde{K}(X) \to \varprojlim_n \tilde{K}(X_n) \to 0,$$

where the "error term", $\varprojlim_n{}^1 \tilde{K}(SX_n)$, is the derived functor of $\varprojlim$, as defined in homological algebra.

We now define the cohomology theory, K-theory, to be the one associated to the BU-spectrum. Thus $\tilde{K}^n(X) := \begin{cases} [X.BU \times \mathbb{Z}] & \text{if } n \text{ is even;} \\ [X, U] & \text{if } n \text{ is odd.} \end{cases}$.

9.1 Calculations

Proposition 9.1.1 *As abelian groups,*

$$\tilde{K}(S^n) \cong \begin{cases} \mathbb{Z} & \textit{if } n \textit{ is even;} \\ 0 & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof:

Bott periodicity reduces the Proposition to the cases $n = 0$ and $n = 1$. Bundles over S^0 are determined by their dimension, so $K(S^0) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $\tilde{K}(S^0) \cong \mathbb{Z}$. Bundles over S^1 are determined by their dimension and a clutching map $S^0 \to U$. However U is connected, so all such clutching maps are trivial and therefore $\tilde{K}(S^1) = 0$. □

The Mayer-Vietoris sequence gives

Corollary 9.1.2 *If X is a finite CW-complex all of whose nonzero cohomology groups are torsion-free and are in even degrees, then as abelian groups, $K^{2n}(X) \cong \oplus_k H^k(X)$ for all n and $K^{\text{odd}}(X) = 0$.*

Proof: Build the space as a CW-complex using only cells in even degrees. □

Corollary 9.1.3 *As abelian groups*

$$K^{\text{even}}(\mathbb{C}P^n) \cong \oplus_{j=0}^n \mathbb{Z} \qquad \textit{and} \qquad K^{\text{odd}}(\mathbb{C}P^n) = 0$$

□

For an unreduced cohomology theory $Y^*(\ )$, the groups $Y^*(*)$ are sometimes called the coefficients of the theory. In our case, since by definition, $K^n(*) := \tilde{K}^m(S^0) = [S^0, Y_n]$ where Y_n is the nth space in the BU-spectrum, we have

$$K^n(*) \cong \begin{cases} \mathbb{Z} & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Let $B^k = \sigma_*^{2k}(1) \in K^{2k}(*) \cong \tilde{K}^{2k}(S^{2k})$ where $1 \in K^0(*) \cong \tilde{K}^0(S^0)$ and σ is the isomorphism in (2) of the definition of a cohomology theory. Because of the way the BU-spectrum is defined, B^{m+k} is the image of B^k under the mth iterate of the Bott periodicity isomorphism.

9.2 Products in K-theory

Tensor product of bundles induces a homomorphism $\mu : K(X) \otimes K(Y) \to K(X \times Y)$. The corresponding reduced version is a homomorphism $\tilde{\mu} : \tilde{K}(X) \otimes \tilde{K}(Y) \to \tilde{K}(X \wedge Y)$. In terms of representing spaces, these maps can be interpreted as coming from the map $BU(n) \times BU(k) \to BU(nk)$ induced from the tensor product $U(n) \times U(k) \to U(nk)$.

We extend our product operation to $\tilde{\mu} : K^i(X) \otimes K^j(Y) \to K^{j+k}(X \times Y)$ as follows. Let $\bar{B}_k \in K^k(*)$ be the generator

$$B_k := \begin{cases} B^r & \text{if } k = 2r; \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

An element of $w \in K^k(W)$ is a pair $w = \left(\tilde{w}, \epsilon(w)\bar{B}_k\right)$ where $\tilde{w} \in \tilde{K}^n(W)$ and $\epsilon(w) \in \mathbb{Z}$. Letting σ_W be as in (2) of the definition of a cohomology theory, writing $a \cdot b$ for $\mu(a, b)$ and letting π_1 and π_2 be the projection maps, we define

$$\begin{aligned}&\big(\tilde{x},\epsilon(x)\bar{B}_i\big)\cdot\big(\tilde{y},\epsilon(y)\bar{B}_j\big)\\&\qquad:=\Big(\sigma^{i+j}_{X\times Y}\big(\sigma^{-i}_X(\tilde{x})\cdot\sigma^{-i}_Y(\tilde{y})\big)+\epsilon(y)\sigma^j_X\big(\pi_1^*(\tilde{x})\big)+\epsilon(x)\sigma^i_Y\big(\pi_2^*(\tilde{y})\big),\epsilon(x)\epsilon(y)\bar{B}_{i+j}\Big).\end{aligned}$$

From the (external) multiplication above, we can define an internal multiplication $K^i(X)\otimes K^j(X)\to K^{i+j}(X)$ by composing the external multiplication with the map $\Delta^*: K^*(X\times X)\to K^*(X)$ induced from the diagonal map $\Delta: X\to X\times X$. We often write ab for $\mu(a,b)$.

Notice that our notation is consistent in that when $X=*$ then $\mu(B^i,B^j)=B^{i+j}$. Therefore as a graded ring, $K^*(*)\cong\mathbb{Z}[B,B^{-1}]$, where $|B|=2$.

In the special case where $X=*$, the product gives us a multiplication $K(*)\otimes K(Y)\to K(Y)$ (after identifying $\{*\}\times Y$ with Y) which gives $K^*(Y)$ the structure of an $K^*(*)$-module. For any $f: X\to Y$, the map $f^*: K^*(Y)\to K^*(X)$ becomes a homomorphism of $K^*(*)$-modules and the exact sequence (3) in the definition of a cohomology theory becomes an exact sequence of $K^*(X)$-modules.

For the Künneth theorem, we will restrict attention to CW-complexes although it is true more generally. The following theorem will suffice for our purposes.

Theorem 9.2.1 *(***Künneth Theorem for K-theory***) Let X and Y be CW-complexes such that $K^*(Y)$ is torsion-free. Suppose that either X is a finite complex or that X is a direct limit of finite complexes X_n such that $\tilde{K}^{\mathrm{odd}}(X_n)=0$ for all n. Then $\tilde{\mu}:\tilde{K}^*(X)\otimes\tilde{K}^*(Y)\to\tilde{K}^*(X\wedge Y)$ is an isomorphism. Equivalently $\mu: K^*(X)\otimes K^*(Y)\to K^*(X\times Y)$ is an isomorphism.*

Remark 9.2.2 There are more general versions in the case where the torsion-free hypothesis does not hold, but we shall not go into them. We have also added a hypothesis on X to avoid discussion of $\varprojlim^1$ terms.

Proof:

Consider first the case where X is a finite complex. The case where X is a sphere follows from the definitions and the computation of $\tilde{K}(S^n)$. Suppose now that it is known that $\tilde{\mu}: K^*(W)\otimes K^*(Y)\to K^*(W\wedge Y)$ holds for all CW-complexes W such that W has fewer cells than X. The CW-complex X is formed as a pushout from some attaching map $f: S^{n-1}\to W$ where W has one cell less than X. In other words, there is a homotopy-cofibration $S^{n-1}\xrightarrow{f}W\to X$, and so taking the smash product with Y gives a homotopy-cofibration $S^{n-1}\wedge Y\xrightarrow{f\wedge 1_Y}W\wedge Y\to X\wedge Y$. The cofibration $S^{n-1}\xrightarrow{f}W\to X$ gives a long exact sequence

$$\ldots\to\tilde{K}^i(X)\to\tilde{K}^i(W)\xrightarrow{f^*}\tilde{K}^i(S^{n-1})\to\tilde{K}^{i+1}(X)\to\ldots$$

Since $\tilde{K}^j(Y)$ is torsion-free, tensoring with it preserves exactness. Mapping the result into the cofibration sequence for $S^{n-1} \wedge Y \xrightarrow{f\wedge 1_Y} W \wedge Y \to X \wedge Y$ gives a commmutative diagram with exact rows

$$\begin{tikzcd}
\cdots \tilde{K}^{i-1}(W) \otimes \tilde{K}^j(Y) \arrow[r] \arrow[d, "\tilde{\mu}"] & \tilde{K}^{i-1}(S^{n-1}) \otimes \tilde{K}^j(Y) \arrow[r] \arrow[d, "\tilde{\mu}"] & \tilde{K}^{i}(X) \otimes \tilde{K}^j(Y) \arrow[r] \arrow[d, "\tilde{\mu}"] & \tilde{K}^{i}(W) \otimes \tilde{K}^j(Y) \cdots \arrow[d, "\tilde{\mu}"] \\
\cdots \tilde{K}^{i-1+j}(W \wedge Y) \arrow[r] & \tilde{K}^{i-1+j}(S^{n-1} \wedge Y) \arrow[r] & \tilde{K}^{i+j}(X \wedge Y) \arrow[r] & \tilde{K}^{i+j}(W \wedge Y) \cdots
\end{tikzcd}$$

so the result follows from the 5-Lemma.

If X is not a finite complex, we must pass to the limit which presents no problem since our hypothesis guarantees that $\tilde{K}^*(X) = \varprojlim \tilde{K}^*(X_n)$.

The unreduced version of the theorem follows from the reduced version. □

9.3 Splitting Principle and Chern Character

In this section we shall restrict ourselves to spaces X which are either compact or can be written as a direct limit of spaces X_k such that $\tilde{H}^{\mathrm{odd}}(X_k) = 0$ for all k. As before, there are more general versions of some of the theorems.

A line bundle over X is determined by its classifying map $f : X \to BU(1) \hookrightarrow BU$. Since $BU(1) = \mathbb{C}P^\infty = K(\mathbb{Z}, 2)$, this says that a line bundle ξ over X is determined by its first Chern class $c_1(\xi) \in H^2(X) \cong [X, \mathbb{C}P^\infty]$. More generally, if $\xi \in K(X)$ is a virtual bundle $\xi' - \xi''$ where each of $\xi' = \oplus_{j=1}^{r'} \eta'_j$ and $\xi'' = \oplus_{j=1}^{r''} \eta''_j$ are sums of line bundles, ξ is determined by the Chern classes $c_1(\{\eta'_j\})$ and $c_1(\{\eta''_j\})$. We make this more precise by using the Chern classes $c_1(\{\eta'_j\})$ and $c_1(\{\eta''_j\})$ to define an element $\mathrm{Ch}(\xi) \in \hat{H}(X; \mathbb{Q}) := \prod_{n=0}^\infty H^{2n}(X; \mathbb{Q})$, called the Chern character of ξ, which determines the element $\xi \in K(X)$. Later will be extend the definition to bundles which are not necessarily formed from line bundles.

If ξ is a line bundle over X, we set $\mathrm{Ch}_L(\xi) := \exp\bigl(c_1^{\mathbb{Q}}(\xi)\bigr)$, where $c_1^{\mathbb{Q}}(\xi) \in H^2(X; \mathbb{Q})$ is image of the first Chern class of X under the map $H^2(X) \to H^2(X; \mathbb{Q})$. Extend the definition to sums and differences of line bundles by linearity. (That is, $\mathrm{Ch}_L(\xi' + \xi'') := \mathrm{Ch}_L(\xi') + \mathrm{Ch}(\xi'')$ and similarly for differences.) We shall sometimes write simply c_j for $c_j^{\mathbb{Q}}$ if it clear that we are working with $\mathbb{Q}$-coefficients.

Notice that $c_1(\xi \otimes \xi') = c_1(\xi) + c_2(\xi')$, so if ξ and ξ' are line bundles then

$$\mathrm{Ch}_L(\xi \otimes \xi') = \exp\bigl(c_1^{\mathbb{Q}}(\xi) + c_1^{\mathbb{Q}}(\xi')\bigr) = \mathrm{Ch}_L(\xi)\,\mathrm{Ch}_L(\xi').$$

Thus Ch_L is a ring homomorphism on the subring $K_L(X)$ of $K(X)$ generated by line bundles.

The structure of $K(\mathbb{C}P^n)\otimes\mathbb{Q}$ as a vector space is determined by Cor. 9.1.3. We can use the above ideas to include the ring structure.

Proposition 9.3.1 *Let $x=\gamma_n^1-1\in\tilde{K}(\mathbb{C}P^n)$, where γ_n^1 is the canonical line bundle over $\mathbb{C}P^n$. Then as rings, $K(\mathbb{C}P^n)\otimes\mathbb{Q}\cong\mathbb{Q}[x]/(x^{n+1})$.*

Proof: Set $t:=c_1^{\mathbb{Q}}(\gamma_n^1)$. Since $K(\mathbb{C}P^n)$ is torsion free by Cor. 9.1.3, the map $K(\mathbb{C}P^n)\to K(\mathbb{C}P^n)\otimes\mathbb{Q}$ is injective. Writing $\hat{H}(\mathbb{C}P^n;\mathbb{Q})\cong\mathbb{Q}[t]/(t^{n+1})$, by definition $\mathrm{Ch}_L(x)=\exp(t)-1=t+t^2/2!+\ldots+t^n/n!$. Upon examination of set of elements $\{\mathrm{Ch}_L(x^j)\}=\{\mathrm{Ch}_L(x)^j\}=\{\exp(t)^j\}$ we see that they are linearly independent in $\mathbb{Q}[t]/(t^{n+1})$, and it follows that $\{x^j\}_{j=1}^n$ are linearly independent in $K(\mathbb{C}P^n)\otimes\mathbb{Q}$. By counting dimensions using Cor. 9.1.3, we see that the images of $\{x_j\}_{j=0}^n$ are a basis for $K^*(\mathbb{C}P^n)\otimes\mathbb{Q}$ and that $\mathrm{Ch}_L: K_L(\mathbb{C}P^n)\to\hat{H}(\mathbb{C}P^n;\mathbb{Q})\cong\mathbb{Q}[t]/(t^{n+1})$, is an injection. Since $t^{n+1}=0$ in $\mathbb{Q}[t]/(t^{n+1})$, we see that $\mathrm{Ch}_L(x^{n+1})=0$ and so $x^{n+1}=0$ in $K(\mathbb{C}P^n)$. $\square$

Let $\xi=p:E\to B$ be a vector bundle. Define the projective bundle $P(\xi)$ associated to ξ to be the bundle $p\big(P(\xi)\big):E\big(P(\xi)\big)\to B$ in which $E\big(P(\xi)\big):=S(\xi)/\!\sim$ where in each fibre F_b, we make the identifications $v\sim w$ if $v=\lambda w$ for some $\lambda\in S^1$. Thus elements of $E\big(P(\xi)\big)$ are lines in F_b for some b. If $\dim\xi=n$, the fibre of the bundle $P(\xi)$ is $\mathbb{C}P^{n-1}$. There is a canonical line bundle γ_ξ over $E\big(P(\xi)\big)$ whose total space consists of pairs (L,v) where $L\in P(\xi)$ and $v\in L$.

Lemma 9.3.2 *Let $\xi=p:E\to B$ be an n-dimensional vector bundle. Then $p\big(P(\xi)\big)^*(\xi)$ decomposes as a Whitney sum $p\big(P(\xi)\big)^*(\xi)\cong\gamma_\xi\oplus\eta$ where η is some bundle of dimension $n-1$.*

Proof: Choose a Riemannian metric on ξ. An element of $E\big(P(\xi)\big)$ consists of a line L in the fibre over some point of B. An element of $E\Big(p\big(P(\xi)\big)^*(\xi)\Big)$ consists of a pair (L,v) where L is a line in the fibre F_b of ξ for some $b\in B$, and $v\in F_b$. Using our chosen Riemannian metric we can write v uniquely as $v=v'\oplus v''$ where $v'\in L$ and $v''\perp L$. The association $(L,v)\mapsto(L,v')$ is a retraction from $p\big(P(\xi)\big)^*(\xi)$ to its canonical line bundle. $\square$

Set $M:=K(\mathbb{C}P^{n-1})$ and $\tilde{M}:=\tilde{K}(\mathbb{C}P^{n-1})$. According to Cor. 9.1.3, if we ignore the grading, $K(\mathbb{C}P^{n-1})\cong H^*(\mathbb{C}P^{n-1})\cong M$. The isomorphism takes $m:=\gamma_{n-1}^1-1\in\tilde{K}(\mathbb{C}P^{n-1})$ to a generator $t\in H^2(\mathbb{C}P^{n-1})\cong\mathbb{Z}$.

Lemma 9.3.3 *Let $\xi=p:E\to X$ be a complex n-dimensional vector bundle. Let $P(\xi)=p:E\to X$ be the associated projective bundle with fibre $\mathbb{C}P^{n-1}$ and let $j:\mathbb{C}P^{n-1}\hookrightarrow E\big(P(\xi)\big)$ denote the inclusion. Let $m=\gamma_n^1-1\in\tilde{K}(\mathbb{C}P^n)$ and let $y=\gamma_\xi-1\in\tilde{K}\Big(P\big(E(\xi)\big)\Big)$. Then $j^*(y)=m$ and the natural map $K^*(X)\otimes M\cong K^*\big(P(E)\big)$ given by $x\otimes m^i\mapsto p^*(x)y^i$ is* *an isomorphism. In particular $p^* : K(X) \to K^*\big(P(E)\big)$ is an injection. Similarly, if we let $t \in H^2(\mathbb{C}P^{n-1};\mathbb{Q}) \cong \mathbb{Q}$ be a generator, then there exists a canonical $y_{\mathbb{Q}} \in H^2\Big(P\big(E(\xi)\big);\mathbb{Q}\Big)$ such that the same statements hold with $K^*(\ )$ replaced throughout by $H^*(\ )$.*

Proof:

Consider the K-theory statement. If $i : U \subset X$ such that $i^*(\xi)$ is a product bundle then the conclusions of the Lemma hold for $p : p^{-1}(U) \to U$ (and for subsets of U) by the Künneth Theorem.

Suppose U and V are subsets of B such that the conclusions of the Lemma hold for U, V and $U \cap V$. Since $K^*(\mathbb{C}P^{n-1})$ is a free abelian group, tensoring with the K-theory Mayer-Vietoris sequence for U, V with $K^*(\mathbb{C}P^{n-1})$ preserves exactness. Mapping the resulting exact sequence into the K-theory Mayer-Vietoris sequence of $p^{-1}(U), p^{-1}(V)$ and applying the 5-Lemma gives that the conclusions of the Lemma fold for $p : p^{-1}(U \cup V) \to U \cup V$.

When X is compact it has a local trivialization with respect to a finite covering and so the result follows by induction. Passing to the limit gives the result for X such that $\tilde{K}^{\mathrm{odd}}(X_n) = 0$ for all n.

The proof for ordinary cohomology will become identical as soon as we have produced the canonical class $y_{\mathbb{Q}}$. We have a diagram of bundles

$$\begin{tikzcd}
S^1 \arrow[r] \arrow[d, equal] & S^{2n-1} \arrow[r] \arrow[d] & \mathbb{C}P^{n-1} \arrow[d, "j"] \\
S^1 \arrow[r] & S\big(E(\xi)\big) \arrow[r] \arrow[d, "p"] & P\big(E(\xi)\big) \arrow[d, "p"] \\
 & X \arrow[r, equal] & X
\end{tikzcd}$$

Naturality of the classifying maps of the S^1-bundles forming the top two rows gives an extension of the diagram to the right

$$\begin{tikzcd}
\mathbb{C}P^{n-1} \arrow[r, hook] \arrow[d, "j"] & BS^1 = \mathbb{C}P^\infty \arrow[d, equal] \\
P\big(E(\xi)\big) \arrow[r, "f"] & BS^1 = \mathbb{C}P^\infty
\end{tikzcd}$$

Let $y_{\mathbb{Q}} = f^*(\tilde{t})$ where $\tilde{t}$ is the image of the generator of $H^2(\mathbb{C}P^\infty) \cong \mathbb{Q}$ which maps to t under the canonical map $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^\infty$. $\square$

Remark 9.3.4 A proof of the existence of $y_{\mathbb{Q}}$ could also be done by showing the Serre spectral sequence of the fibration $\mathbb{C}P^{n-1} \to P\big(E(\xi)\big) \to X$ collapses. To show that the differential $d(t)$ is 0 use the fact that $t^n = 0$ and $d(t^{n-1}) = d(t) \otimes t^{n-1} \neq 0$ in $E_2^{2,2n-1} = H^2(X) \otimes H^{n-1}(\mathbb{C}P^{n-1})$.

By induction from Lemma 9.3.2 and Lemma 9.3.3 we get

Corollary 9.3.5 *(***Splitting Principle***) Let $\xi = p : E \to X$ be a vector bundle Then there exists a space* $\mathrm{Flag}(\xi)$ *and a map* $\mathrm{Flag}(p) : \mathrm{Flag}(\xi) \to X$ *such that*

- $\mathrm{Flag}(p)^* : K^*(X) \to K^*\big((\mathrm{Flag}(\xi)\big)$ *is an injection.*
- $\mathrm{Flag}(p)^* : H^*(X;\mathbb{Q}) \to H^*\big((\mathrm{Flag}(\xi);\mathbb{Q}\big)$ *is an injection.*
- $\mathrm{Flag}(p)^*(\xi)$ *splits into a Whitney sum of line bundles.*

$\square$

Notice that in the case where ξ is the universal bundle γ_n over $G_n^{\mathbb{C}}$, the space $\mathrm{Flag}(\xi)$ becomes $\big(BU(1)\big)^n$ with the map $\mathrm{Flag}(p)^*(\xi)$ classifying $(\gamma^1)^n$ giving the injection described earlier in (the complex analogue of) Prop 4.3.1.

We now extend our definition of the Chern character, defined so far only on $K_L(X)$ to a natural ring homomorphism $\mathrm{Ch}_X : K(X) \to \hat{H}(X;\mathbb{Q})$. Let ξ be an n-dimensional complex vector bundle over X. Set

$$M := \hat{H}(\mathbb{C}P^{n-1};\mathbb{Q}) \otimes \hat{H}(\mathbb{C}P^{n-2};\mathbb{Q}) \otimes \cdots \otimes \hat{H}(\mathbb{C}P^1;\mathbb{Q}) = \hat{H}(F;\mathbb{Q})$$

where F is the fibre of $p : \mathrm{Flag}(\xi) \to X$. Write $M = \tilde{M} \oplus \mathbb{Z}$ and let $\epsilon : M \to \mathbb{Z}$ be the augmentation. By Lemma 9.3.3 and induction we have isomorphisms $K^*(X) \otimes M \cong K^*\big(\mathrm{Flag}(\xi)\big)$ and $\hat{H}(X;\mathbb{Q}) \otimes M \cong \hat{H}\big(\mathrm{Flag}(\xi);\mathbb{Q}\big)$ taking $x \otimes 1$ to $p^*(x)$. Define $\mathrm{Ch}_X(x) := s\Big(\mathrm{Ch}_{L,\mathrm{Flag}(\xi)}\big(p^*(x)\big)\Big)$, where $s : \hat{H}\big(\mathrm{Flag}(\xi);\mathbb{Q}\big) \to \hat{H}(X;\mathbb{Q})$ is given by $s(y \otimes m) := \epsilon(m)y$. Notice that $s\big((\exp(c)\big) = \exp\big(s(c)\big)$ and so Ch_X is a natural ring homomorphism. Let $j : F \to \mathrm{Flag}(\xi)$ denote the inclusion of the fibre into the total space. Notice that $H^2\big(\mathrm{Flag}(\xi);\mathbb{Q}\big) \cong H^2(X;\mathbb{Q}) \oplus H^2(F;\mathbb{Q})$, with the isomorphism given by $c \mapsto \big(s(c), i^*(c)\big)$. If x is already a line bundle, then $j^*\Big(p^*\big(c_1^{\mathbb{Q}}(x)\big)\Big) = 0$ and so $s\Big(p^*\big(c_1^{\mathbb{Q}}(x)\big)\Big) = c_1^{\mathbb{Q}}$. Therefore Ch_X agrees with $\mathrm{Ch}_{L,X}$ on $K_L(X)$.

If $A \to X \to Y$ is a cofibration sequence then it induces long exact sequences on $K(\ ) \otimes \mathbb{Q}$ and $\hat{H}(\ );\mathbb{Q}$. Since the connecting homomorphisms are induced by a map $Y \to SA$, as discussed in Section 3, by naturality, Ch induces a map between these long exact sequences.

Theorem 9.3.6 *Let X be a CW-complex which is either finite or the direct limit of finite complexes X_k such that $H^{\text{odd}}(X_k) = 0$ for all k. Then the Chern character is a natural ring isomorphism*

$$\operatorname{Ch} : K(X) \otimes \mathbb{Q} \cong \hat{H}(X; \mathbb{Q}).$$

Proof:

Consider first the case where $X = S^{2n}$. We already know that $K(S^{2n}) \cong \mathbb{Z} \oplus \mathbb{Z} \cong \hat{H}(S^{2n})$ so $K(S^{2n}) \otimes \mathbb{Q} \cong \hat{H}(S^{2n}; \mathbb{Q})$. The issue is whether or not Ch is an isomorphism. Let y be a nonzero element of $\tilde{K}(S^{2n}) \otimes \mathbb{Q} \cong \mathbb{Q}$. Let $q : \mathbb{C}P^n \to S^{2n}$ denote the map which pinches $\mathbb{C}P^{n-1}$ to a point. From the calculation of $K(\mathbb{C}P^n)$ (Cor. 9.1.3) we know that $q^*(y) \neq 0$. According to Prop. 9.3.1, $K(\mathbb{C}P^n) \otimes \mathbb{Q}$ is generated by line bundles and so $\operatorname{Ch}\bigl(q^*(y)\bigr) = \operatorname{Ch}_L\bigl(q^*(y)\bigr) \neq 0 \in \hat{H}(\mathbb{C}P^n; \mathbb{Q})$. Since $q^* : \hat{H}(S^{2n}) \to \hat{H}(\mathbb{C}P^n; \mathbb{Q})$ is injective, naturality gives $\operatorname{Ch}(y) \neq 0$. Thus $\operatorname{Ch} : K(S^{2n}) \otimes \mathbb{Q} \to \hat{H}(S^{2n}; \mathbb{Q})$ is an isomorphism.

Now suppose that X is a finite complex with m cells and assume by induction that the theorem is known for all complexes with fewer than m cells. Let $f : S^r \to W$ be the attaching map by which X is formed from a subcomplex W having $m-1$ cells. Then $S^r \to W \to X$ is a cofibration sequence, so naturality and the 5-Lemma gives that Ch_X is an isomorphism.

Finally, is X is the direct limit of finite complexes X_k for which $H^{\text{odd}}(X_k) = 0$ for all k then $K(X) \otimes \mathbb{Q} = \varprojlim K(X_k) \otimes \mathbb{Q}$. $\hat{H}(X) = \varprojlim \hat{H}(X_k)$ so passing to the limit shows that Ch_X is an isomorphism.

□

Remark 9.3.7 On a large class of spaces including all CW-complexes X which are either connected or H-space topological groups there is a natural functor $X \to X_{(0)}$ called "rationalization" is defined. The rationalization of spaces has the effect of algebraic rationalization of the homotopy and homology groups: $\pi_*(X_{(0)}) \cong \pi_*(X) \otimes \mathbb{Q}$ and $H_*(X_{(0)}) \cong H_*(X) \otimes \mathbb{Q}$ with the $X \to X_{(0)}$ inducing the natural map $A \to A \otimes \mathbb{Q}$, and of abelian groups. Noting that $\mathbb{Q}$ is the direct limit of the system

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{4} \mathbb{Z} \longrightarrow \cdots$$

For a sphere S^m, one construction of the rationalization $S^m_{(0)}$ is the infinite mapping telescope of the sequence

$$S^m \xrightarrow{\underline{2}} S^m \xrightarrow{\underline{3}} S^m \xrightarrow{\underline{4}} S^m \longrightarrow \cdots$$

, where $\underline{k} : S^m \to S^m$ denotes a degree k map. Having constructed $S^m_{(0)}$, they can be used as building blocks, replacing S^m to construct $X_{(0)}$ for CW-complexes X.

We say that $f : X \to Y$ is a rational equivalence if $f_{(0)} : X_{(0)} \to Y_{(0)}$ is a homotopy equivalence, which turns out to be equivalent to saying that $f_* : \pi_*(X) \otimes \mathbb{Q} \cong \pi_*(Y) \otimes \mathbb{Q}$ is an isomorphism which is also equivalent to saying that $f_* : H_*(X;\mathbb{Q}) \cong H_*(Y;\mathbb{Q}) \otimes \mathbb{Q}$ is an isomorphism.

In terms of the representing spaces, the Chern isomorphism can be interpreted as follows. Using the fact that the only nontorsion homotopy group in S^{2n+1} is $\pi_{2n+1}(S^{2n+1}) \cong \mathbb{Z}$ we see that the natural map $S^{2n+1} \to K(Q, 2n+1)$ induces a rational isomorphism on homotopy groups after tensoring with $\mathbb{Q}$ so it is a rational equivlance. Also one can show that $U(n)_{(0)} \simeq \prod_{k=1^n} S^{2k-1}_{(0)}$. Indeed, supposing by induction that $U(n)_{(0)} \simeq \prod_{k=1^{n-1}} S^{2k-1}_{(0)}$. the clutching map in the bundle $U(n-1) \to U(n) \to S^{2n-1}$ is an element of $\pi_{2n-2}U(n-1)$ and by induction $\pi_{2n-2}U(n-1) \otimes \mathbb{Q} \cong \prod_{k=1^n} \pi_{2n-2}(S^{2k-1}_{(0)}) \otimes \mathbb{Q} = 0$. since the groups are all torsion, as noted above. Thus after rationalizing, the bundle splits to give $U(n)_{(0)} \simeq \prod_{k=1^n} S^{2k-1}_{(0)}$. Passing to the limit, we have $U_{(0)} \simeq \prod_{k=1^\infty} S^{2k-1}_{(0)}$. Looping and applying Bott periodicity gives $BU_{(0)} \times \mathbb{Q} \simeq \prod_{k=1^\infty} \Omega S^{2k-1}_{(0)} \cong \prod_{k=1^\infty} \Omega\big(K(\mathbb{Q}, 2k-1)\big)_{(0)} \cong \prod_{k=1^\infty} \big(K(\mathbb{Q}, 2k-2)\big)_{(0)}$. In other words the representing spaces for $K(\ )$ and $\prod_n H^{2n}(\ )$ become the same after rationalization. Thus $K(X) \otimes \mathbb{Q} \cong \hat{H}(X;\mathbb{Q})$.

9.4 Calculations and the K-theory Thom isomorphism

We refine Prop 9.1.1 to include the ring structure.

Proposition 9.4.1 *As rings,* $K(S^{2n}) \cong \Lambda(x)$.

Proof: The only part that is not immediate from the group calculation of $K(S^{2n})$ is that $x^2 = 0$. Since $K(S^{2n})$ is torsion free, $K(S^{2n}) \to K(S^{2n}) \otimes \mathbb{Q}$ is an injection, so $x^2 = 0$ can verified by applying Ch and noting that the corresponding statement holds in $\hat{H}(S^{2n};\mathbb{Q})$. □

Let $f_1 : S^2 = \mathbb{C}P^2 \to \mathbb{C}P^\infty = BU(1) \to BU$ be the canonical map. Let $f_n : S^{2n} \to BU$ denote the generator of $\pi_{2n}(BU)$ given inductively by $S^{2n} \xrightarrow{S^2 f_{2n-2}} S^2BU \xrightarrow{\beta} BU$ where β is the adjoint of $BU \cong \Omega^2 BU \times \mathbb{Z} \to \Omega^2 BU$. For connectivity reasons, f_{2n} lifts to a map $S^{2n} \to BU(n)$ which we also denote by f_{2n}. By Bott periodicity, the bundles $\{1, f^*_{2n}(\gamma_n)\}$ generate $K(S^{2n}) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Let $j_k : BU(k) \to BU(1)^k$ be the classifying map of the bundle $\gamma_1 \oplus \gamma_1 \oplus \ldots \oplus \gamma_1$.

Proposition 9.4.2 *The coefficient of* c_n *in* $\mathrm{Ch}(\gamma_n) \in \hat{H}\big(BU(n)\big)$ *is* $1/(n-1)!$.

Proof:

$$\begin{tikzcd}
K\big(BU(n)\big)\otimes\mathbb{Q} \arrow[r, "j_n^*"] \arrow[d, "\mathrm{Ch}_{BU(n)}"] & K\big(BU(1)^n\big)\otimes\mathbb{Q} \arrow[d, "\mathrm{Ch}_{BU(1)^n}"] \\
\hat{H}\big(BU(n);\mathbb{Q}\big) \arrow[r, "j_n^*"] & \hat{H}\big(BU(1)^n;\mathbb{Q}\big)
\end{tikzcd}$$

□

Then using naturality and the Newton polynomial we get

$$\begin{aligned}
j_n^*\big(\,\mathrm{Ch}(\gamma_n)\big) &= \mathrm{Ch}(\oplus_{k=1}^n \gamma_1^n) = \sum_{k=1}^n \exp(t_i) = \sum_{k=1}^n \frac{\psi_k(t_1,\ldots,t_n)}{k!} \\
&= \frac{ns_n(t_1,\ldots,t_n)}{n!} + \text{ other terms} = \frac{s_n(t_1,\ldots,t_n)}{(n-1)!} + \text{ other terms.}
\end{aligned}$$

Since, by definition, the image of $c_n \in H^{2n}\big(BU(n)\big)$ under the injection j^* is $s_n(t_1,\ldots,t_n)$, the result follows. □

Proposition 9.4.3 (Bott) *The image under* $\mathrm{Ch}_{S^{2n}}$ *of* $K(S^{2n})\otimes 1$ *is the subgroup* $\hat{H}(S^{2n};\mathbb{Z})$ *of* $\hat{H}(S^{2n};\mathbb{Q})$*. Also the top Chern class* $c_n\big(f_{2n}^*(\gamma_n)\big)$ *is* $(n-1)!$ *times a generator of* $H^{2n}(S^{2n})\cong\mathbb{Z}$*.*

Proof: Since $K(S^2)=K(\mathbb{C}P^1)$ is generated by line bundles, it is trivial that the Proposition holds for $n=1$. Suppose by induction that it holds for S^{2n-2}. Let $q: S^{2n-2}\times S^2 \to S^{2n}$ be the canonical map which pinches the $(2n-2)$-skeleton to a point. By construction (i.e. Bott periodicity) $\alpha(f_{n-1}\times f_1) = f_n\circ q : S^{2n-2}\times S^2 \to BU(n)$ where $\alpha: BU(n-1)\times BU(1)\to BU(n)$ is induced by direct sum.

$$\begin{tikzcd}
K\big(BU(n-1)\times BU(1)\big)\otimes\mathbb{Q} \arrow[r, "(f_{n-1}\times f_1)^*"] \arrow[d, "\mathrm{Ch}_{BU(n-1)\times BU(1)}"] & K(S^{2n-2}\times S^2)\otimes\mathbb{Q} \arrow[d, "\mathrm{Ch}_{S^{2n-2}\times S^2}"] & K(S^{2n})\otimes\mathbb{Q} \arrow[l, "q^*"'] \arrow[d, "\mathrm{Ch}_{S^{2n}}"] \\
\hat{H}\big(BU(n-1)\times BU(1);\mathbb{Q}\big) \arrow[r, "(f_{n-1}\times f_1)^*"] & \hat{H}(S^{2n-2}\times S^2;\mathbb{Q}) & \hat{H}(S^{2n};\mathbb{Q}) \arrow[l, "q^*"']
\end{tikzcd}$$

As in the proof of the preceding proposition, we calculate that the coefficient of $c_{n-1}\times c_1$ of $\mathrm{Ch}(\gamma_{n-1}\times\gamma_1)\in\hat{H}\big(BU(n-1)\times BU(1)\big)$ is $1/(n-2)!$. Writing simply f_k for the bundle $f_k^*(\gamma_k)$, the

Whitney product formula gives $c_n(f_{n-1} \times f_1) = \sum_{i=1}^{n} c_i(f_{n-1}) \times c_{n-i}(f_1) = c_{n-1}(f_{n-1}) \times c_1(f_1)$, which, by the induction hypothesis, is $(n-2)!$ times a generator of $H^{2n}(S^{2n-2} \times S^2)$. Therefore the left square shows that $\mathrm{Ch}(f_{n-1} \times f_1)$ is a generator of $H^{2n}(S^{2n-2} \times S^2; \mathbb{Z})$. Thus the right square shows that $\mathrm{Ch}(f_n)$ is a generator of $H(S^{2n}; \mathbb{Z})$. The diagram

$$\begin{tikzcd} K\big(BU(n)\big) \otimes \mathbb{Q} \arrow[r, "f_n^*"] \arrow[d, "\mathrm{Ch}_{BU(n)}"] & K(S^{2n}) \otimes \mathbb{Q} \arrow[d, "\mathrm{Ch}_{S^{2n}}"] \\ \hat{H}\big(BU(n); \mathbb{Q}\big) \arrow[r, "f_n^*"] & \hat{H}(S^{2n}; \mathbb{Q}) \end{tikzcd}$$

and Proposition 9.4.2 now shows that $c_n(f_n)$ is $(n-1)!$ times a generator. $\square$

We can now give the integral version of Prop. 9.3.1.

Corollary 9.4.4 *Let $x = \gamma_n^1 - 1 \in \tilde{K}(\mathbb{C}P^n)$, where γ_n^1 is the canonical line bundle over $\mathbb{C}P^n$. Then as rings, $K(\mathbb{C}P^n) \cong \mathbb{Z}[x]/(x^{n+1})$.*

Proof: We know that the results holds after tensoring with $\mathbb{Q}$. To see that it holds over the integers, suppose by induction that the theorem has been proved for $\mathbb{C}P^{n-1}$. Consider the cofibration sequence $\mathbb{C}P^{n-1} \xrightarrow{j} \mathbb{C}P^n \xrightarrow{q} S^{2n}$ which was used in the computation of the group struction of $K(\mathbb{C}P^n)$. To finish the proof we must show that it is not possible to write x^n as $x^n = \lambda y$ for some $y \in K^*(\mathbb{C}P^n)$ and some $\lambda \in \mathbb{Z}$ with $|\lambda| > 1$. Therefore assume to the contrary that there exist $y \in K^*(\mathbb{C}P^n)$ and $\lambda \in \mathbb{Z}$ with $|\lambda| > 1$ such that $x^n = \lambda y$. Then $\lambda j^*(y) = j^*(x^n) = 0$ and so $j^*(y) = 0$, since $K^*(\mathbb{C}P^{n-1})$ is torsion-free. Thus there exists $\bar{y} \in K(S^{2n})$ such that $q^*(\bar{y}) = y$. Using the diagram

$$\begin{tikzcd} K\big(\mathbb{C}P^n\big) \otimes \mathbb{Q} \arrow[r, "q^*"] \arrow[d, "\mathrm{Ch}_{\mathbb{C}P^n}"] & K(S^{2n}) \otimes \mathbb{Q} \arrow[d, "\mathrm{Ch}_{S^{2n}}"] \\ \hat{H}\big(\mathbb{C}P^n; \mathbb{Q}\big) \arrow[r, "f_n^*"] & \hat{H}(S^{2n}; \mathbb{Q}) \end{tikzcd}$$

we calculate that $\mathrm{Ch}(\bar{y})$ is $1/\lambda$ times a generator of $H^{2n}(S^{2n}; \mathbb{Z})$, contradicting Theorem 9.4.3. $\square$

Example 9.4.5 In the case $n = 1$ we could deduce the ring structure for $K(\mathbb{C}P^1) = K(S^2)$ without using the Chern character as follows. Write L for the line bundle γ_1^1. Since γ_1^1 is the pullback to $\mathbb{C}P^1$ of the universal bundle over $\mathbb{C}P^\infty$, its classifying map is the inclusion $\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty = G_1^{\mathbb{C}} = BU(1)$. Tensor products of bundles are induced by the maps $BU(m) \times BU(k) \to BU(mk)$ coming from the tensor product $U(m) \times U(k) \to U(mk)$. In the case of line bundles, this is the map $BU(1) \times BU(1) \to BU(1)$ coming from the group multiplication $U(1) \times U(1) \to U(1)$. Notice that the homomorphism $i_1 : U(1) \to U(2)$ and $i_2 : U(1) \to U(2)$ given by $i_1(A) := A \oplus 1$ and $i_2(A) := 1 \oplus A$ are homotopic. That is, define $H_t : U(1) \to U(2)$ by

$$H_t(A) := \begin{pmatrix} \cos(\pi t) & \sin(\pi t) \\ -\sin(\pi t) & \cos(\pi t) \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\pi t) & -\sin(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{pmatrix}.$$

In the case of $\mathbb{C}P^1 = S^2$ we can cover with only 2 charts and for L their unique transition function is the identity function $S^1 \to U(1) = S^1$. The transition function for $L \otimes L$ is $z \mapsto z^2$. Using our homotopy H_t we get, for all t, isomorphic bundles with transition functions $zH^tAz(H^t)^{-1}$. Comparing $t = 0$ to $t = 1$ this shows that 2-dimensional bundle with transition functions $\begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix}$ is homotopic to the one with transition functions $\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$. In other words $L^2 + 1 = 2L$, or equivalently $(L - 1)^2 = 0$. This gives the desired relation $x^2 = 0$ for $x = \gamma_1^1 - 1$.

It is much harder to give a direct proof of this sort for higher n, since the number of charts has grown and constructing appropriate homotopies which preserve the relation $g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\alpha\gamma}$ required by transition functions, is much more difficult.

Taking the limit of the results for $K(\mathbb{C}P^n)$ as $n \to \infty$ we get

Corollary 9.4.6 *$K(\mathbb{C}P^\infty) \cong \mathbb{Z}[[x]]$ as (ungraded) rings and $K^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[[x]] \otimes \mathbb{Z}[B, B^{-1}]$ as rings and $K(*)$-modules, where $|B| = 2$ and $|x| = 0$.*

Remark 9.4.7

In cohomology, $H^*(\mathbb{C}P^\infty) = \varprojlim H^*(\mathbb{C}P^n)$ and also in K-theory $K(\mathbb{C}P^\infty) = \varprojlim K(\mathbb{C}P^n)$, however we get $\mathbb{Z}[x]$ in the first case and $\mathbb{Z}[[x]]$ in the second. That is because in the case of cohomology the inverse limit is taken in the category of graded rings, while in K-theory we are discussing only the ungraded ring $K(X)$ and the inverse limit is taken the category of ungraded rings.

□

Let $\eta = p : E \to B$ be a k-dimensional complex vector bundle. Let $\epsilon := \epsilon^1$ denote the trivial 1-dimensional bundle over B. Form the projective bundle $\tilde{p} : P(\eta) \to B$ with fibres $\mathbb{C}P^{k-1}$ as discussed before Lemma 9.3.2. Applying Lemma 9.3.2 gives a canonical splitting

$$\tilde{p}^*(\eta) \cong \gamma_\eta \oplus \eta' \tag{9.4.1}$$

for some $(k-1)$-dimensional bundle η' over $P(\xi)$. Let L_η be the dual line bundle $L_\eta := \gamma_\eta *$. Then $L_\eta \otimes \gamma_\eta \cong \epsilon'$ where ϵ' denotes the trivial 1-dimensional bundle over $P(\eta)$. Tensoring equation 9.4.1 with L_η gives

$$L_\eta \otimes \tilde{p}^*(\eta) \cong \epsilon' \oplus (L_\eta \otimes \eta').$$

In particular, we get a canonical nowhere zero cross-section

$$s_\eta : P(\eta) \to E\big(L_\eta \otimes \tilde{p}^*(\eta)\big)$$

Let $\xi = p : E \to X$ be an n-dimensional complex vector bundle. Let $\epsilon := \epsilon^1$ denote the trivial 1-dimensional bundle over X. There is a pushout of bundles

$$\begin{tikzcd} S(\xi) \arrow[r] \arrow[d] & D(\xi) \arrow[d, "\bar{c}"] \\ P(\xi) \arrow[r, "j"] & P(\xi \oplus \epsilon) \end{tikzcd}$$

corresponding to the pushout

$$\begin{tikzcd} S^{2n-1} \arrow[r] \arrow[d] & D^{2n} \arrow[d, "c"] \\ \mathbb{C}P^{n-1} \arrow[r] & \mathbb{C}P^n \end{tikzcd}$$

on each fibre. More precisely, the map c is given by $c(tv) = [(tv_0, tv_1, \ldots tv_{2n-1}, \sqrt{1-t^2})]$ for $v = (v_0, v_1, \ldots, v_{2n-1}) \in S^{2n-1}$ and $t \in [0,1]$. If $\mu : U_\alpha \cap U_\beta \to U(n)$ is a transition function for ξ, then the action of $\mu(x)$ is length-preserving for all x, and so leaves t invariant. Therefore under the fibrewise quotient map given by c, the transition functions for $D(\xi)$ project to those of $P(\xi \oplus \epsilon)$ producing a bundle map $\bar{c}$.

The pushout diagram shows that $P(\xi \oplus \epsilon)/P(\xi) = D(\xi)/S(\xi) = \mathrm{Th}(\xi)$. Let $q : P(\xi \oplus \epsilon) \to \mathrm{Th}(\xi)$ denote the quotient map.

Applying the preceding discussion to the bundle $\xi \oplus \epsilon$ gives a canonical canonical nowhere zero cross-section

$$s : P(\xi \oplus \epsilon) \to E\big((L \otimes \tilde{p}^*(\xi \oplus \epsilon)\big) = E\Big(\big(L \otimes \tilde{p}^*(\xi)\big) \oplus L\Big),$$

where $L = \gamma^*_{\xi\oplus\epsilon}$. Composing with the projection $E\Big(\big(L\otimes\tilde{p}^*(\xi)\big)\oplus L\Big) \to E\big(L\otimes\tilde{p}^*(\xi)\big)$ gives a canonical cross-section s' of $L\otimes\tilde{p}^*(\xi)$, although s' may have zeros. The composition

$$P(\xi) \xrightarrow{j} P(\xi\oplus\epsilon) \xrightarrow{s'} E\big(L\otimes\tilde{p}^*(\xi)\big)$$

is the canonical nowhere zero cross section s_ξ of $L_\xi\otimes\bar{p}^*(\xi)$, where $\bar{p} : P(\xi) \to X$ is the projective bundle of ξ, and we have used the isomorphism $j^*(L) \cong L_\xi$. Applying Proposition 6.0.22 to s' gives a canonical element $u_\xi \in K\big(P(\xi\oplus\epsilon)/P(\xi)\big) = K\big(\mathrm{Th}(\xi)\big)$ such that $q^*(u_\xi) = \sum_{i=0}^{n}\lambda_i[L\otimes\tilde{p}^*(\xi)]$. We refer to u_ξ as the K-Theory Thom class of ξ. Since s and λ_i are natural, (i.e. commute with bundle maps,) so is u_ξ. That is, if $f : \alpha \to \beta$ is a bundle map then $f^*(u_\beta) = u_\alpha$.

Theorem 9.4.8 *(K-Theory Thom Isomorphism Theorem) Let $\xi = p : E \to X$ be an n-dimensional complex vector bundle. Then the map $x \mapsto p^*(x)u_\xi$ is an isomorphism $K^*(E) \cong \tilde{K}^*\big(\mathrm{Th}(\xi)\big)$.*

Note: $p^*(x)u_\xi$ means the relative product $K(E)\otimes K\big(\mathrm{Th}(\xi)\big) \to K\big(\mathrm{Th}(\xi)\big)$.

Proof:
As usual, it suffices to consider the special case of the universal bundle and the Splitting Principle shows that it suffices to consider the special case where ξ is a line bundle. In other words, it suffices to consider the special case where $\xi = \gamma^1$. Since $S(\gamma^1) \simeq S^\infty$ is contractible, we get $\mathrm{Th}(\gamma^1) \cong E(\gamma^1) \cong B(\gamma^1) = \mathbb{C}P^\infty$, so the statement to be proved becomes

$$\tilde{K}^*\big(\mathbb{C}P^\infty\big) \cong \ker\big(K^*(\mathbb{C}P^\infty) \to K^*(*)\big).$$

After applying Cor 9.4.4, this becomes the fact that multiplication by t is an isomorphism from $\mathbb{Z}[[t]]$ to the principal ideal (t). $\square$

Chapter 10

Adams Operations

A *cohomology operation* is a natural transformation $H^q(\ ;G_1) \to H^{q'}(\ ;G_2)$ for some q and q' and some groups G_1 and G_2. More generally, if $Y^*(\ )$ and $Y'^*(\ )$ are cohomology theories, then a (generalized) cohomology operation is a natural transformation $Y^q(\ ) \to Y'^{q'}(\ )$ for some q and q'.

If the functors $Y^q(\ )$ and $Y'^{q'}(\ )$ are representable, then any map between the representing objects yields a cohomology operation. Conversely, if there exist representing objects B and B' such $Y^q(X) = [X, B]$ and $Y'^{q'}(X) = [X, B']$, then any cohomology operation $\eta : Y^q(\ ) \to Y'^{q'}(\ )$ arises in this way, since $\eta_B(1_B) \in [B, B']$ gives the map $B \to B'$ which induces η. This is a special case of the corresponding statement for arbitrary representable functors known as "Yoneda's Lemma".

Usually one is interested in "stable cohomology operations" by which we mean a family of operations $\eta_q : Y^q(\ ) \to Y'^{q'+k}(\ )$ with the property that the diagrams

$$\begin{tikzcd}
\tilde{Y}^q(X) \arrow[r, "\eta_X"] \arrow[d, "\sigma", "\cong"'] & \tilde{Y'}^{q'+k}(X) \arrow[d, "\sigma", "\cong"'] \\
\tilde{Y}^{q+1}(SX) \arrow[r, "\eta_X"] & \tilde{Y'}^{q'+1+k}(SX)
\end{tikzcd}$$

commute for all q. Stable cohomology operations between representable theories correspond to morphisms of homotopy classes of spectra between the representing objects, where we will not go into the details of the precise definition of what a such a morphism is.

Thus in the case of K-theory, an operation would correspond to an element of $\tilde{K}^*(BU)$, while a stable operation would correspond to an element of the K^*-theory of the BU spectrum, defined using a direct limit.

The Adams operation, ψ^k is an unstable K-theory operation which is characterized by the facts that $\psi^k(L) = L^{\otimes k}$ whenever L is a line bundle, and $\psi^k(\xi + \eta) = \psi^k(\xi) + \psi^k(\eta)$. Its value on other bundles can be determined from this via the splitting principle, but it would be convenient to have a more direct method of computing $\psi^k(\xi)$.

If $\xi = L_1 \oplus \ldots \oplus L_n$ is a sum of line bundles, then $\psi^k\xi := (L_1)^{\otimes k} \oplus (L_n)^{\otimes k}$. If x_1, …, x_n are bases for one-dimensional vector spaces V_1, …, V_n, then the jth elementary symmetric polynomial $s_j(x_1, \ldots x_n)$ is a basis for the jth component $\Lambda_j(V_1 \oplus \ldots \oplus V_n)$ of the exterior algebra $\Lambda(V_1 \oplus \ldots \oplus V_n)$. Since $x_1^k + \ldots + x_n^k$ is a symmetric polynomial it can be written as $N(s_1, \ldots s_n)$ for some polynomial N. Therefore, for a line bundle ξ, the Adams operation $\psi^k(\xi)$ is given by $\psi^k(\xi) = N_k\big(\Lambda_1(\xi), \ldots, \Lambda_n(\xi)\big)$. The right hand side of this formula makes sense for any ξ and can be treated as an alternate definition of $\psi^k(\xi)$. (This was, in fact, Adams original definition.)

10.1 Properties of Adams Operations and Solution of the Hopf Invariant Problem

Theorem 10.1.1 *The Adams operations satisfy the following properties. For any $x, y \in K(X)$:*

1. $\psi^k(x + y) = \psi^k(x) + \psi^k(y)$
2. $\psi^k(xy) = \psi^k(x)\psi^k(y)$
3. $\psi^{km}(x) = \psi^k\big(\psi^m(x)\big) = \psi^m\big(\psi^k(x)\big)$
4. *If p is a prime then* $\psi^p(x) \equiv x^p \pmod p$

Proof:

Suppose $x, y \in K(X)$ and set $\bar{x} := c_1^{\mathbb{Q}}(x)$. Then ψ^k is determined by the conditions:

- $\mathrm{Ch}(\psi^k(x + y)) = \mathrm{Ch}(\psi^k(x)) + \mathrm{Ch}(\psi^k(y))$ for all x and y;
- $\mathrm{Ch}\big(\psi^k(x)\big) = \exp(k\bar{x})$ whenever x is a line bundle.

Parts (1) and (3) are immediate from the definition, and (2) follows from the fact that for line bundles $(xy)^{\otimes k} = x^{\otimes k} \otimes y^{\otimes k}$ together with the additivity (part (1)). Because of additivity together with the fact that $(a+b)^p \equiv a^p + b^p \pmod p$, it suffices to check part (4) for line bundles. The statement to be proved then becomes $\exp(p\bar{x}) \equiv \exp(\bar{x}) \pmod p$, which is trivial. □

Proposition 10.1.2 *The action of the Adams operations on $K(\mathbb{C}P^n) \cong \mathbb{Z}[x]/(x^{n+1})$ is determined by $\psi^k(x) = (x+1)^k - 1$.*

Proof:
We have $x = L - 1$ where $L = \gamma_n^1$. Also, $\psi^k(L) = L^k$, since L is a line bundle. Therefore

$$\psi^k(x) = \psi^k(L-1) = \psi^k(L) - \psi^k(1) = L^k - 1 = (x+1)^k - 1.$$

$\square$

Proposition 10.1.3 *Let y be a generator of $\tilde{K}(S^{2n})$. Then $\psi^k(y) = k^n y^k$.*

Proof:
Let $q : \mathbb{C}P^n \to S^{2n}$ by the map which pinches $\mathbb{C}P^{n-1}$ to a point. We calculated that $K(\mathbb{C}P^n) = \mathbb{Z}[x]/(x^{n+1})$. From our calculation of $K(\mathbb{C}P^n)$ as an abelian group (9.1.3) we know that $q^*(y) = x^n$. Therefore

$$q^*\big(\psi^k(y)\big) = \psi^k(x^n) = \big(\psi^k(x)\big)^n = \big((x+1)^k - 1\big)^n.$$

However $\big((x+1)^k - 1\big)^n = k^n x^n$ in $K(\mathbb{C}P^n)$ since $x^{n+1} = 0$. Therefore $q^*\big(\psi^k(y)\big) = k^n x^n = q^*(k^n y)$, which implies $\psi^k(y) = k^n y^k$, since q^* is injective. $\square$

Let $f : S^{2m-1} \to S^m$ and let X be the 2-cell complex with cells in degree m and $2m$ with f as the attaching map. Then as abelian groups,

$$H^q(X) = \begin{cases} 0 & \text{if q=0,m,2m;} \\ 0 & \text{otherwise.} \end{cases}$$

Let x and y be the generators in degree m and $2m$ respectively. Then $x^2 = \mu y$ for some μ and called the "Hopf invariant" of f. Only the absolute value of μ is significant, since the sign depends on the choice of generators. The Hopf maps $S^3 \to S^2$, $S^7 \to S^4$ and $S^{15} \to S^8$ are the attaching maps in the projective planes $\mathbb{C}P^2$, $\mathbb{H}P^2$ and $\mathbb{O}P^2$ of the complexes, quaternians, and octonians (Cayley numbers) respectively, They have Hopf invariant 1. The Hopf Invariant One Problem is to determine the set of integers m for which there exists a function $f : S^{2m-1} \to S^m$ having Hopf invariant 1 and solves several other problems such as

- For which m does S^{m-1} have an H-space structure
- For which m is there a continuous multiplication without zero divisors on $\mathbb{R}^m$
- For which m are S^{m-1} and $\mathbb{R}P^{m-1}$ parallelizable

- For which m is there an identity which in any ring expresses a product of two elements which are sums of m-squares as a sum of m-squares

Theorem 10.1.4 *There is exists an function $f : S^{2m-1} \to S^m$ having Hopf invariant 1 if and only if $m = 2$, 4, or 8.*

Proof:

Hopf's maps show that there do indeed exists functions having Hopf invariant 1 when $m = 2$, 4, or 8.

Suppose now that $f : S^{2m-1} \to S^m$ has Hopf invariant 1. It is easy to see that m must be even, since $x^2 = 0$ whenever x is a torsion-free cohomology class in odd degree. Write $m = 2n$. By Thm. 9.1.2, our knowledge of $H^*(X)$ tells us that $K(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ as groups. Choose $x \in K(X)$ such that $j^*(x)$ is a generator of $\tilde{K}(S^{2n})$. Then $j^*(x^2) = 0$ so x^2 is a multiple of the generator of $\operatorname{Im} q^*$. Applying the Chern character shows that this multiple is the same the corresponding multiple on cohomology — i.e. the Hopf invariant of f. Thus under our hypothesis, x^2 is a generator of $\operatorname{Im} q^*$.

Naturality gives $\psi^k(x^2) = k^{2n}x^2$ and $\psi^k(x) = k^n x + \lambda_k x^2$ for some integer λ_k, where $\lambda_k \equiv 1 \pmod{k}$ by Property (3) of Thm. 10.1.1. Let $a = \lambda_2$ and $b = \lambda_3$. Notice that Property (4) of Thm. 10.1.1 implies that $\lambda_2 \equiv 1 \pmod{2}$ Then

$$\psi^6(x) = \psi^3\big(\psi^2(x)\big) = \psi^3(2^n x + ax^2) = 6^n x + 2^n bx^2 + 3^{2n}ax^2$$

and also

$$\psi^6(x) = \psi^2\big(\psi^3(x)\big) = \psi^2(3^n x + bx^2) = 6^n x + 3^n ax^2 + 2^{2n}bx^2.$$

Therefore $2^n b + 3^{2n}a = 2^{2n}b + 3^n a$ or equivalently $2^n(2^n - 1)b = 3^n(3^n - 1)a$. Since $a = \lambda_2 \equiv 1 \pmod{2}$, this gives 2^n divides $3^n - 1$.

Let $\nu(n)$ denote the number of factors of 2 in $3^n - 1$. It is easy to see by induction that if n is odd then $\nu(n) = 1$ and if n is even then $\nu(n) > 1$. Furthermore, if $n = 2r$ is even, then $3^n = (2^{\nu(r)}u + 1)^2 = 2^{2\nu(r)} + 2^{\nu(r)+1}u + 1$ where u is odd. Therefore $\nu(2r) = \nu(r) + 1$ holds whenever $\nu(r) > 1$. It is easy to see that this implies that 2^n divides $3^n - 1$ happens only when $n = 1$, 2, or 4.

□

Bibliography

[1] F. Adams, “On the nonexistence of elements of Hopf invariant 1”; Ann. of Math. **72** (1960)

[2] F. Adams, “Vector Fields on Sphere”; Ann. of Math. **77** (1965); 20–104

[3] M. Atiyah, “K-theory”; (1964); Benjamin

[4] D. Husemoller, “Fiber Bundles”; Springer Graduate Texts in Mathematics; Springer (1970)

[5] M. Karoubi, “K-theory”;

[6] P. May, “Classifying Spaces and Fibrations”; Memoirs AMS **155** (1975)

[7] J. Milnor, “Morse Theory”; Annals of Math. Studies **51** (1969); Princeton University Press

[8] J. Milnor and J. Stasheff, Characteristic Clases”; Annals of Math. Studies **76** (1974); Princeton University Press; **155** (1975)

[9] A. Pressley and G. Segal, “Loop Groups”; Oxford Mathematical Monographs; (1986); Oxford University Press

[10] P. Selick, “Introduction to Homotopy Theory”; Fields Institute Monograph Series **9** (1997)