Complex Analysis

0. Preliminaries and Notation

For a complex number $z = x + iy$, we set $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$, its real and imaginary components. Any nonzero complex number $z = x + iy$ can be written uniquely in the polar form $z = re^{i\theta}$ where $r = |z| = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$. r is called the norm, modulus, or absolute value of z and θ is called the argument of z, written $\arg z$. Notice that $\arg z$ is well defined only up to multiples of 2π.

Euler's formulas $e^{iz} = \cos(z) + i\sin(z)$ and $e^{-iz} = \cos(z) - i\sin(z)$ give

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \qquad \text{and} \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

Complex numbers can sometimes be used an intermediate step to solve problems which are given entirely in terms of real numbers. In fact, if this were not true there would be little reason to define complex numbers.

Example. Let $n \geq 2$ be an integer. Show that $\prod_{k=1}^{n-1} \sin(\frac{k\pi}{n}) = \frac{n}{2^{n-1}}$.

Solution. Set $\omega := e^{\frac{2\pi i}{n}}$. Thus $\omega^{n/4} = e^{\pi i/2} = i$ and $\omega^n = 1$.

$$\sin\left(\frac{k\pi}{n}\right) = \frac{e^{k\pi i/n} - e^{k\pi i/n}}{2i} = \frac{\omega^{k/2} - \omega^{-k/2}}{2i} = \frac{\omega^k - 1}{2i\omega^{k/2}}$$

Therefore

$$\begin{aligned}\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) &= \prod_{k=1}^{n-1} \frac{\omega^k - 1}{2i\omega^{k/2}} = \frac{\prod_{k=1}^{n-1}(\omega^k - 1)}{2^{n-1}i^{n-1}\omega^{(1+2+\ldots+n-1)/2}} = \frac{\prod_{k=1}^{n-1}(\omega^k - 1)}{2^{n-1}i^{n-1}\omega^{n(n-1)/4}} \\ &= \frac{\prod_{k=1}^{n-1}(\omega^k - 1)}{2^{n-1}i^{n-1}(\omega^{n/4})^{n-1}} = \frac{\prod_{k=1}^{n-1}(\omega^k - 1)}{2^{n-1}i^{n-1}i^{n-1}} = \frac{\prod_{k=1}^{n-1}(\omega^k - 1)}{2^{n-1}(-1)^{n-1}}\end{aligned}$$

Since $(\omega^k)^n = 1$ for each k, each factor in the numerator satisfies the equation $(z+1)^n = 1$. In other words, they satisfy $z^n + nz^{n-1} + \binom{n}{2}z^{n-2} + \ldots + nz = 0$. Since none are zero, they satisfy $p(z) := z^{n-1} + nz^{n-2} + \binom{n}{2}z^{n-3} + \ldots + n = 0$. Therefore they are all the roots of the degree $n-1$ polynomial $p(z)$. In the general, the product of all the roots of a degree d polynomial is $(-1)^d$ times the constant term. Thus in this case, the numerator is $(-1)^{n-1}n$. $\square$

1. Complex Differentiation

$f : \mathbb{C} \to \mathbb{C}$.

We shall assume that f is defined on a domain D which is open and path connected (meaning that any two points in D can be joined by a path within D.) Using real and imaginary components we can write $z = x+iy$ and $f(z) = u(x,y)+iv(x,y)$ for real-valued functions u and v, and in this way, when convenient, we can regard f as a function $f(x,y) = \big(u(x,y), v(x,y)\big) : \mathbb{R}^2 \to \mathbb{R}^2$. For example, when $c = a+bi \in \mathbb{C}$ the function $z \mapsto cz : \mathbb{C} \to \mathbb{C}$ corresponds to the function $(x,y) \mapsto (ax - by, ay + bx) : \mathbb{R}^2 \to \mathbb{R}^2$ or equivalently, the function given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The (complex) derivative of f is defined by

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

if the limit exists.

The assumption that the limit exists implies that the same value is obtained if the limit is taken as z approaches z_0 along any line. Approaching along the x-axis gives

$$f'(x_0 + iy_0) = \lim_{x \to x_0} \frac{f(x+iy_0) - f(x_0+iy_0)}{(x+iy_0) - (x_0+iy_0)} = (\partial_x f)(x_0,y_0) = (\partial_x u + i\partial_x v)(x_0,y_0). \quad (1)$$

Similarly approaching along the y-axis gives

$$f'(x_0 + iy_0) = \frac{1}{i}(\partial_y u + i\partial_y v)(x_0,y_0). \quad (2)$$

Equating real and imaginary parts in (1) and (2) gives

$$\text{Cauchy-Riemann Equations:} \qquad \partial_x u = \partial_y v, \qquad \partial_x v = -\partial_y u$$

Conversely, suppose $f : \mathbb{C} \to \mathbb{C}$ is differentiable when regarded as a function $\mathbb{R}^2 \to \mathbb{R}^2$ and also satisfies the Cauchy-Riemann Equations. The Jacobian matrix for the derivative is given by

$$Df = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

which, upon substitution from the Cauchy-Riemann equations, becomes the matrix corresponding to $z \mapsto (u_x + iv_x)z$. It follows that f is complex differentiable with derivative $f'(z) = u_x + iv_x$.

Summing up, we have:

Theorem. *Let $f : \mathbb{C} \to \mathbb{C}$ be differentiable when regarded as a function from $\mathbb{R}^2 \to \mathbb{R}^2$. Then f is differentiable as a complex function if and only if it satisfies the Cauchy-Riemann equations.* $\square$

A function which is differentiable at every point in a domain D is called holomorphic on D.

In polar form, the CR-equations are as follows. Set $x = r\cos\theta$, $y = r\sin\theta$. Then

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta = \frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

and similarly

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

Example. Let $D = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$. Recall that $\arg z$ is well defined only up to multiples of 2π. Define $f : D \to \mathbb{C}$ by $f(z) = \log(|z|) + i\arg(z)$ where we choose the value of $\arg(z)$ which lies in $(-\pi, \pi)$.

$$\frac{\partial u}{\partial x} = \frac{\partial \log(r)}{\partial x} = \frac{1}{r}\frac{\partial r}{\partial x} = \frac{1}{r}\frac{x}{\sqrt{x^2+y^2}} = \frac{1}{r}\frac{x}{r} = \frac{x}{r^2}$$

and

$$\frac{\partial v}{\partial x} = \frac{\partial \theta}{\partial x} = \frac{\partial \theta}{\partial x} = \frac{\partial \tan^{-1}(y/x)}{\partial x} = \frac{(-y/x^2)}{1+(y/x)^2} = \frac{-y}{x^2+y^2} = \frac{-y}{r^2}$$

Similarly we can calculate $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ and verify that the Cauchy-Riemmann equations are satisfied. Therefore $f(z)$ is differentiable.

According to (1), $\frac{\partial f}{\partial z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ and so

$$\frac{df}{dz} = \frac{x - iy}{r^2} = \frac{1}{x+iy} = \frac{1}{z}$$

□

A matrix of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

corresponds to the linear transformation consisting of the composition of rotation by $\cos^{-1}\big(a/(a^2+b^2)\big)$ and scalar multiplication by $a^2 + b^2$. Thus the Jacobian matrix for Df corresponds to the composition of rotation by $\arg\big(f'(z)\big)$ and multiplication by $|f'(z)|$.

2. Complex Integration

For a differentiable parameterized curve $\gamma(t) = x(t) + iy(t) : [a, b] \to \mathbb{C}$ let $\gamma'(t) \in \mathbb{C}$ denote the derivative $\gamma'(t) := x'(t) + iy'(t)$. As discussed in MATB42, $\gamma'(t)$ gives the tangent vector to the curve $\gamma(t)$, corresponding to the velocity at time t of a point moving along curve with position $\gamma(t)$ at time t. Let $f : B \to \mathbb{C}$ be continuous where B is a domain containing the curve γ. (That is, $\gamma([a, b] \subset B$.) Assume that $\gamma'(t)$ is a continuous function of t. Define $\int_\gamma f(z)\,dz$ by

$$\int_\gamma f(z)\,dz := \int_a^b f\big(\gamma(t)\big)\gamma'(t)\,dt$$

where the right hand side is defined by taking integrals of the real and imaginary components. In other words,

$$\int_\gamma f(z)\,dz := \int_a^b \mathrm{Re}\Big(f\big(\gamma(t)\big)\gamma'(t)\Big)\,dt + i\int_a^b \mathrm{Im}\Big(f\big(\gamma(t)\big)\gamma'(t)\Big)\,dt.$$

More generally, for a piecewise differentiable curve γ, the integral can be defined by adding the integrals on the subintervals on which γ is differentiable. It is also possible to relax the condition that f be continuous, although we shall not need to consider such cases.

As in line integrals in MATB42, the sign of the answer depends upon the orientation of the curve γ which is determined by the given parameterization.

Example. Compute $\int_\gamma z\,dz$ where γ is the straight line joining 0 to $1 + i/2$.

Solution. Parameterize γ by $\gamma(t) = t + it/2$, $0 \le t \le 1$.

$$\int_\gamma z\,dz = \int_0^1 (t+it/2)d(t+it/2) = \int_0^1 (t+it/2)(1+i/2)\,dt = (1+i/2)^2\int_0^1 t\,dt = \frac{1}{2}(1+i/2)^2.$$

□

Example. Compute $\int_C z^2\,dz$ where C is the unit circle, oriented counterclockwise.

Solution. Parameterize C by $C(t) = \cos(t) + i\sin(t)$, $0 \le t \le 2\pi$.

$$\int_C z^2\,dz = \int_0^{2\pi} \big(\cos(t) + i\sin(t)\big)^2 d\big(\cos(t) + i\sin(t)\big)\,dt = \frac{\big(\cos(t) + i\sin(t)\big)^3}{3}\Bigg|_{t=0}^{t=2\pi} = 0.$$

□

Example. Compute $\int_C \frac{1}{z}\,dz$ where C is the circle of radius R, oriented counterclockwise.

Solution. Parameterize C by $C(t) = R\cos(t) + iR\sin(t)$, $0 \le t \le 2\pi$.

$$\int_C \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{R\cos(t) + iR\sin(t)}\big(-R\sin(t) + iR\cos(t)\big)\,dt = \int_0^{2\pi} i\,dt = 2\pi i.$$

The calculation could equivalently be expressed by writing $C(t) = Re^{it}$, $0 \le t \le 2\pi$, giving

$$\int_C \frac{1}{z}\,dz = \int_0^{2\pi} \frac{d(Re^{it})}{Re^{it}} = \int_0^{2\pi} i\,dt = 2\pi i.$$

Note in particular that the answer is independent of the radius of C. □

Proposition. *For $f = u + iv$,*

$$\int_\gamma f(z)\,dz = \int_\gamma \big(u(x,y)\,dx - v(x,y)\,dy\big) + i\int_\gamma \big(v(x,y)\,dx + u(x,y)\,dy\big)$$

Proof.

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_a^b \Big(u\big(x(t),y(t)\big) + iv\big(x(t),y(t)\big)\Big)\big(x'(t) + iy'(t)\big)\,dt \\
&= \int_a^b \Big(u\big(x(t),y(t)\big)x'(t) - v\big(x(t),y(t)\big)y'(t)\Big) \\
&\quad + i\int_a^b \Big(v\big(\big(x(t),y(t)\big)x'(t) + u\big(x(t),y(t)\big)y'(t)\Big) \\
&= \int_\gamma \big(u(x,y)\,dx - v(x,y)\,dy\big) + i\int_\gamma \big(v(x,y)\,dx + u(x,y)\,dy\big)
\end{aligned}$$

□

Recall (MATB42) that a differential form ω is called closed if $d\omega = 0$. Given $f = u+iv$, let $\omega = u(x,y)\,dx - v(x,y)\,dy + iv(x,y)\,dx + u(x,y)\,dy$. Then $d\omega = \big(-\partial_y u - \partial_x v + i(-\partial_y v + \partial_x u)\big)dx \wedge dy$. Therefore, if f is holomorphic, the Cauchy-Riemann equations imply that $d\omega = 0$.

Lemma. *Suppose $|f(z)| \le M$ for z on γ. Then $\left|\int_\gamma f(z)\,dz\right| \le M$ (arc length of γ).*

Proof. Let $\gamma(t)$, $a \le t \le b$ be a parameterization of γ.

$$(\text{arc length of } \gamma) = \int_\gamma 1\,ds = \int_a^b \|\gamma'(t)\|\,dt.$$

Therefore

$$\begin{aligned}
\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f\big(\gamma(t)\big)\gamma'(t)\,dt\right| \le \int_a^b \big|f\big(\gamma(t)\big)\gamma'(t)\big|\,dt \le \int_a^b M|\gamma'(t)|\,dt \\
= M\,(\text{arc length of } \gamma).
\end{aligned}$$

□

Theorem (Cauchy — Preliminary Version). *Let B be a closed region whose boundary ∂B consists of piecewise differentiable curves. Let $f(z)$ be holomorphic in a domain containing B. Suppose $f'(z)$ is continuous throughout B. Then $\int_{\partial B} f(z)\,dz = 0$.*

Proof. Given $f = u + iv$, let $\omega = u(x,y)\,dx - v(x,y)\,dy + iv(x,y)\,dx + u(x,y)\,dy$. Since f is holomorphic, $d\omega = 0$, as above. Since $f'(z)$ is continous, the partial derivatives of u and v are continuous. Therefore using Stokes' Theorem (MATB42) gives $\int_{\partial B} f(z)\,dz = \int_{\partial B} \omega \overset{(\text{Stokes' Thm.})}{=} \int_B d\omega = 0$. □

Note: The preceding theorem does not require that ∂B be connected. For example, if B is an annular-shaped region lying between an outer curve C_2 and an inner curve C_1 (each oriented in the same direction, say counterclockwise) then the theorem gives $\int_{C_2} f\,dz - \int_{C_1} f\,dz = 0$ so $\int_{C_2} f\,dz = \int_{C_1} f\,dz$. (As in Stokes' Theorem, the signs are determined by picking an orientation on D and using the induced orientations on C_2 and C_1.) If f is holomorphic not only on B but also on the interior of C_1 (i.e. the "hole" in the annular region B) then applying the theorem to that region gives the stronger statement that $\int_{C_1} f\,dz = 0$ and $\int_{C_2} f\,dz = 0$. Thus, in particular, if γ is any simple closed curve which goes once counterclockwise around the origin then the earlier example gives $\int_\gamma \frac{1}{z}\,dz = \int_C \frac{1}{z}\,dz = 2\pi i$.

The preceeding proof used Green's Theorem (special case of Stokes' theorem) so required the hypothesis that $f'(z)$ be continuous. The proof can be refined to eliminate this hypothesis.

Theorem (Cauchy-Goursat). *Let B be a closed region whose boundary ∂B consists of piecewise differentiable curves. Let $f(z)$ be holomorphic in a domain containing B. Then $\int_{\partial B} f(z)\,dz = 0$.*

Proof. As in the proof of Green's Theorem, it suffices to consider the case where B is a rectangle.

Bisect the sides of the rectangle B to subdivide it into four (conguent) subrectangles B_1, B_2, B_3, B_4 whose sidelengths are 1/2 the sidelengths of B. Then

$$\int_{\partial B} f(z)\,dz = \int_{\partial B_1} f(z)\,dz + \int_{\partial B_2} f(z)\,dz + \int_{\partial B_3} f(z)\,dz + \int_{\partial B_4} f(z)\,dz,$$

the integrals over the interior lines cancelling out.

The preceding equation implies that the average value of $\left|\int_{\partial B_j} f(z)\,dz\right|$ is at least $\left|\int_{\partial B} f(z)\,dz\right|/4$, so $\left|\int_{\partial B_j} f(z)\,dz\right| \ge \left|\int_{\partial B} f(z)\,dz\right|/4$ for (at least) one j.

Pick $B^{(1)}$ to be B_1, B_2, B_3, or B_4 such that $\left|\int_{\partial B^{(1)}} f(z)\,dz\right| \ge \left|\int_{\partial B} f(z)\,dz\right|/4$.

Applying the same procedure to $B^{(1)}$, pick $B^{(2)} \subset B^{(1)}$ such that the sidelengths of $B^{(2)}$ are half those of $B^{(1)}$ and $\left|\int_{\partial B^{(2)}} f(z)\,dz\right| \ge \left|\int_{\partial B^{(1)}} f(z)\,dz\right|/4$.

Continuing, for each n, find $B^{(n)}$ such that

$$B^{(n)} \subset B^{(n-1)} \subset \ldots \subset B^{(1)} \subset B$$

and the sidelengths $B^{(n)}$ are (sidelengths of $B/2^n$) and $\left|\int_{\partial B^{(n)}} f(z)\,dz\right| \ge \left|\int_{\partial B} f(z)\,dz\right|/4^n$.

Since $\lim_{n\to\infty} \operatorname{diam}(B^{(n)}) = 0$, by the Cantor Intersection Theorem (`MATB43`), $\cap_n B_n$ is a single point.

Let $w = \cap_n B_n$. Since f is differentiable at w, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left|\frac{f(z) - f(w)}{z - w} - f'(w)\right| \le \epsilon$$

whenever $0 < |z - w| < \delta$.

Equivalently
$$|f(z) - f(w) - f'(w)(z - w)| \leq \epsilon |z - w|$$
whenever $0 < |z - w| < \delta$.

Pick n such that $\mathrm{diam}(B_n) < \delta$. Then for all $z \in B_n$, $|z - w| < \delta$ and so

$$|f(z) - f(w) - f'(w)(z - w)| \leq \epsilon \ \mathrm{diam}(B_n) = \epsilon \ \mathrm{diam}(B)/2^n$$

for all $z \in B_n$.

Since 1 and $z - w$ are differentiable with continous derivatives, by the prelimary version of Cauchy's theorem $\int_{\partial B_n} f(w)\, dz = f(w) \int_{B_n} 1\, dz = 0$ and $\int_{\partial B_n} f'(w)(z-w)\, dz = f'(w) \int_{B_n} (z - w)\, dz = 0$.

Therefore

$$\int_{\partial B_n} f(z)\, dz = \int_{\partial B_n} \big(f(z) - f(w) - f'(w)(z - w)\big)\, dz.$$

Thus

$$\begin{aligned}\left| \int_{\partial B} f(z)\, dz \right| &\leq 4^n \left| \int_{\partial B_n} \big(f(z) - f(w) - f'(w)(z - w)\big)\, dz \right| \\ &= 2^n \epsilon \ \mathrm{diam}(B)(\text{Perimiter of } B_n) \\ &= 2^n \epsilon \ \mathrm{diam}(B)(\text{Perimiter of } B)/2^n = \epsilon \ \mathrm{diam}(B)(\text{Perimiter of } B)\end{aligned}$$

Since this is true for all $\epsilon > 0$, $\int_{\partial B} f(z)\, dz = 0$. □

Theorem (Cauchy Integral Formula). *Let f be holomorphic on a domain containing a simple closed counterclockwise-oriented curve γ together with its interior. Then for any z_0 in the interior of γ,*
$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z_0}\, dw.$$

Proof. By translation (i.e. the change of variable $\tilde{z} := z - z_0$), it suffices to consider the special case $z_0 = 0$. Let D be the interior of γ. The closure of D is $\bar{D} = D \cup \partial D = D \cup \gamma$, the union of D with its boundary. Set

$$F(w) := \begin{cases} \frac{f(w) - f(0)}{w} & \text{if } w \neq 0; \\ f'(0) & \text{if } w = 0. \end{cases}$$

Since $2\pi i f(0) = f(0) \int_\gamma \frac{1}{w}\, dw$ (earlier example) we need to show that $\int_\gamma F(w)\, dw = 0$.

It is clear that F is continuous throughout $\bar{D}$ and differentiable in the interior except possible at $w = 0$. (It is actually also differentiable at $w = 0$ but this is not so obvious and we do not need it.) Since $\bar{D}$ is closed and bounded (i.e. compact in the terminology of MATB43) continuity of F implies (MATB43) that F is bounded on $\bar{D}$. That is, there exists $M \in \mathbb{R}$ such that $|F(w)| \leq M$ for all $w \in \bar{D}$. Let C be a circle within D centred at 0, oriented counterclockwise. The preceding theorem implies that $\int_\gamma F(w)\, dw = \int_C F(w)\, dw$.

Parameterize C by $C(t) = R\cos(t) + iR\sin(t)$, $0 \leq t \leq 2\pi$, where R is the radius of C. Then $|C'(t)| = |-R\sin(t) + iR\cos(t)| = R$, so

$$\begin{aligned}\left|\int_\gamma F(w)\,dw\right| &= \left|\int_C F(w)\,dw\right| = \left|\int_0^{2\pi} F\big(C(t)\big)C'(t)\,dt\right| \leq \int_0^{2\pi} \left|F\big(C(t)\big)C'(t)\right| dt \\ &\leq \int_0^{2\pi} MR\,dt = 2\pi MR.\end{aligned}$$

Since this inequality holds for arbitrarily small R, $\int_\gamma F(w)\,dw = 0$ as desired. $\square$

The special case when $f(z) = 1$ says that if γ is any simple closed counterclockwise-oriented curve about z_0, then $\frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_0}\,dz = 1$. more generally, if γ is a closed curve with $z_0 \notin \gamma$, we define the *index*, or *winding number* of γ about z_0, denoted $I(\gamma, (z_0))$, by $I(\gamma, z_0) := \frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_0}\,dz$. It is always an integer, and represents the number of times γ circles around z_0, where curves oriented in the clockwise direction are considered to circle a negative number of times.

By translation, consider the special case $z_0 = 0$ in which case we write simply $I(\gamma)$ for $I(\gamma, 0)$. If we write $z = x + iy$, then

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}$$

so by our earlier Proposition,

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{1}{z}\,dz &= \frac{1}{2\pi i}\int_\gamma \left(\frac{x}{x^2+y^2}\,dx + \frac{y}{x^2+y^2}\,dy\right) + \frac{1}{2\pi}\int_\gamma \left(\frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\right) dy \\ &= \frac{1}{4\pi i}\int_\gamma \left(d(\log(x^2+y^2)\right) + \frac{1}{2\pi}\int_\gamma \left(\frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\right) dy \\ &= 0 + \frac{1}{2\pi}\int_\gamma \left(\frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy\right) \\ &= \frac{1}{2\pi}\int_\gamma \left(\frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\right) dy\end{aligned}$$

so this agrees with the MATB42 definition of winding number.

Another geometrical interpretation of $I(\gamma)$ is as follows. Start at some point w_0 on γ and let $\theta_0 = \arg w_0$, normalized to lie in $[0, 2\pi)$. As w moves around the curve γ, $\arg(w)$ changes continuously, returning to $\theta_0 + 2\pi n$ when we get back to w_0.
Claim: $I(\gamma) = n$.
Proof. Paramterize γ by $\gamma(\theta) = r(\theta)e^{i\theta}$, $0 \leq \theta \leq 2\pi n$. Then

$$\begin{aligned}&\frac{1}{2\pi i}\int_\gamma \frac{1}{z}\,dz = \frac{1}{2\pi i}\int_0^{2\pi n} \frac{1}{r(\theta)e^{i\theta}}\left(\frac{\partial r}{\partial\theta}e^{i\theta} + ir(\theta)e^{i\theta}\right) d\theta \\ &= \frac{1}{2\pi i}\int_0^{2\pi n} \frac{1}{r(\theta)}\frac{dr}{d\theta}d\theta + \frac{1}{2\pi}\int_0^{2\pi n} d\theta = \frac{1}{2\pi i}\log\big(r(\theta)\big)\Big|_{\theta=0}^{\theta=2\pi n} + \frac{1}{2\pi}\int_0^{2\pi n} d\theta = 0 + n = n.\end{aligned}$$

$\square$

We noted earlier that for holomorphic $f(z)$, the formula $\int_{\partial B} f(z)\,dz = 0$ does not require that ∂B be connected. We can similarly generalize Cauchy's Integral Formula so that it applies to cases where ∂B is not a single simple closed curve, but perhaps a disjoint union of curves.

Corollary. *Let B be a closed region whose boundary ∂B consists of piecewise differentiable curves. Let f be holomorphic in a domain containing B. Then for any z_0 in B,*

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial B} \frac{f(w)}{w - z_0}\,dw.$$

Proof. Pick a simple closed curve γ in B such that z_0 lies in the interior of γ and let $\tilde{B} = B - \{\text{interior of } \gamma\}$ and orient it in the counterclockwise direction. Then $g(w) := f(w)/(w - z_0)$ is holomorphic throughout $\tilde{B}$ so $\int_{\partial \tilde{B}} g(w)\,dw = 0$. The boundary of $\tilde{B}$ is the disjoint union of ∂B and γ, and so, taking orientation into account, we get

$$\int_{\partial B} \frac{f(w)}{w - z_0}\,dw = \int_{\gamma} \frac{f(w)}{w - z_0}\,dw = 2\pi i f(z_0).$$

$\square$

Notice that Cauchy's Integral Formula implies that for a holomorphic function f the values of f on γ completely determine the values of f at any point inside γ. In particular,

Corollary. *Let f be holomorphic on a domain containing the closed ball $B = B_R[z_0]$. Then*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta})\,d\theta.$$

Proof.

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial B} \frac{f(w)}{w - z_0}\,dw = \frac{1}{2\pi i} \int_{\partial B} \frac{f(z_0 + Re^{i\theta})}{Re^{i\theta}} iRe^{i\theta}\,d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta})\,d\theta.$$

$\square$

In other words, the value of f at the centre of a circle is the average of its values on the circle.

Corollary (Maximum Modulus Principle). *Let f be holomorphic on a domain containing a closed bounded set B. Then the maximum value M of $|f|$ occurs on ∂B and unless f is constant, $|f(z)| < M$ for all z in the interior of B.*

Proof. A continous function on a closed bounded set attains a maximum (MATB43). The average value of a function can equal its maximum value only if the function is constant. Therefore if $|f(z_0)| = M$ for some z_0 in the interior then $|f(z)| = M$ in some neighbourhood of z_0. Since our domains are connected, this implies $|f(z)| = M$ for all z in the domain. Therefore it suffices to show

Lemma. *If $|f(z)|$ is constant then $f(z)$ is constant.*

Proof. Let $f = u + iv$ and suppose $|f(z)| = M$ for all z in the domain. If $M = 0$ then $f = 0$ so suppose $M > 0$. Differentiating $M^2 = |f(z)|^2 = u^2 + v^2$ gives $2uu_x + 2vv_x = 0$ and $2uu_y + 2vv_y = 0$. After substituting from the Cauchy-Riemann equations we get $uu_x + vu_y = 0$ and $uu_y - vu_x = 0$. Therefore $u^2u_x = -uvu_y = -v^2u_x$ and $v^2u_y = -uvu_x = -u^2u_y$. Thus $(u^2 + v^2)u_x = (u^2 + v^2)u_y = 0$. Since $u^2 + v^2 = M^2 \neq 0$, we get $u_x = u_y = 0$ so u is constant, and similarly v is constant. $\square$

Theorem (Cauchy Integral Formula for higher derivatives). *Let f be holomorphic on a domain containing simple closed counterclockwise-oriented curve γ together with its interior. Suppose that f is holomorphic throughout the interior of γ. Then f is infinitely differentiable on the interior of γ with and for any z_0 in the interior of γ the nth derivative is given by*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w - z_0)^{n+1}}\, dw.$$

Proof. Suppose by induction that the theorem is known for $f^{(n)}$, the case $n = 0$ being the preceding theorem. Thus to prove the theorem it suffices to prove the following lemma which completes the induction.

Lemma. *Let f be holomorphic on a domain containing a simple closed counterclockwise-oriented curve γ. For z in the complement of γ define functions $F_n(z)$ by $F_n(z) := \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{n+1}}\, dw$. Then F_n is differentiable on the complement of γ with derivative equal to F_{n+1}.*

For future reference, notice that the Lemma is stronger than needed in the proof of the theorem in that the definition of the functions F_n and the proof that they are differentiable do not require f to be defined throughout the interior of γ and the conclusion also applies to points outside γ.

Proof. Let z_0 lie in the complement of γ.

$$\begin{aligned}
\frac{F_n(z) - F_n(z_0)}{z - z_0} - \frac{(n+1)!}{2\pi i}\int_\gamma \frac{f(w)}{(w - z_0)^{n+2}}\, dw \\
&= \frac{1}{z - z_0}\left(\frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w - z)^{n+1}}\, dw - \frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w - z_0)^{n+1}}\, dw\right) \\
&\quad - \frac{(n+1)!}{2\pi i}\int_\gamma \frac{f(w)}{(w - z_0)^{n+2}}\, dw \\
&= \frac{n!}{2\pi i}\int_\gamma \left(\frac{f(w)}{(w - z)^{n+1}} - \frac{f(w)}{(w - z_0)^{n+1}} - (n+1)\frac{f(w)}{(w - z_0)^{n+1}}\right) dw
\end{aligned}$$

For any a and b, applying $\frac{(x^{k+1}-y^{k+1})}{x-y} = \sum_{j=0}^{k} x^j y^{k-j}$ with $x = 1/a$ and $y = 1/b$ gives

$$\begin{aligned}
\frac{1}{b-a}\left(\frac{1}{a^{n+1}} - \frac{1}{b^{n+1}}\right) - (n+1)\frac{1}{b^{n+2}} &= \frac{1}{b-a}\left(\frac{1}{a} - \frac{1}{b}\right)\sum_{j=0}^{n} \frac{1}{a^j b^{n-j}} - (n+1)\frac{1}{b^{n+2}} \\
&= \frac{1}{ab}\sum_{j=0}^{n} \frac{1}{a^j b^{n-j}} - (n+1)\frac{1}{b^{n+2}} \\
&= \sum_{j=0}^{n} \frac{1}{a^{j+1} b^{n-j+1}} - (n+1)\frac{1}{b^{n+2}} \\
&= \sum_{j=0}^{n} \left(\frac{1}{a^{j+1} b^{n-j+1}} - \frac{1}{b^{n+2}}\right) \\
&= \sum_{j=0}^{n} \frac{1}{b^{n-j+1}}\left(\frac{1}{a^{j+1}} - \frac{1}{b^{j+1}}\right) \\
&= \sum_{j=0}^{n} \frac{1}{b^{n-j+1}}\left(\frac{1}{a} - \frac{1}{b}\right)\sum_{i=0}^{j} \frac{1}{a^i b^{j-i}} \\
&= (b-a)\sum_{j=0}^{n}\sum_{i=0}^{j} \frac{1}{a^{i+1} b^{n-i+2}}
\end{aligned}$$

Setting $a = w - z$ and $b = w - z_0$ gives

$$\int_\gamma \left(\frac{f(w)}{(w-z)^{n+1}} - \frac{f(w)}{(w-z_0)^{n+1}} - (n+1)\frac{f(w)}{(w-z_0)^{n+1}}\right) dw = (z - z_0)\int_\gamma g(z,w)\, dw$$

where

$$g(z) = \sum_{j=0}^{n}\sum_{i=0}^{j} \frac{f(w)}{(w-z)^{i+1}(w-z_0)^{n-i+2}}.$$

Thus

$$\left|\frac{F_n(z) - F_n(z_0)}{z - z_0} - \frac{(n+1)!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{n+2}}\, dw\right| \le \frac{n!\,|z - z_0|}{2\pi i}\left|\int_\gamma g(z,w)\, dw\right|$$

Since $g(z,w)$ is continuous on the closed bounded set γ it is bounded (MATB43), and so $\int_\gamma g(z,w)\, dw$ is bounded, showing $\lim_{z\to z_0} |z - z_0| \left|\int_\gamma g(z,w)\, dw\right| = 0$. Hence

$$F_n'(z_0) - \frac{(n+1)!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{n+2}}\, dw = 0.$$

$\square$

Corollary (Liouville). *If f is holomorphic throughout $\mathbb{C}$ and f is bounded then f is constant.*

Proof. Suppose $|f(z)| < M$ for all z. Then given $z_0 \in \mathbb{C}$, letting applying the Cauchy Integral formula for the first derivative to the circle of radius R about z_0 gives $|f'(z_0)| = \left|\frac{1}{2\pi i}\int_{\partial B}\frac{f(z_0+Re^{i\theta})}{(Re^{i\theta})^2} iRe^{i\theta}\,d\theta\right| \leq \frac{1}{2\pi}\int_0^{2\pi}\frac{M}{R^2}R\,d\theta = M/R$. Since R is arbitrary, this implies $f'(z_0) = 0$ for all z_0 and so f is constant. $\square$

The converse to Cauchy's theorem also holds.

Theorem (Moreira). *Let $f : D \to \mathbb{C}$ be a continuous function such that $\int_\gamma f\,dz = 0$ for any closed curve $\gamma \subset D$. Then there exists a holomorphic function $g(z)$ on D such that $g'(z) = f(z)$. In particular, f is holomorphic.*

Proof. Let $f = u+iv$, and set $\omega_1 := u(x,y)\,dx - v(x,y)\,dy$ and $\omega_2 := v(x,y)\,dx + u(x,y)\,dy$. Since $\int_\gamma f\,dz = \int_\gamma \omega_1 + i\int_\gamma \omega_2$, it follows that $\int_\gamma \omega_1 = 0$ and $\int_\gamma \omega_2 = 0$ for any closed curve $\gamma \subset D$. According to a theorem from `MATB42`, this means that there exists "potential functions" for ω_1 and ω_2 in D. That is, there exist functions $g_1(x,y)$ and $g_2(x,y)$ such that $dg_1 = \omega_1$ and $dg_2 = \omega_2$. In other words, $\partial_x g_1 = u$, $\partial_y g_2 = -v$, $\partial_x g_2 = v$, and $\partial_y g_2 = u$.

Set $g(z) := g_1 + ig_2$. Since f is continuous, the partial derivatives of the components of $g(z)$ are continuous, so g is differentiable as a function from $\mathbb{R}^2 \to \mathbb{R}^2$. Furthermore, the formulas above show that the Cauchy-Riemann equations are satisfied, so $g(z)$ is holomorphic in D.

The derivative of $g(z)$ is given by $g'(z) = \partial_x g_1 + i\partial_x g_2 = u + iv = f$. Since $g(z)$ is holomorphic in D, according to the Cauchy Integral Theorem for Derivatives, its derivative $f(z)$ is also holomorphic in D. $\square$

Corollary. *If $f(z)$ is holomorphic throughout a simply connected domain D then there exists a holomorphic $g(z)$ on D such that $g'(z) = f(z)$.*

Proof. Since $f(z)$ is holomorphic and D is simply connected, integrals of $f(z)$ over curves in D are independent of the path. In particular, $\int_\gamma f(z) = 0$ for any closed curve γ in D. $\square$

This implies that a holomorphic function has "local" antiderviatives as follows.

Corollary. *If $f(z)$ is holomorphic throughout a domain D then for each $z_0 \in D$ there exists and open neighbourhood of z_0 throughout which f has an antidervative.*

Proof. Since our domains are assumed to be open, there exists an open ball B about z_0 within D. Since B is simply connected, applying the previous Corollary shows that $f(z)$ has an antiderivative in B. $\square$

Notice that this does **not** say that there is a single function $g(z)$ defined throughout all of the domain of $f(z)$ such that $g'(z) = f(z)$.

3. Power Series Expansions

Recall the following from MATB43.

1) A sequence of functions $f_n(z)$ is said to converge uniformly to $f(z)$ if $\forall\ \epsilon > 0\ \exists N$ (independent of z) such that $\forall\ n \geq N\ |f(z) - f_n(z)| < \epsilon$ for all z in the domain.
2) If f_n continuous on D and f_n converges uniformly to f then f is continuous on D and $\lim_{n\to\infty} \int_D f_n(z)\,dz = \int_D f(z)\,dz$.
3) To any power series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ there is an associated radius of convergence $R \in [0, \infty]$ such that $f(z)$ converges (absolutely) for $|z| < R$ and diverges for $|z| > R$. Within its radius of convergence, $f(z)$ is differentiable and integrable (on bounded regions) with $f'(z) = \sum_{n=0}^{\infty} c_n n z^{n-1}$ and anti-derivative $\sum_{n=0}^{\infty} \frac{c_n}{n+1} z^{n+1}$ and the radius of convergence of the differentiated and integrated series are R.
4) A continuous function $f(z)$ on a closed bounded domain has both a minimum and a maximum. In particular $|f(z)|$ is bounded.

For any curve γ our assumption that $\gamma'(t)$ is continous implies (by (4)) that it is bounded. It follows that:

Proposition. *If $f_n(z)$ converges uniformly to $f(z)$ on a domain including the image of γ then $f_n\big(\gamma(t)\big)\gamma'(t)$ converges uniformly to $f\big(\gamma(t)\big)\gamma'(t)$ and so applying (1) gives $\lim_{n\to\infty} \int_\gamma f_n(z)\,dz = \int_\gamma f(z)\,dz$.* □

For differentiable functions of a real variable, even if $f_n(x)$ converges uniformly to $f(x)$ in the vicinity of a point a, it need not be true that $f(x)$ is differentiable at a. However for functions of a complex variable, we can use the fact that corresponding statement for integration (the preceding proposition) to deduce the result for differentiation.

Theorem. *Suppose $f_n(z)$ is a sequence of functions which converge uniformly to $f(z)$ on D. If $f_n(z)$ is holomorphic on D for all n, then $f(z)$ is holomorphic on D.*

Proof. For evey closed curve γ in D,

$$\int_\gamma f(z) = \lim_{n\to\infty} \int_\gamma f_n(z) = \lim_{n\to\infty} 0 = 0.$$

Therefore $f(z)$ is holomorphic in D by Moreira's theorem, □

From (3) we know that a function which is representable by a power series is differentiable within its radius of convergence. For complex differentiable functions we show that the converse is true.

Observe that $\frac{1}{a-x} = \frac{1}{a}\frac{1}{1-x/a} = \frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{x}{a}\right)^n = \sum_{n=0}^{\infty}\frac{x^n}{a^{n+1}}$ within its radius of convergence $|x| < a$.

Suppose that $f(z)$ is differentiable on a domain D and let z_0 belong to D. Since D is open, the closed ball $B_r[z_0] := \{z \in \mathbb{C} \mid |z - z_0| \leq r\}$ is contained in D for sufficiently small r. Let C be the circle $C := \partial B_r[z_0] := \{z \in \mathbb{C} \mid |z - z_0| = r\}$, oriented counterclockwise. Cauchy says $f(z) = \int_C \frac{f(w)}{w-z}\,dw$ for all z in the open ball

$B_r(z_0) := \{z \in \mathbb{C} \mid |z - z_0| < r\}$. For simplicity we will use translation to consider expansions about $z_0 = 0$. Then for $z \in B_r(0)$ we have $|z| < |w|$ for any $w \in C$ and so

$$f(z) = \int_C \frac{f(w)}{w-z}\,dw = \int_C \sum_{n=0}^{\infty} \frac{f(w)z^n}{w^{n+1}}\,dw = \sum_{n=0}^{\infty} z^n \int_C \frac{f(w)}{w^{n+1}}\,dw = \sum_{n=0}^{\infty} z^n \frac{f^{(n)}(0)}{n!}.$$

Thus within C, the Taylor series of $f(z)$ converges to $f(z)$. Since r was arbitrary (subject to the condition that $B_r(z_0)$ be contained in D) the Taylor series expansion is valid for any z whose distance to z_0 is less than the distance of z_0 to the boundary of D. In particular, the radius of convergence of the Taylor series of $f(z)$ about z_0 is at least as large as the distance from z_0 to the boundary of D.

It is also clear that the power series expansion of $f(z)$ about any point is unique. That is, if $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ is any series which converges to $f(z_0)$ in some neighbourhood of z_0 then by successively differentiating and evaluating at $z = z_0$ we find that $a_n = f^{(n)}(z_0)/n!$ so the series is the Taylor series of $f(z)$.

A function with the property that at every point in its domain, the Taylor series of the function converges to the function within some sufficiently small radius of the point is called analytic. Therefore for complex functions, "differentiable" and "analytic" are equivalent. This is in contract to functions of a real varable where if $f(x)$ is differentiable:

a) f not need be differentiable more than once so the Taylor series of f might not be defined.
b) Even if f is infinitely differentiable, the Taylor series of f need not converge to f.

Since the $\sum_{n=0}^{\infty} z^n \frac{f^{(n)}(0)}{n!}$ converges within C, its radius of convergence is at least r. Thus within C we can define a function $g(z)$ by the convergent series

$$g(z) := \sum_{n=0}^{\infty} z^{n+1} \frac{f^{(n)}(0)}{(n+1)!}$$

and it will have the property that it is differentiable within C with $g'(z) = f(z)$.

In summary, unlike functions of a real variable, complex differentiable functions have the following properties:

Theorem.
1) $f(z)$ is analytic at every point in its domain
2) f is locally the derivative of some complex function. That is, at every point of the domain of f there a function $g(z)$ defined in some neighbourhood of that point such that $g'(z) = f(z)$.
3) If $f_n(z)$ is a sequence of differentiable functions which converge uniformly to $f(z)$ on D then $f(z)$ is differentiable on D. □

Notice that (2) does **not** say that there is a single function $g(z)$ defined throughout all of the domain of $f(z)$ such that $g'(z) = f(z)$.

Laurent Series

Consider a function $f(z)$ which is holomorphic throughout an annular region

$$A := \{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}$$

where $0 \leq R_1 < R_2 \leq \infty$. By translation, assume $z_0 = 0$. Given $p \in A$, choose concentric circles C_1 and C_2 about 0 such that $R_1 <$ radius of $C_1 < |p| <$ radius of $C_2 < R_2$ and orient them in the counterclockwise direction. By the Corollary to Cauchy's Integral Formula,

$$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w - z}\, dw - \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w - z}\, dw.$$

Set $f_1(z) := \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}\, dw$ and $f_2(z) := \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z}\, dw$ so that $f(z) = f_2(z) - f_1(z)$. According to the lemma in the proof of Cauchy's Integral Formula, the functions f_1 and f_2 are defined and analytic on the complement of C_1 and C_2 respectively. Hence f_2 is analytic throughout the interior of C_2, and given in this region by the power series expansion $f_2(z) = \sum_{n=0}^{\infty} a_n z^n$ where $a_n = f_2^{(n)}(0)/n! = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w^{n+1}}\, dw$, which holds, in particular, at the point p.

For the analogous statement concerning the function f_1 make the change of variable $\zeta := 1/z$. That is, set $\tilde{f}_1(\zeta) := f_1(1/\zeta) = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-(1/\zeta)}\, dw$. Letting $\tilde{C}_1$ be the circle (with counterclockwise orientation) whose radius is the reciprocal of the radius of C_1, the change of variable $w := 1/\omega$ gives

$$\tilde{f}_1(\zeta) = \frac{1}{2\pi i}\int_{\tilde{C}_1} \frac{f\left(\frac{1}{\omega}\right)}{(1/w) - (1/\zeta)}\left(-\frac{1}{\omega^2}\right) d\omega = \frac{1}{2\pi i}\int_{\tilde{C}_1} \frac{\zeta}{\omega}\frac{f\left(\frac{1}{\omega}\right)}{\omega - \zeta}\, d\omega = \frac{\zeta}{2\pi i}\int_{\tilde{C}_1} \frac{g(\omega)}{\omega - \zeta}\, d\omega$$

where $g(\omega) = \frac{f\left(\frac{1}{\omega}\right)}{\omega}$. As with f_2, we use the fact that $h(\zeta) := \frac{1}{2\pi i}\int_{\tilde{C}_1} \frac{g(\omega)}{\omega-\zeta}\, d\omega$ is analytic in the interior of $\tilde{C}_1$ to obtain the expansion $h(\zeta) = \sum_{n=0}^{\infty} c_n \zeta^n$ where $c_n = \frac{1}{2\pi i}\int_{\tilde{C}_1} \frac{g(\omega)}{\omega^{n+1}}\, d\omega$. Therefore $\tilde{f}_1(\zeta) = \zeta h(\zeta) = \sum_{n=0}^{\infty} c_n \zeta^{n+1} = -\sum_{n=1}^{\infty} b_n \zeta^n$ where

$$\begin{aligned} b_n = -c_{n-1} &= -\frac{1}{2\pi i}\int_{\tilde{C}_1} \frac{g(\omega)}{\omega^n}\, d\omega = -\frac{1}{2\pi i}\int_{\tilde{C}_1} \frac{f\left(\frac{1}{\omega}\right)}{\omega^{n+1}}\, d\omega \\ &= -\frac{1}{2\pi i}\int_{C_1} f(w) w^{n+1}\left(-\frac{1}{w^2}\right) dw = \frac{1}{2\pi i}\int_{C_1} f(w) w^{n-1}\, dw. \end{aligned}$$

The expansion $\tilde{f}_1(\zeta) = -\sum_{n=1}^{\infty} b_n \zeta^n$ is valid in the interior of $\tilde{C}_1$ which contains $1/p$. Therefore $f_1(z) = \tilde{f}_1(\zeta) = -\sum_{n=1}^{\infty} b_n \zeta^n = -\sum_{n=1}^{\infty} b_n / z^n$ holds at p.

Using the Corollary to Cauchy's theorem, if we choose a curve γ about p which lies within the annulus and give it the counterclockwise orientation, we can write our coefficents as $a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w^{n+1}}\, dw$ and $b_n = \frac{1}{2\pi i}\int_\gamma f(w) w^{n-1}\, dw$.

To summarize,

Theorem. *Let $f(z)$ be holomorphic in an annular region $R_1 < |z - z_0| < R_2$ where $0 \leq R_1 < R_2 \leq \infty$. Then*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

where $a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w - z_0)^{n+1}}\, dw$ and $b_n = \frac{1}{2\pi i} \int_\gamma f(w)(w - z_0)^{n-1}\, dw$ for any curve γ within the annulus such that p lies in the interior of γ. □

This is called the Laurent expansion of $f(z)$ in the region. If $f(w)$ is holomorphic within the entire ball of radius R_2 about z_0, then the b_n's are all 0 and the Laurent expansion of $f(z)$ reduces to its Taylor expansion.

4. Analytic Continuation

Theorem. *Suppose that $f(z)$ are $g(z)$ are analytic within a domain D. For any $z_0 \in D$, if there exists a sequence of points $z_n \neq z_0$ with $z_n \to z$, such $f(z_n) = g(z_n)$ for all n then $f(z) = g(z)$ for all z in D.*

Note: Recall that our definition of "domain" was an open path connected set.

Proof. By substraction, it suffices to consider the special case where $g(z) = 0$. Let d be the distance from z_0 to the boundary of D. Let $f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$ be the Taylor series expansion of $f(z)$, which, as noted earlier, converges to $f(z)$ throughout $B := B_d(z_0)$. Since $f(z)$ is differentiable at z_0 it is continuous so $f(z_0) = \lim_{n\to\infty} f(z_n) = 0$. Therefore substituting into the power series gives $a_0 = 0$. Write $f(z) = (z - z_0)h(z)$ where

$$h(z) = \begin{cases} f(z)/(z - z_0) & \text{if } z \neq z_0; \\ f'(z_0) & \text{if } z = 0. \end{cases}$$

The series $\sum_{k=0}^{\infty} a_{k+1}(z - z_0)^n$ converges to $h(z)$ for $z \neq z_0$, so by continuity it also converges to $g(z) = f'(z_0)$ when $z = z_0$. Since $z_n \neq z_0$, $h(z_n) = f(z_n)/(z_n - z_0) = 0$ so substituting into the power series gives $a_1 = 0$. Proceeding, we inductively conclude that $a_k = 0$ for all k and therefore $f(z) = 0$ throughout B.

Now let p be an arbitrary point in D. Let $X = \{z \in D \mid f(z) = g(z)\}$. Since we assumed that D is path connected, there exists a path $\gamma : [0, 1] \to D$ joining z_0 to p. Set $\tilde{t} := \sup\{t \in [0, 1] \mid \gamma(t) \in X\}$. Since the definition of supremum implies that there is a sequence of points in X converging to $\gamma(\tilde{t})$, the preceding shows that X contains an open ball about $\gamma(\tilde{t})$. This can happen only if $\tilde{t} = 1$, and so p lies in X. □

An analytic function g which extends an analytic f to a larger domain is called an analytic continuation of f. That is, if $f(z)$ and $g(z)$ are analytic functions with $D :=$ (domain of f) $\subset \tilde{D} :=$ (domain of g) and $f(z) = g(z)$ for all $z \in D$ then g is called an analytic continuation of f to $\tilde{D}$.

The preceding theorem implies that any two analytic continuations $g_1(z)$, $g_2(z)$ of $f(z)$ to the same domain $\tilde{D}$ are equal. However this does **not** imply that analytic continuations of $f(z)$ to **different** domains must agree on the intersection of those domains. In other words it might be possible that $f(z)$ has analytic continuations $g_1(z)$ defined on $D_1 \supset D$ and $g_2(z)$ defined on $D_2 \supset D$ for which there exists $p \in D_1 \cap D_2$ for which $g_1(p) \neq g_2(p)$. This can happen only if there is no path in the intersection $D_1 \cap D_2$ joining p to a point in D. Indeed, if $D_1 \cap D_2$ contains a path γ joining p to a point q in D, we can form an open connected subset $\tilde{D}$ of $D_1 \cap D_2$ containing γ. Then $\tilde{D}$ is a domain and since by hypothesis $g_1(z)$ and $g_2(z)$ agree in a neighbourhood of q (where both are given by $f(z)$) applying the preceding to the domain $\tilde{D}$ shows that $g_1(z)$ and $g_2(z)$ agree on $\tilde{D}$ and in particular at p.

Example. Let $D_1 = \mathbb{C} - \{(x, 0) \mid x \leq 0\}$ and $D_2 = \mathbb{C} - \{(0, y) \mid y \leq 0\}$. Recall that $\arg z$ is well defined only up to multiples of 2π. Define $g_1 : D_1 \to \mathbb{C}$ by $g_1(z) = \log(|z|) + i \arg(z)$ where we choose the value of $\arg(z)$ which lies in $(-\pi, \pi)$. Define $g_2 : D_2 \to \mathbb{C}$ by $g_2(z) = \log(|z|) + i \arg(z)$ where we choose the value of $\arg(z)$ which lies in $(-\pi/2, 3\pi/2)$. We showed earlier that g_1 and g_2 are differentiable with $g_1'(z) = g_2'(z) = 1/z$.

Since $z = |z|e^{i \arg z}$ by definition, $e^{g_1(z)} = e^{\log(|z|)}e^{i \arg z} = z$. Similarly $e^{g_2(z)} = z$. Therefore each of g_1 and g_2 might deserve the name $\log(z)$. They are called "branches" of the logarithm function. If we let

$$D = \text{open 1st quadrant} = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0 \text{ and } \operatorname{Im} z > 0\}$$

and define $f : D \to \mathbb{C}$ by $f(z) = \log(|z|) + i \arg(z)$ where we choose the value of $\arg(z)$ which lies in $(0, \pi/2)$, then each of g_1 and g_2 are analytic continuations of $f(z)$. However although the points in the 3rd quadrant are in $D_1 \cap D_2$, the values of $g_1(z)$ and $g_2(z)$ on these points differ by 2π.

Proposition. *Let R be the radius of convergence of $f(z) = \sum_{n=0}^{\infty} a_n(z-p)^n$ and suppose $0 < R < \infty$. Then there must be at least one point q with $|q - p| = R$ such that f cannot be analytically continued to any domain containing q.*

Proof. Let $B = B_R(p)$ be the open ball of radius R about p, and let ∂B be the boundary circle of B. Suppose that for each point $q \in \partial B$, there exists an analytic continuation $g_q(z)$ of $f(z)$ to a domain D_q containing q. Let $\hat{D} = \cup_{q \in \partial B} D_q$. For any q_1, q_2 on the boundary, applying the preceding uniqueness theorem to $D_{q_1} \cap D_{q_2}$ shows that $g_{q_1}(z)$ and $g_{q_2}(z)$ agree on $D_{q_1} \cap D_{q_2}$. Therefore the functions $\{g_q(z)\}_{q \in \partial B}$ piece together to produce a well defined function on $g(z)$ on $\hat{D}$. The function g_z is differentiable since in the neighbourhood of each point of $\hat{D}$ it equals some differentiable function $g_q(z)$. Since $\hat{D}$ is open and ∂B is closed, the distance d of ∂B to the boundary of $\hat{D}$ is positive, which means that the distance $R+d$ from p to the boundary of $\hat{D}$ is greater than R. But according to our earlier discussion (Section 3), the radius of convergence of the Taylor series of $g(z)$ about p (which is the same as $f(z)$) is at least as large as the distance of p to the boundary of $\hat{D}$, contradicting the definition of R. □

Let $f(z)$ be analytic throughout D. At any point p in D we can expand $f(z)$ into a power series $f(z) = a_n(z-p)^n$ whose radius of convergence R is at least as large as the distance d from p to the boundary of D (where R and d depend upon p). If there is a point p in D at which $d < R$, then we can create an analytic continuation $g(z)$ of $f(z)$ to $D \cup B_R(p)$ by defining

$$g(z) := \begin{cases} f(z) & \text{if } z \in D; \\ a_n(z-p)^n & \text{if } z \in B_R(p). \end{cases}.$$

By applying this procedure to more points in the extended domain one might be able to extend the domain still further.

Recall (MATB42):

a) two curves γ_0, γ_1, from p to q in a subset X of $\mathbb{R}^n$ are called homotopic in X, written $\gamma_0 \simeq \gamma_1$, if one can be continuously deformed into the other within the subset X. More precisely, $\gamma_0 \simeq \gamma_1$ if there exists a continuous function $H : [0,1] \times [0,1] \to X$ such that $H(0,t) = \gamma_0(t)$, $H(1,t) = \gamma_1(t)$ and $H(s,0) = p$ for all s and $H(s,1) = q$ for all q. In other words, the family of curves defined by $\gamma_s(t) := H(s,t)$ interpolates continuously from γ_0 to γ_1.
b) a connected subset X is called simply connected if any two curves in X with the same endpoints are homotopic in X.

For subsets X of $\mathbb{R}^2$, this is equivalent to saying that X is simply connected iff for every closed curve γ lying in X, the interior of γ also lies in X.

Theorem (Monodromy). *Let $f(z)$ be analytic on a domain D. Let X be a simply connected domain containing D. Suppose that for every curve $\gamma \in X$ there is an analytic continuation of $f(z)$ to some domain containing γ. Then there exists a unique analytic continuation of $f(z)$ to X.*

Proof. If the analytic continuation exists, it is unique by an earlier theorem.

Pick a point $p \in D$. Given $q \in X$, choose a path γ_0 joining p to q. By hypothesis, there exists an analytic continuation $g_0(z)$ to some domain D_0 containing γ_0. We wish to show that $g_0(q)$ is independent of the choices involved. It is clear from the uniqueness theorem that the same value of $g_0(q)$ would be obtained if we choose a different domain containing γ_0, but what happens if we choose a different path from p to q.

Suppose that γ_1 is another path joining p to q. Since X is simply connected, there exists a homotopy $H : [0,1] \times [0,1] \to X$ from γ_0 to γ_1 and set $\gamma_s(t) := H(s,t)$. Let d be the distance of $\operatorname{Im} H$ to the boundary of X. $d > 0$, since $\operatorname{Im} H$ is compact (closed and bounded) and X is open. Suppose h is some point in H and suppose $k_h(z)$ is any analytic continuation of $f(z)$ to some domain containing h. The straight line joining h to any point whose distance to h is less than d lies in X, and therefore by hypothesis there is an analytic continuation of $f(z)$ to some domain containing that line. By the preceding proposition, this means that the radius of convergence of the Taylor series of $k_h(z)$ about h must be at least d.

Let $S = \{s \in [0,1] \mid \exists$ an analytic continuation of $f(z)$ to some domain containing $\gamma_s\}$ and let $\hat{s} = \sup S$. We show that $\hat{s} = 1$. Choose $s_0 \in S$ such that $\hat{s} - s_0 < d$. Let $g_{s_0}(z)$ be an analytic continuation of $f(z)$ to a domain D_{s_0} containing γ_{s_0}. According to the preceding discussion, for each $h \in \gamma_{s_0}$, the radius of convergence of the Taylor series of $g_{s_0}(z)$ about h is at least d, so $f(z)$ has an analytic continuation $g_h(z)$ to $D_{s_0} \cup B_d(h)$. According to the uniqueness theorem, these functions (for various $h \in \gamma_{s_0}$) agree whenever their domains overlap, so they piece together to produce a well defined analytic continuation $\hat{g}(z)$ on $\hat{D} := D_{s_0} \cup \bigcup_{h \in \gamma_{s_0}} B_d(h)$. Since $s_0 + d > \hat{s}$, unless $\hat{s} = 1$, $\hat{D}$ contains γ_s for some $s > \hat{s}$, contradicting the definition of $\hat{s}$. Therefore $\hat{s} = 1$ and $\hat{g}(z)$ is an analytic continuation of $f(z)$ to a domain containing all of H and in particular contains both γ_0 and γ_1. Therefore the value $g_0(q)$ obtained using the path γ_0 equals the value $g_1(q)$ obtained using the path γ_1.

The conclusion is that there is a well defined extension of $f(z)$ to a function $g(z)$ defined on X where for any $q \in X$, $g(q)$ is defined by chosing any path γ from p to q, and setting $g(q) := g_\gamma(q)$ where $g_\gamma(z)$ is any analytic continuation of $f(z)$ to a domain containing $\gamma(z)$. The resulting function $g(z)$ is differentiable at each point q since it equals some differentiable function in the neighbourhood of q. $\square$

Theorem (Schwarz Reflection Principle). *Let D be a domain which is symmetrical about the x-axis. (i.e. $z \in D$ if and only if $\bar{z} \in D$.) Let $I = D \cap x-$axis. Let $U = \{z \in D \mid |\operatorname{Im} z > 0\}$ (the upper half of D) and let $\hat{U} = U \cup I$. Let $f : \hat{U} \to \mathbb{C}$ be a continuous function such that the restriction $f\big|_U$ is holomorphic and the restriction $f\big|_I$ is real-valued. Then f has a holomorphic extension to D.*

Proof. Extend f to $D - \hat{U}$ by setting $f(z) := \overline{f(\bar{z}}$ for $z \in D - \hat{U}$. Since f is continuous on $\hat{U}$, by symmetry $f\big|_{D-\hat{U}}$ has a continuous extension to I and since $f(p) = \overline{f(p)}$ for $p \in I$, the symmetry in the definition of f implies that this extension agrees with $f(I)$. Thus f is continous on D. By Moreira's theorem, it suffices to show that $\int_\gamma f(z) = 0$ for any closed curve $\gamma \subset D$. For a curve which is entirely contained in U, this is clear, since $f(z)$ is holomorphic on U and for a curve contained entirely in the refection $D - \hat{U}$ of U it is also clear, by symmetry. Therefore consider a curve γ which intersects I. We may write γ as a union of curves each of which lies entirely in the upper half plane or the lower half plane so this reduces the problem to showing that $\int_\gamma f(z) = 0$ for a curve γ which lies entirely in one the two half-planes. By slightly perturbing the portion of γ running along the x-axis, such a curve can be approximately as closely as desired by a curve $\tilde{\gamma}$ which lies entirely within U or $D - \hat{U}$. Since, as noted above, $\int_{\tilde{\gamma}} f(z)\,dz = 0$, and we can choose the $\tilde{\gamma}$ so as to make the difference $|\int_{\tilde{\gamma}} f(z)\,dz - \int_\gamma f(z)\,dz|$ arbitrarily small, it follows that $\int_\gamma f(z)\,dz = 0$. $\square$

5. Residues

Proposition. *Let γ be a simple closed counterclockwise-oriented curve about a point p. Then* $\int_\gamma (w-p)^n\,dw = \begin{cases} 2\pi i & \text{if } n=-1; \\ 0 & \text{otherwise.} \end{cases}$

Proof. If $n \geq 0$, $(z-p)^n$ is holomorphic everywhere so the integral is 0 by Cauchy's theorem. According to Cauchy's Integral Formula for higher derivatives,

$$\int_\gamma \frac{1}{(w-p)^m}\,dw = \frac{2\pi i}{m!} g^{(m-1)}(p),$$

where $g(z) = 1$, and applying this with $m := -n$ gives the result when $n < 0$. $\square$

Let $f(z)$ be homorphic in an annular region $0 < |z-p| \leq R$ about some point p. Let $B := B_R[p]$ be the closed ball of radius R about p and let $B' = B - \{p\}$ be the punctured ball obtained by removing the point p. Let $f(z) = \sum_{n=0}^\infty a_n(z-p)^n + \sum_{n=1}^\infty \frac{b_n}{(z-p)^n}$ be the Laurent expansion of $f(z)$ in B. Then applying the proposition gives

$$\frac{1}{2\pi i}\int_{\partial B} f(z)\,dz = \frac{1}{2\pi i}\sum_{n=-\infty}^{\infty} c_n \int_{\partial B} (z-p)^n\,dz = c_{-1}.$$

The coefficient c_{-1} in the Laurent expansion of $f(z)$ about p is called the *Residue* of f and p, written $\operatorname{Res}_p f(z)$.

Notice that if γ is any simple closed counterclockwise-oriented curve about a point p and $f(z)$ is holomorphic in a domain containing γ together with all of its interior except possibly p then for any circle C about p contained in γ, applying Cauchy's Theorem to the region between γ and C gives $\frac{1}{2\pi i}\int_\gamma f(z) = \frac{1}{2\pi i}\int_C f(z)$, so the answer is again given by $\operatorname{Res}_p f(z)$.

Consider now a simple closed counterclockwise-oriented curve γ and a function $f(z)$ which is holomorphic on a domain containing γ together will all of its interior except possibly a finite number of points $p_1, p_2, \ldots, p_k$. Carve the interior of γ into subregions $B_1, B_2, \ldots, B_k$ where B_j contains p_j but none of the other p's. Then, assuming all the curves are given the counterclockwise orientation, $\int_\gamma f(z)\,dz = \sum_{j=0}^k \int_{\partial B_j} f(z)\,dz$, since the integrals over the extra curves in the RHS introduced by the division into subregions each appear twice with opposite directions and cancel out. Therefore

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \frac{1}{2\pi i}\sum_{j=0}^{k}\int_{\partial B_j} f(z)\,dz = \sum_{j=0}^{k} \operatorname{Res}_{p_j} f(z).$$

Suppose $f(z)$ is a function which is holomorphic in a punctured neighbourhood of a point p. (That is, there is a ball $B = B_R(p)$ about p such that $f(z)$ is holomorphic on the punctured ball $B - \{p\}$.) Then p is called an "isolated singularity" of f. Let $f(z) = \sum_{n=0}^\infty a_n(z-p)^n + \sum_{n=1}^\infty \frac{b_n}{(z-p)^n}$ be the Laurent expansion of $f(z)$ in the neighbourhood of an isolated singular p. If $b_n = 0$ for all n then p is called a "removable singularity" of f.

If there exists an integer N such that $b_n = 0$ for all $n > N$, then p is called a "pole" of f of order k, where k is the largest integer for which $b_k \neq 0$. A pole of order 1 is called a "simple pole". If p is not a pole of f (i.e. $b_n \neq 0$ for infinitely many n), then p is called an "essential singularity" of f. If f is holomorphic on a region A everywhere except for a finite number of islated singularties none of which are essential singularities, then f is called "meromorphic" on A. Note that the uniqueness theorem in the analytic continuation section implies that the zeros of a nonconstant meromorphic function are isolated.

Our previous result can be stated as

Residue Theorem. *Let γ be a simple closed curve counterclockwise-oriented curve and let $f(z)$ be holomorphic, aside from isolated singularties, on domain D containing γ and its interior, with no singularities on γ. Then*

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \sum_{j=0}^{k} \operatorname{Res}_{p_j} f(z)$$

where p_1, p_2, …, p_k are the singularities of f within the interior of γ. □

Proposition. *Let p be an isolated singularity of $f(z)$. Then p is a removable singularity if and only if any of the following conditions hold (in which case all must hold):*
1) $f(z)$ is bounded in a deleted neighbourhood of p.
2) $\lim_{z\to p} f(z)$ exists
3) $\lim_{z\to p}(z-p)f(z) = 0$

Proof. If p is a removable singularity then in some deleted neighbourhood of p, $f(z) = \sum_{n=0}^{\infty} a_n(z-p)^n$. In this case we can extend $f(z)$ to an analytic function in the neighbourhood of p by setting $f(p) := a_0$. It follows that all three conditions are satisfied. Conversely, if either conditions (1) or (2) holds then obviously so does (3), so it suffices to show that condition (3) implies that the singularity is removable. Let C_r denote the circle of radius r around p. $b_k = \frac{1}{2\pi i}\int_{C_r} f(w)(w-p)^{k-1}\,dw$. Given $\epsilon > 0$, condition 3 says that there exists r such that $|z-p|\,|f(z)| < \epsilon$ whenever $|z-p| \leq r$. In particular, if z lies on C_r then $r|f(z)| < \epsilon$. Choosing a smaller r if necessary, we may assume $r < 1$. Therefore $|b_k| \leq \frac{1}{2\pi}\int_{C_r} r^{k-1}\epsilon/r = \frac{1}{2\pi}(2\pi r r^{k-1}\epsilon/r) = r^{k-1}\epsilon \leq \epsilon$ for every $\epsilon > 0$ and so $b_k = 0$. □

Computation of residues can be complicated, but there are some tricks which handle many common situations. If $f(z)$ has a removable singularity at p then $\lim_{z\to p} f(z) = a_0$ exists and the domain of f can be extended to include p by setting $f(p) := a_0$. Conversely, it is clear from the Laurent expansion that if $\lim_{z\to p} f(z)$ exists then the singularity must removable with the limit equalling a_0. If the singularity of f at p is a pole of order k then $(z-p)^k f(z)$ has a removable singularity at p. In particular, if $k = 1$ then $\lim_{z\to p}(z-p)f(z)$ exists and equals the residue $\operatorname{Res}_p f(z)$ and conversely if $\lim_{z\to p}(z-p)f(z)$ exists then $k = 1$ and $\operatorname{Res}_p f(z) = \lim_{z\to p}(z-p)f(z)$.

As a consequence we have

Proposition. *Let $f(z) = g(z)/h(z)$ where $g(z)$ and $h(z)$ are holomorphic at p with $h(p) = 0$ and $h'(p) \neq 0$. Then $\operatorname{Res}_p f(z) = g(p)/h'(p)$.*

Proof. $h'(p) = \lim_{z\to p} \frac{h(z)-h(p)}{z-p} = \lim_{z\to p} \frac{h(z)}{z-p}$ and so $\lim_{z\to p} \frac{z-p}{h(z)} = \frac{1}{h'(p)}$. Thus

$$\lim_{z\to p}(z-p)f(z) = \lim_{z\to p} g(z)\frac{z-p}{h(z)} = \frac{g(p)}{h'(p)}.$$

As above, the existence of the limit shows $\operatorname{Res}_p f(z) = \lim_{z\to p}(z-p)f(z) = g(p)/h'(p)$. ☐

Corollary.

a) If f has a zero of order k at p, then $\operatorname{Res}_p\big(f'(z)/f(z)\big) = k$.

b) If f has a pole of order k at p, then $\operatorname{Res}_p\big(f'(z)/f(z)\big) = -k$.

Proof. Use the convention that a pole of order k can also be called a zero of order $-k$. With this convention, let m be the order of the zero of f at p, where m may be either positive or negative. In a punctured neighbourhood of p, set $g(z) := f'(z)/(z-p)^{m-1}$ and $h(z) := f(z)/(z-p)^{m-1}$. Since $f(z)$ has a zero of order m at p, $f'(z)$ has a zero of order $m-1$ at p, and so both $g(z)$ and $h(z)$ have removable singularites at p and thus have extensions to holomorphic functions at p. Therefore the proposition applies to give

$$\begin{aligned}
\operatorname{Res}_p\big(f'(z)/f(z)\big) &= \operatorname{Res}_p\big(g(z)/h(z)\big) = g(p)/h'(p) \\
&= \lim_{z\to p} \frac{f'(z)/(z-p)^{m-1}}{f'(z)/(z-p)^{m-1} - (m-1)f(z)/(z-p)^m} \\
&= \lim_{z\to p} \frac{f'(z)}{f'(z) - (m-1)f(z)/(z-p)} \\
&= \lim_{z\to p} \frac{(z-p)f'(z)/f(z)}{(z-p)f'(z)/f(z) - (m-1)} \\
&= \frac{m}{m-(m-1)} = m
\end{aligned}$$

using that $\lim_{z-p} = \frac{(z-p)f'(z)}{f(z)}$ is the order of the zero at p. ☐

Corollary (Argument Principle). *Let γ be a simple closed counterclockwise-oriented curve and let $f(z)$ be meromorphic on a domain containing γ with no zeros or poles on γ. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \text{total number of zeros of } f(z) \text{ within } \gamma$$

where the zeros are counted with multiplicity and poles are consider to be zeros of negative multiplicity. ☐

Since making the change of variable $w = f(z)$ gives $\int_{f\circ\gamma} \frac{1}{w}\,dw = \int_\gamma \frac{f'(z)}{f(z)}\,dz$ the argument principle can be rewritten in terms of the index $I(f\circ\gamma)$ of the curve $f\circ\gamma$ about 0 as follows:

Corollary (Argument Principle). *Let γ be a simple closed counterclockwise-oriented curve and let $f(z)$ be meromorphic on a domain containing γ with no zeros or poles on γ. Then*

$$I(f \circ \gamma) = \text{total number of zeros of } f(z) \text{ within } \gamma$$

where the zeros are counted with multiplicity and poles are consider to be zeros of negative multiplicity. □

Example. How many zeros does the function $f(z) = z^6 + 6z + 10$ have in the first quadrant?

Solution. Let C be the portion of a large circle $|z| = R$ which lies in the first quadrant. Let γ be the curve consisting of the line segment $L_1 := [0, R]$ on the x-axis followed by the curve C from 0 to R, followed by the line segment $L_2 := [iR, 0]$ on the y-axis from iR to 0.

The restriction of $f(z)$ to L_1 is $f(x) = x^6 + 6x + 10$ which is a positive real number for all x and thus $\arg\big(f(z)\big)$ remains constant at 0 along L_1.

For large R, the value of $f(z)$ is approximately the same as the value of z^6 and in particular, the change in the argument of $f(z)$ along C is the same as the change in the argument of z^6 on C. Over a complete circle, $\arg(z^6)$ changes from 0 to 12π so on the quarter circle C it changes from 0 to 3π.

The restriction of $f(z)$ to L_2 is $f(iy) = (-y^6 + 10) + 6iy$, from which we see that, along L_2, $f(z)$ begins in the second quadrant, slightly above the x-axis, ($\arg(f(z)$ slightly less than 3π,) moves into the first quadrant as we pass through $y = \sqrt[6]{10}$ finishing at $(10, 0)$ on the x-axis. Thus along L_2, $\arg(z)$ decreases from approximately 3π to 2π. Putting it all together, $\arg\big(f(z)\big)$ stays at 0 along L_1, increases from 0 to slighly less than 3π along C and then decreases by around π, going from from approximately 3π to 2π along L_2.

Hence, for large R, the index $I(f \circ \gamma)$ is 1 and so $f(z)$ has 1 zero in the first quadrant. □

Example. How many zeros does the function $f(z) = z^3 + 5z^2 + 8z + 6$ have in the first quadrant?

Solution. Again let C be the portion of a large circle $|z| = R$ which lies in the first quadrant and let γ be the curve consisting of the line segment $L_1 := [0, R]$ on the x-axis followed by the curve C from 0 to R, followed by the line segment $L_2 := [iR, 0]$ on the y-axis from iR to 0.

The restriction of $f(z)$ to L_1 is $f(x) = x^3 + 5x^2 + 8x + 6$ which is a positive real number for all x and thus $\arg\big(f(z)\big)$ remains constant at 0 along L_1.

For large R, the value of $f(z)$ is approximately the same as the value of z^3 and in particular, the change in the argument of $f(z)$ along C is the same as the change in the argument of z^3 on C. Over a complete circle, $\arg(z^6)$ changes from 0 to 6π so on the quarter circle C it changes from 0 to $3\pi/2$.

The restriction of $f(z)$ to L_2 is $f(iy) = u(y) + iv(y)$ where $u(y) = -y^2 + 6$ and $v(y) = -y^3 + 8y$. Along L_2, $\arg\big(f(z)\big) = \tan^{-1}\left(\frac{v}{u}\right)$. $u(y) > 0$ for $y < \sqrt{6/5}$ and $u(y) < 0$ for $y > \sqrt{6/5}$. $v(y) > 0$ for $0 < y < \sqrt{8}$ and $v(y) < 0$ for $y > \sqrt{8}$. Therefore as we traverse L_2, $\arg\big(f(z)\big)$ begins in the third quadrant (at approximately $3\pi/2$), moves into the second quadrant as we pass $y = \sqrt{6/5}$, and then into the first quadrant as we pass $y = \sqrt{8}$, finishing at 0.

Hence, for large R, the index $I(f \circ \gamma)$ is 0 and so $f(z)$ has no zeros in the first quadrant. $\square$

In some cases, determination of the number of zeros can sometimes be done more easily by comparison with a function whose number of zeros in known, according to the following corollary of the Argument Principle.

Corollary (Rouché's Theorem). *Let γ be a simple closed counterclockwise-oriented curve. Let $f(z)$ and $g(z)$ be holomorphic on a domain containing γ and its interior, with no zeros or poles on γ. Suppose that $|f(z) - g(z)| < |f(z)|$ for all z on γ. Then*

$$\text{total number of zeros of } f(z) \text{ within } \gamma = \text{total number of zeros of } g(z) \text{ within } \gamma$$

where the zeros are counted with multiplicity.

Proof. Set $h(z) = g(z)/f(z)$. The hypothesis implies that $|h(z) - 1| < 1$ for all z on γ. Therefore the curve $h \circ \gamma$ is contained within the open disk of radius 1 about 1 and in particular does not contain 0 in its interior. Thus $I(h \circ \gamma) = 0$. Applying the Argument Principle gives

$$0 = \int_\gamma \frac{h'(z)}{h(z)}\, dz = \int_\gamma \frac{f(z)g'(z) - g(z)f'(z)}{\big((f(z)\big)^2} \frac{f(z)}{g(z)}\, dz = \int_\gamma \frac{g'(z)}{g(z)}\, dz - \int_\gamma \frac{f'(z)}{f(z)}\, dz$$

so another application of the Argument Principle gives that the result $\square$

Example. How many zeros does the function $k(z) = e^z - 4z^7$ have inside the unit circle?

Solution. Let $f(z) = -4z^7$. Then $k(z) - f(z) = e^z$. If z lies on the unit circle then $|k(z) - f(z)| = |e^z| = |e^{\mathrm{Re}(z)}| \le e$. However for z on the unit circle $|f(z)| = |-4z^7| = 4|z|^7 = 4 > e$. Thus $|f(z) - k(z)| < |f(z)|$ for all z on the unit circle. It follows that the number of zeros within the unit circle of $k(z)$ is the same as that of $f(z)$, which is 7. $\square$

Theorem. *Let p be a zero of order k of a nonconstant holomorphic function $f(z)$. Then there exist an open neighbourhood U of p such that $f(U)$ is an open neighbourhood of 0 and every point in $f(U)$ aside from 0 has precisely k-preimages under f.*

Proof. Since $f(z)$ is nonconstant, neither $f(z)$ nor $f'(z)$ is identically zero. Therefore we can choose a closed ball B in the domain of $f(z)$ containing p but no other zeros of f and no zeros of f' except possibly p. Let M be the minimum value of the restriction of $|f(z)|$ to ∂B. $M > 0$ by choice of B. Set $U := f^{-1}\big(B_M(0)\big) \cap \text{Interior of } B$.

For $q \in B_M(0)$ define $g(q)$ by $g(q) := \frac{1}{2\pi i} \int_{\partial B} \frac{f'(z)}{f(z)-q}\, dz$. The zeros of the function $h_q(z) := f(z) - q$ count, with multiplicity, the number of preimages in B of q under f. By the argument principle

$$\#\text{ of preimages of } q = \frac{1}{2\pi i} \int_{\partial B} \frac{h_q'(z)}{h_q(z)}\, dz = g(q).$$

The function g takes on only integer values, so by continuity $g(q) = g(0) = k$. If $u \neq p$ lies in U, then by choice of B, $h'_{f(u)}(u) = f'(u) \neq 0$ and so u is a zero of $h_{f(u)}$ of multiplicity 1. Since the multiplicity of each preimage of any point $q \neq 0$ is 1, each such point must have k pre-images in B. Note also that since every point in $B_M(0)$ has preimages in U, so that $f(U) = B_M(0)$ and is, in particular, an open neighbourhood of 0. $\square$

Corollary (Complex Inverse Function Theorem). *Let $f(z)$ be holomorphic at p with $f'(p) \neq 0$. Then there exists an open neighbourhood U of p such that $V := f(U)$ is an open neighbourhood of $f(p)$ and the restriction $f : U \to V$ is a bijection. The inverse function $f^{-1} : V \to U$ is also holomorphic with derivative given by $\frac{1}{f'(z)}$.*

Proof. Apply the preceding theorem to $g(z) := f(z) - f(p)$. Since $g'(p) = f'(p) \neq 0$, the multiplicity of p as a zero of g is 1. Therefore $k = 1$, and choosing U as in the theorem, every point in $g(U)$ (including 0) has precisely one preimage in U. Equivalently, $f : U \to V$ is a bijection.

Let $g : V \to U$ be the inverse to f. If γ is any closed curve in V, then making the change of variable $z = f(w)$ gives $\int_\gamma g(z)\,dz = \int_{g(\gamma)} wf'(w)\,dw = 0$ since it is the integral of the holomorphic function $wf'(w)$ over the closed curve $g(\gamma)$. Therefore g is holomorphic by Moreira's theorem, and its derivative is determined by applying the chain rule to $z = f \circ g(z)$.

$\square$

Residues can be used to evaluate integrals (MATC34). We will now examine how this process can sometimes be usefully reversed.

The function $\sin(\pi z)$ has zeros precisely at the integers. We can sometimes make use of this to compute $\sum_{n=-\infty}^{\infty} f(n)$ in cases where $f(n)$ is the restriction to the integers of some meromorphic function.

Suppose $f(z)$ is a meromorphic function. Then $\frac{f(z)}{\sin(\pi z)}$ is meromorphic with poles at the integers in addition to the poles of $f(z)$. The poles at the integers which are not singularities of $f(z)$ are simple poles, while those at singularities of $f(z)$ have higher order. If $f(n)$ is not a singularity of f, according to our earlier formula,

$$\mathrm{Res}_n\left(\frac{f(z)}{\sin(\pi z)}\right) = \frac{f(n)}{d\big(\sin(\pi z)\big)/dz|_{z=n}} = \frac{f(n)}{\pi\cos(\pi n)} = (-1)^n\frac{f(n)}{\pi}$$

This is sometimes useful in evaluating alternating series, but for series of positive terms it is more useful to replace $\sin \pi z$ in the preceding discussion by $\tan \pi z$ and consider the function $\pi\frac{f(z)}{\tan(\pi z)} = \pi\cot(\pi z)f(z)$. Since $d\big(\tan(\pi z)\big)/dz|_{z=n} = \pi\sec^2(\pi n) = 1$, repeating the preceding calculation gives

$$\mathrm{Res}_n\, \pi\cot(\pi z)f(z) = f(n).$$

To apply the method we will need to assume that $zf(z)$ is bounded as $|z| \to \infty$. That is, there exists R and M such $|zf(z)| \le M$ for all $|z| \ge R$.

Consider a large square γ_N centred at the origin and having side length $2N+1$ for a large integer N. Then

$$\begin{aligned}\int_{\gamma_N} \pi\cot(\pi z)f(z)\,dz &= \sum \text{all residues inside } \gamma_N \text{ of } \pi\cot(\pi z)f(z)\\ &= \sum_{n=-N}^{N} \{f(n) \mid n \text{ is not a singularity of } f\} \qquad (*)\\ &\quad + \sum \text{residues of } \pi\cot(\pi z)f(z) \text{ at singularities of } f(z) \text{ inside } \gamma_N\end{aligned}$$

We will show that our boundedness assumptions on $f(z)$ imply that

$$\lim_{N\to\infty} \int_{\gamma_N} \pi\cot(\pi z)f(z)\,dz = 0$$

so that we might get a formula for $\sum_{n=-\infty}^{\infty} f(n)$.

Lemma. *Let $f(z)$ be a meromorphic function such that $zf(z)$ is bounded as $z \to \infty$. Then*

$$\lim_{N\to\infty} \int_{\gamma_N} \cot(\pi z)f(z)\,dz = 0$$

where γ_N is the square of side length $2N+1$ centred at the origin.

Proof. Euler's formulas $e^{iz} = \cos(z) + i\sin(z)$ and $e^{-iz} = \cos(z) - i\sin(z)$ give $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$ and $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$. Therefore $\cot(z) = \frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}} = \frac{e^{2iz}+1}{e^{2iz}-1}$.

On the vertical sides of γ_N, $z = \pm(2N+1)/2 + iy$ so

$$|\cot(\pi z)| = \left|\frac{e^{\pm 2\pi i N}e^{\pm i\pi}e^{-2\pi y}+1}{e^{\pm 2\pi i N}e^{\pm i\pi}e^{-2\pi y}-1}\right| = \left|\frac{1(-1)e^{-2\pi y}+1}{1(-1)e^{-2\pi y}-1}\right| = \left|\frac{-e^{-2\pi y}+1}{-e^{-2\pi y}-1}\right| = \left|\frac{1-e^{-2\pi y}}{1+e^{-2\pi y}}\right| \leq 1$$

On the horizontal sides of γ_N, $z = x \pm i(2N+1)/2$ so

$$|\cot(\pi z)| = \left|\frac{e^{2\pi i x}e^{-\pm\pi(2N+1)}+1}{e^{2\pi i x}e^{-\pm\pi(2N+1)}-1}\right| = \left|\frac{e^{2\pi i x}+e^{\pm\pi(2N+1)}}{e^{2\pi i x}-e^{\pm\pi(2N+1)}}\right|$$

The right hand side is a continuous periodic function of x so it is bounded. (A continuous function on a closed bounded set is bounded (MATB43) and in the case of a periodic function we may restrict attention to one period, which is a bounded set.) In fact, it represents the ratio of the distances of some point on the unit circle to the points w and to $-w$ for some fixed w and is bounded, for example, by 2. Thus 2 is an upper bound for $|\cot(\pi z)|$ on γ_N.

Our assumption on $f(z)$ is that there exists R and M such that $|zf(z)| \leq M$ for all $|z| \geq R$. Let $w = 1/z$. Set $g(w) := zf(z)$ for $0 < |w| \leq 1/R$ or equivalently $|z|/geR$. Since $g(w)$ is bounded on $0 < |w| < 1/R$, its singularity at 0 is removable so it extends to a holomorphic function on $|w| < 1/R$. Let $g(w) = a_0 + a_1w + a_2w^2 + a_3w^3\ldots$ be the Taylor series of $g(w)$ about 0.

SubLemma. *$z^2\left(f(z)-\frac{a_0}{z}\right)$ is bounded as $z \to \infty$.*

Proof. Since $\lim_{w\to 0}\frac{g(w)-a_0}{w} = g'(0)$ exists, $\frac{g(w)-a_0}{w}$ has a removable singularity at 0 so it represents an analytic, thus continuous, function on $|w| \le 1/R$. Therefore (MATB43) it has a bound on any closed set in its domain. Let K be a bound for $\left|\frac{g(w)-a_0}{w}\right|$ on $|w| \le 1/(2R)$. If $|z| \ge 2R$ then $w \le 1/(2R)$ and so

$$\left|z^2\left(f(z)-\frac{a_0}{z}\right)\right| = |z(zf(z)-a_0)| = \left|\frac{g(w)-a_0}{w}\right| \le K$$

for $|z| \ge 2R$. □

Proof of Lemma (cont.). If $2N+1 > 2R$,

$$\begin{aligned}\left|\int_{\gamma_N}\cot(\pi z)\left(f(z)-\frac{a_0}{z}\right)dz\right| &\le \int_{\gamma_N}|\cot(\pi z)|\left|\frac{z^2\big(f(z)-a_0/z\big)}{z^2}\right|dz \le \int_{\gamma_N}\left|\frac{2K}{z^2}\right|dz\\ &\le \int_{\gamma_N}\frac{2K}{(2N+1)^2}\,dz \le \frac{4(2N+1)2K}{(2N+1)^2} = \frac{8K}{2N+1}\end{aligned}$$

Therefore

$$\lim_{N\to\infty}\int_{\gamma_N}\cot(\pi z)f(z)\,dz = \lim_{N\to\infty}\int_{\gamma_N}a_0\frac{\cot(\pi z)}{z}\,dz = a_0\sum \text{Residues of } \frac{\cot(\pi z)}{z}\text{inside } \gamma_N$$

We noted earlier if that $f(z)$ does not have a zero at n then $\operatorname{Res}_n\cot(\pi z)f(z) = f(n)/\pi$. Applying this in the case $f(z) = 1/z$ shows $\operatorname{Res}_n\frac{\cot(\pi z)}{z} = 1/(\pi n)$. Therefore, in the above sum, for $n \ne 0$, the residues at n and $-n$ cancel out leaving

$$\lim_{N\to\infty}\int_{\gamma_N}\cot(\pi z)f(z)\,dz = a_0\operatorname{Res}_0\left(\frac{\cot(\pi z)}{z}\right) = 0$$

since the pole of $\frac{\cot(\pi z)}{z}$ at 0 has order 2. □

Substituting this into equation (*) and taking the limit as $N \to \infty$ gives

Theorem. *Let $f(z)$ be a meromorphic function such that $zf(z)$ is bounded as $z \to \infty$. Then*

$$\sum_{n=-N}^{N}\{f(n) \mid n \textit{ is not a singularity of } f\}$$

$$= -\sum \text{residues of } \pi\cot(\pi z)f(z) \text{ at singularities of } f(z)$$

Note: $\lim_{N\to\infty}\sum_{n=-N}^{N}f(n)$ is not exactly the same as $\sum_{n=-\infty}^{\infty}f(n)$ since the former might exist in cases where the latter does not converge. For example, if $f(z)$ is some function for which $f(n) = 1/n$ for $n \ne 0$ then, because of the cancellation, the limit on the left exists and equals $f(0)$, but the series on the right does not converge.

Example. Let $f(z)=1/z^2$. Then $zf(z)=1/z$ is bounded as $z\to\infty$. The only singularity of $f(z)$ is at 0. Therefore $\lim_{N\to\infty}\sum_{n=-1}^{-N}\frac{1}{n^2}+\sum_{n=1}^{N}\frac{1}{n^2}=-\operatorname{Res}_0\left(\pi\frac{\cot(\pi z)}{z^2}\right)$. In this case we know that the two series on the left converge individually and are equal by symmetry. Thus we get

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n^2} &= -\frac{\pi}{2}\operatorname{Res}_0\left(\frac{\cot(\pi z)}{z^2}\right)=-\frac{\pi}{2}\operatorname{Res}_0\left(\frac{1-(\pi z)^2/2+\ldots}{z^2(\pi z-(\pi z)^3/6+\ldots}\right)\\
&= -\frac{\pi}{2}\operatorname{Res}_0\left(\frac{(1-\pi^2z^2/2+\ldots)(1+\pi^2z^2/6+\ldots}{\pi z^3}\right)=-\frac{\pi}{2}\operatorname{Res}_0\left(\frac{1-\pi^2z^2/3\ldots}{\pi z^3}\right)\\
&= -\frac{\pi}{2}\left(\frac{-\pi}{3}\right)=\frac{\pi^2}{6}
\end{aligned}
$$

□

Example. Suppose p is not an integer and let $f(z)=1/(z-p)$. Then $zf(z)=z/(z-p)$ is bounded as $z\to\infty$. The only singularity of $f(z)$ is at p. Therefore

$$
\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{n-p}=-\operatorname{Res}_p\left(\pi\frac{\cot(\pi z)}{z-p}\right)=-\pi\cot(\pi p).
$$

In other words, for every point in the domain of $\cot(\pi z)$ we have the identity

$$
\begin{aligned}
\pi\cot(\pi z) &= -\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{n-z}=\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{z-n}\\
&= \frac{1}{z}+\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{z+n}+\sum_{n=1}^{N}\frac{1}{z-n}\right)=\frac{1}{z}+\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{z+n}+\frac{1}{z-n}\right)\\
&= \frac{1}{z}+\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{z-n+z+n}{z^2-n^2}\right)=\frac{1}{z}+\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{2z}{z^2-n^2}\right)\\
&= \frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}
\end{aligned}
$$

Notice that series in the final answer converges for each z, but it would not have been correct to write $\pi\cot(\pi z)=-\sum_{n=-\infty}^{\infty}\frac{1}{n-z}$ since the right hand side does not converge.

6. Harmonic Functions

A real-valued function $h : \mathbb{R}^n \to \mathbb{R}$ is called *harmonic* if it satisfies Laplace's equation $\nabla^2 h = 0$, where $\nabla^2 h$, the *Laplacian* of h, is defined by $\nabla^2 h = \sum_{k=0}^{n} \partial^2 h/\partial x_i^2$. The notation is suggested by the fact that $\nabla^2 h = \nabla \cdot (\nabla h)$, where ∇h is the gradient of h and $\nabla \cdot V$ denotes the divergence of a vector field V. When $n = 2$, we will see that there is a very close connection between harmonic functions and holomorphic functions.

Let $f = u+iv$ be holomorphic throughout a domain D. A consequence of the Cauchy-Riemann equations is that ∇u is always perpendicular to ∇v. This means that the family of curves $u = C$ (the solutions to the differential equation $\nabla u = 0$) are the othogonal trajectories (MATB44) of the family $v = \tilde{C}$. Also

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial}{\partial y}\frac{\partial u}{\partial y} = -\frac{\partial^2 u}{\partial y^2}$$

and so u (and similarly v) is harmonic. Thus a holomorphic function on D determines a harmonic function u on D.

To what extent can this process be reversed? Let u be a harmonic function on a domain D. Can we find a v such that $u + iv$ is holomorphic? We reformulate the question in terms of differential forms. Recall (MATB42) that a differential k-form ω is called *closed* if $d\omega = 0$ and called *exact* if there exists an $(k-1)$-form η such that $d\eta = \omega$. The fact that exact implies closed is a trivial consequence of the formula $d^2 = 0$. We know from MATB42 that for differential forms defined throughout simply connected regions, the converse also holds: if ω is defined and closed throughout a simply connected region D then there exists η defined throughout D such that $d\eta = \omega$.

Given harmonic u, we wish to find a function v such that $\nabla v = (-u_y, v_x)$ so that the Cauchy-Riemann equations will be satisfied. Define a 1-form ω by $\omega := -u_y\, dx + u_x\, dy$. Then

$$d\omega = -u_{yy}\, dy \wedge dx + u_{xx}\, dx \wedge dy = (u_{xx} + u_{yy})\, dx \wedge dy = 0,$$

since u is harmonic. Therefore in any simply connected region (for example a ball about any point in domain) there exists a 0-form (i.e. a function) v such that $dv = \omega$, which is equivalent to saying that $\nabla v = (-u_y, v_x)$. The solution is (locally) unique up to a constant, since if $\tilde{v}$ also satisfies $d\tilde{v} = \omega$ then $d(v - \tilde{v}) = 0$ so $v - \tilde{v}$ is constant. To summarize,

Theorem. *Given a harmonic function $u(x, y)$, in any simply connected neighbourhood N of any point in the domain of u, there exists a harmonic function v, unique up to a constant, such that the function $f(z) = u + iv$ is homolomorphic throughout N.*

Thus, locally, a harmonic function u determines, uniquely up to a constant, a holomorphic function whose real part is u. Similarly a holomorphic function is uniquely determined up to a constant by its imaginary part. Therefore the study of the local properties of holomorphic functions is equivalent to the study of the local properties of harmonic functions. Note again that the preceding does **not** say that given harmonic u on D we can necesarily find a holomorphic function throughout D whose real part is u (unless D happens to be simply connected). For example, if $u(x, y) = \log(x^2 + y^2) = \log(|z|)$, then u is harmonic on the domain $\mathbb{R}^2 - \{0\}$, and, although we have branches of the function $\log(z)$ whose real part is locally $u(x, y)$, there is no holomorphic function defined on all of $\mathbb{R}^2 - \{0\}$ whose real part is u.

Example. Let $u(x,y) = \frac{x^2}{2} + xy - \frac{y^2}{2}$. Observe that u is harmonic. In this case the domain is u is all of $\mathbb{R}^2$ which is simply connected so there exists a holomorphic function $f(z)$ (unique up to a constant) whose real part is u. To find such an f we could proceed as follows. Since differentiation shows that $u = C$ is the solution of the differential equation $(x+y)\,dx + (x-y)\,dy = 0$. we wish to find a solution v of the differential equation $(y-x)\,dx + (x+y)\,dy = 0$. Equivalently, we need to solve $dv = \omega$ where $\omega = (y-x)\,dx + (x+y)\,dy = 0$. This equation is equivalent to

$$\frac{\partial v}{\partial x} = y - x \tag{1}$$

$$\frac{\partial v}{\partial y} = x + y \tag{2}$$

(1) implies $v = \int (y-x)\,dx + h(y) = xy - x^2/2 + h(y)$ for some function $h(y)$. Differentiate to get $\frac{\partial v}{\partial y} = x + \frac{dh}{dy}$ which, upon comparison with (2) yields $\frac{dh}{dy} = y$ and so $h = y^2/2 + C$ for some contant C. Therefore $v = xy - x^2/2 + h(y) = xy - x^2/2 + y^2/2 + C$. Choosing $C = 0$ gives the solution $f(x,y) = u + iv = x^2/2 + xy - y^2/2 + i(xy - x^2/2 + y^2/2)$, which, by inspection can be written equivalently as $f(z) = z^2/2$.

Proposition. *Let $u(z)$ be harmonic and let $f(z)$ be holomorphic. Then $u \circ f$ is harmonic.*

Proof. Find a holomorphic function $g(z)$ such that $\operatorname{Re} g = u$. Then $g \circ f$ is holomorphic and its real part is $u \circ f$. (Of course, this Proposition can also be proved directly by differentiating the composition.) $\square$

Let u be a harmonic function on a domain containing a disk B. Since the disk is simply connected, there is a holomorphic function $f(z) = u + iv$ defined a domain containing B. Since the values of f in the interior of B are determined according to Cauchy's integral formula by the values of f on the boundary circle of B, the values of u on the interior of B must also be determined by the values of u on the boundary of B. The formula for harmonic functions corresponding to the Cauchy Integral Formula in this case is called the "Poisson Integral Formula". By translation, we might as well consider the case where the ball B is centred at the origin.

Theorem (Poisson Integral Formula). *Let $u(x,y)$ be harmonic on a domain containing the closed ball $B = B_R(0)$ of radius R about 0. Then for any z_0 in the interior of B,*

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) \frac{R^2 - |z_0|^2}{|Re^{i\theta} - z_0|^2}\, d\theta.$$

Proof. Consider first the special case where $z_0 = 0$. According to the preceding discussion, there is a function $v(x,y)$ such that $f(z) := u + iv$ is holomorphic on B. Applying the Cauchy Integral Formula to f on the circle $C = \partial B$ of radius R about 0, gives

$$f(0) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w}\, dw = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(Re^{i\theta})}{Re^{i\theta}} iRe^{i\theta}\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\theta})\, d\theta.$$

Taking the real part yields

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta})\,d\theta.$$

which is the Poisson formula in the case $z_0 = 0$.

Given arbitrary z_0 in the interior of B, define $g(z)$ by $g(z) := \frac{z+z_0}{R^2+\overline{z_0}z}R^2$ and set $\tilde{u} := u \circ g$. The inverse of g is given by $h(z) = \frac{z-z_0}{R^2-\overline{z_0}z}R^2$ and so $u = \tilde{u} \circ h$. Observe that $\tilde{u}(0) = u\big(g(0)\big) = u(z_0)$. We can check that $g(C) = C$ as follows. Suppose $|z| = R$. Then

$$\begin{aligned}|g(z)|^2 = \left(\frac{z+z_0}{R^2+\overline{z_0}z}\right)\left(\frac{\bar{z}+\overline{z_0}}{R^2+z_0\bar{z}}\right)R^4 &= \left(\frac{|z|^2+z_0\bar{z}+\overline{z_0}z+|z_0|^2}{R^4+R^2z_0\bar{z}+R^2\overline{z_0}z+|z_0|^2|z|^2}\right)R^4\\ &= \left(\frac{R^2+z_0\bar{z}+\overline{z_0}z+|z_0|^2}{R^4+R^2z_0\bar{z}+R^2\overline{z_0}z+|z_0|^2R^2}\right)R^4 = R^2.\end{aligned}$$

Notice that since u is harmonic and g is holomorphic, the real valued function $\tilde{u}$ is harmonic, since it is the real part of the holomorphic function $f \circ g$, where f is a holomorphic function whose real part is u. Applying the special case the formula to the harmonic function $\tilde{u}$ gives

$$\begin{aligned}u(z_0) = \tilde{u}(0) = \frac{1}{2\pi}\int_0^{2\pi}\tilde{u}(Re^{i\theta})\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}u\big(g(Re^{i\theta})\big)\,d\theta\\ = \frac{1}{2\pi i}\int_0^{2\pi}u\big(g(Re^{i\theta})\big)\frac{iRe^{i\theta}}{Re^{i\theta}}\,d\theta = \frac{1}{2\pi i}\int_C\frac{u\big(g(w)\big)}{w}\,dw.\end{aligned}$$

Since

$$h'(z) = \frac{(R^2-\overline{z_0}z)(1)-(z-z_0)(-\overline{z_0})}{(R^2-\overline{z_0}z)^2}R^2 = \frac{R^2-\overline{z_0}z+z\overline{z_0}-|z_0|^2}{(R^2-\overline{z_0}z)^2}R^2 = \frac{R^2-|z_0|^2}{(R^2-\overline{z_0}z)^2}R^2,$$

making the change of variable $w = h(\zeta)$ gives

$$\begin{aligned}\frac{1}{2\pi i}\int_C\frac{u\big(g(w)\big)}{w}\,dw &= \frac{1}{2\pi i}\int_C\frac{u(\zeta)}{h(\zeta)}h'(\zeta)\,d\zeta\\ &= \frac{1}{2\pi i}\int_C\frac{u(\zeta)}{\big((\zeta-z_0)/(R^2-\overline{z_0}\zeta)\big)R^2}\frac{(R^2-|z_0|^2)R^2}{(R^2-\overline{z_0}\zeta)^2}\,d\zeta\\ &= \frac{1}{2\pi i}\int_C\frac{u(\zeta)}{(\zeta-z_0)}\frac{(R^2-|z_0|^2)}{(R^2-\overline{z_0}\zeta)}\,d\zeta\\ &= \frac{1}{2\pi i}\int_0^{2\pi}\frac{u(Re^{it})}{(Re^{it}-z_0)}\frac{(R^2-|z_0|^2)}{(R^2-\overline{z_0}Re^{it})}iRe^{it}\,dt\\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{u(Re^{it})}{(Re^{it}-z_0)}\frac{(R^2-|z_0|^2)}{(Re^{-it}-\overline{z_0})}\,dt\\ &= \frac{1}{2\pi}\int_0^{2\pi}u(Re^{it})\frac{(R^2-|z_0|^2)}{|Re^{it}-z_0|^2}\,dt\end{aligned}$$

as desired. $\square$

We note in particular the special case where z_0 is the centre of the circle where the formula reads

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + Re^{i\theta})\, d\theta.$$

indicating that as in the case of holomorphic functions, for harmonic functions the value of u at the centre of a circle is the average of its values on the circle. And as we holomorphic functions we have

Corollary (Maximum Modulus Principle). *Let u be harmonic on a domain containing a closed bounded set B. Then the maximum value M of $|u|$ occurs on ∂B and unless u is constant, $|u(z)| < M$ for all z in the interior of B.* □

7. Conformal Maps

Many important problems lead to differential equations whose solutions are bounded harmonic functions. For example consider the flow of heat in a region B. Let $u(z,t)$ the temperature at the point z at time t. The total thermal energy $\int_B c_1 u\, dV$ (where c_1 is a constant depending on the units or measurement) changes only as heat enters or leaves through the boundary. Heat "flows" from hot to cold and in particular at any point it flows in the direction of most rapid decrease of the temperature function, which (MATB41) is the direction of $-\nabla u$. At any time, the rate of change of heat loss through the boundary is given by $\int_{\partial B} c_2 \nabla u \cdot \mathbf{n}\, dS$. Therefore, $\frac{d}{dt} \int_B c_1 u\, dV = \int_{\partial B} c_2 \nabla u \cdot \mathbf{n}\, dS$. By Gauss' (Divergence) Theorem (MATB42), $\int_{\partial B} c_2 \nabla u \cdot \mathbf{n}\, dS = \int_B c_2 \nabla \cdot (\nabla u)\, dV$. Therefore u is a bounded function satisfying the "heat equation" $\nabla^2 u = k u_t$, where $\nabla^2 u = \sum_{j=1}^n u_{x_j x_j}$ and k is a constant. If the temperature of the surroundings (i.e. the boundary) do not change with time, then the value of $u(z,t)$ at any point z will stabilize over time and the function $u(z,t)$ will approach a "steady-state solution" $u(z)$ which is independent of t. Since in steady state, $u_t = 0$, the steady-state solution is a bounded harmonic function and thus determined by its values at the boundary. For example, if the temperature at the boundary is a constant C, then the steady-state solution will be $u = C$. (e.g. in the absence of other influences, a hot object will eventually cool to the temperature of its surroundings.) It should be noted that in recent years the study of the heat equation has played a central role in the emerging field of financial mathematics, since it has been discovered that in addition to providing a mathematical model for the flow of heat, it also provides a good model for the "flow" of money. Because of applications to the heat equation and others, an important problem in mathematics is:

Dirichlet Problem. *Given a real-valued function u on the boundary of a region B, find the harmonic function on B (if there is one) which extends the given function to the interior.*

The preceding discussion applies in any number of dimensions, but the methods are complex analysis are particularly suited to handle the case $n = 2$.

Example. A large (effectively infinite) sheet of metal occupies the right half-plane $x \geq 0$. The upper y-axis is maintained at room temperature ($20°$) but ice placed along the negative y-axis keeps the temperature at $0°$ there. Suppose that the interior of the sheet is insulated above and below, so that heat can flow only in the xy-plane. Find the steady-state temperature at the point (x, y).

Solution. Let $u(x, y)$ be the steady-state temperature at (x, y). Then $u(x, y)$ is the bounded harmonic function for which

$$u(0, y) = \begin{cases} 20 & \text{if } y > 0; \\ 0 & \text{if } y < 0. \end{cases}$$

The temperature would not actually jump discontinuously from 0 to 20 at the origin, but would change very quickly so that our model would be a good approximation except in the immediate vicinity of the origin.

Let $f(z)$ be the branch of $\log z$ defined on $\mathbb{C} - \{(x, y) \mid x \le 0\}$ by

$$f(z) = \log(|z|) + i \arg(z)$$

with $\arg(z)$ chosen in the range $(-\pi, \pi)$. Then the imaginary part of f is $\begin{cases} \pi/2 & \text{if } y > 0; \\ -\pi/2 & \text{if } y < 0. \end{cases}$ Therefore if we set $g(z) := \frac{20}{\pi}\bigl(-if(z)+\pi/2\bigr)$ then $u(z) = \operatorname{Re} g(z)$ is the solution. Explicitly,

$$u(x, y) = \frac{20}{\pi}\bigl(\arg(z) + \pi/2\bigr) = \frac{20}{\pi} \tan^{-1}(y/x) + 10$$

where we are using the branch of $\tan^{-1}(\;)$ which takes values in $[-\pi/2, \pi/2]$. □

If $u(z)$ is harmonic and $f(z)$ is holomorphic then $u \circ f$ is harmonic. If we know how to solve the Dirichlet problem on some region B then one way to solve it for some other region B' would be to find a holomorphic function f whose domain U contains B and which has the property that f is a bijection from U to $V := f(U)$ taking B to B'. The inverse function $f^{-1} : V \to U$ will then also be holomorphc (with derivative $\frac{1}{f'(z)}$) and $u \circ f^{-1}$ solves the Dirichlet problem on B'.

A holomorphic bijection $f : U \to V$ is called a *conformal mapping* from U to V. We say that U and V are *conformally equivalent* if there exists a conformal map between them. The derivative of a bijection must be nowhere zero, and conversely, we showed early that given a holomorphic function $f(z)$ for which $f'(p) \neq 0$ there exists open neighbourhoods U of p and V of $f(p)$ such that $f : U \to V$ is conformal.

The word "conformal" means "angle-preserving". As we noted earlier, the Jacobian matrix for $f'(z)$ corresponds to rotation by $\arg\bigl(f'(z)\bigr)$ followed by multiplication by $|f'(z)|$. Therefore the angle at which two curves cross (defined as the angle between their tangent vectors) is preserved under the application of any holomorphic function with nonzero derivative. Thus conformal mappings are indeed angle-preserving.

Example. A large (effectively infinite) sheet of metal, insulated above and below, occupies the first quadrant. The y-axis is maintained at room temperature ($20°$) but ice placed along x-axis keeps the temperature at $0°$ there. Find the steady-state temperature at the point (x, y).

Solution. Let Q be the first quadrant and let H be the right half plane. The function $f(z) = \frac{z^2}{i}$ is a conformal mapping from $Q - \{0\}$ to $H - \{0\}$ taking the upper y-axis (as a unit, not point-by-point) to itself and taking the x-axis to the lower y-axis. Therefore the solution is given by $u \circ f$ where u solves the corresponding problem in the right half plane. Since that solution was found earlier to be $u(x,y) = \frac{20}{\pi}\arg(z) + 10$, the solution to the present problem is

$$\begin{aligned} h(x,y) = \frac{20}{\pi}\arg(z^2/i) + 10 &= \frac{20}{\pi}\big(\arg(z^2) - \pi/2\big) + 10 = \frac{20}{\pi}\arg(z^2) \\ &= \frac{20}{\pi}\arg(x^2 - y^2 + 2ixy) = \frac{20}{\pi}\tan^{-1}\left(\frac{2xy}{x^2 - y^2}\right). \end{aligned}$$

□

Example. A sheet of metal, insulated above and below, occupies the unit disk. The upper arc of the boundary circle is maintained at room temperature (20°) but ice placed along the lower arc keeps the temperature at 0° there. Find the steady-state temperature at the point (x,y).

Solution. Let B be the unit disk. The function $f(z) = \frac{1+z}{1-z}$ is a conformal mapping from $B - \{1\}$ to the right half plane taking the boundary circle ∂B to the y-axis. Therefore the solution is given by $u \circ f$ where u solves the corresponding problem in the right half plane. Since that solution was found earlier to be $u(x,y) = \frac{20}{\pi}\arg(z) + 10$, the solution to the present problem is

$$\begin{aligned} h(x,y) &= \frac{20}{\pi}\arg\left(\frac{1+z}{1-z}\right) + 10 = \frac{20}{\pi}\arg\left(\frac{1+x+iy}{1-x-iy}\right) + 10 \\ &= \frac{20}{\pi}\arg\left(\frac{(1+x+iy)(1-x+iy)}{(1-x)^2+y^2}\right) + 10 = \frac{20}{\pi}\arg\left(\frac{1-x^2-y^2+2iy}{(1-x)^2+y^2}\right) + 10 \\ &= \frac{20}{\pi}\tan^{-1}\left(\frac{2y}{1-x^2-y^2}\right) + 10 \end{aligned}$$

Using the identity

$$\tan^{-1}(v) + \tan^{-1}(1/v) = \begin{cases} \pi/2 & \text{if } v > 0; \\ -\pi/2 & \text{if } v < 0, \end{cases}$$

the final answer can also be written as $h(x,y) = \frac{20}{\pi}\tan^{-1}\left(\frac{1-x^2-y^2}{-2y}\right)$ using the branch of $\tan^{-1}(\ )$ which takes values in $[0,\pi]$. □

The Poisson Integral Formula gives a solution to the Dirichlet problem in the case where the region B is a closed disk, although even in this case one might hope for a more explicit formula for the solution (as in the preceding example) rather than the integral, which might be hard to compute. More precisely:

Solution to Dirichlet Problem on a disk. *Let $u(z)$ be a continuous function on the boundary circle of the closed ball $B_R[p]$ of radius R about p. The function defined by*

$$h(z) = \begin{cases} \frac{1}{2\pi}\int_0^{2\pi} u(p + Re^{i\theta})\frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2}\,d\theta & \text{if } z \text{ lies in interior } B; \\ u(z) & \text{if } z \text{ lies on } \partial B. \end{cases}$$

solves the Dirichlet problem on B.

It is clear from Poisson's formula that if the problem has any solution then this is it, but there are two technical points which need to be resolved.

(1) Does the resulting function $h(z)$ satisfy $\nabla^2 h = 0$ throughout the interior of B?

(2) Is h continuous?

It is not so hard to demonstrate that h satisfies (1), but showing that the limit as we approach the boundary is the original function $u(z)$ is not so easy. We shall not go into the details except to observe that this a place where the properties of harmonic functions differ from the analogous properties of holomorphic functions.

Example. Let B be the unit disk $B_1[0]$. Let $g : \partial B \to \mathbb{C}$ be given by $g(z) = 1/z$. Define $f : B \to \mathbb{C}$ by

$$f(z) = \begin{cases} \frac{1}{2\pi i} \int_{\partial B} \frac{g(w)}{w-z}\, dw & \text{if } z \text{ lies in interior } B; \\ g(z) & \text{if } z \text{ lies on } \partial B. \end{cases}$$

For z in the interior of B, $f(z) = \frac{1}{2\pi i} \int_{\partial B} \frac{1}{w(w-z)}\, dw$. The function $h(w) = \frac{1}{w(w-z)}$ is meromorphic on B with simple poles at 0 and z. Therefore

$$\begin{aligned} f(z) &= \operatorname{Res}_0 h(w) + \operatorname{Res}_z h(w) \\ &= \left(\frac{1}{w-z}\Big|_{w=0} \right) \left(\operatorname{Res}_0 \frac{1}{w} \right) + \left(\frac{1}{w}\Big|_{w=z} \right) \left(\operatorname{Res}_z \frac{1}{w-z} \right) = \frac{-1}{z} + \frac{1}{z} = 0. \end{aligned}$$

But $f\big|_{\partial B} \neq 0$, so f is not continuous. $\square$

The examples above suggest the following questions:

1) Given regions B and B', how can we tell if there exists a conformal mapping between B and B'?
2) If f is conformal, how does it behave as we approach the boundary of its domain?
3) Assuming a conformal mapping $f : B \to B'$ exists, how do we find it?

For simply connected regions, we can give a complete answer to question (1):

Riemann Mapping Theorem. *Let U be a simply connected domain such that $U \neq \mathbb{C}$. Then there exists a conformal map $f : U \to D$ where $D = \{z \mid |z| < 1\}$ is the unit disk. Furthermore, given any $p \in U$ and $q \in D$, there exists a unique such f such that $f(p) = q$ and $f'(p)$ is a positive real number.*

The proof is difficult and we shall not go into it. The Riemann Mapping Theorem tells us that, aside from $\mathbb{C}$ itself any two simply connected domains are conformally equivalent since both are conformally equivalent to D. Notice that $\mathbb{C}$ is not conformally equivlent to D since D is bounded and therefore by Liouville's Theorem there cannot exist a holomorphic function $f : \mathbb{C} \to D$ whose image contains more than one point. It follows that $\mathbb{C}$ is not conformally equivalent to any other domain, since any conformal image of $\mathbb{C}$ would be simply connected and thus conformally equivalent to D. Determining whether or not two non-simply-connected domains are conformally equivalent is not easy in general.

For bounded simply connected regions, information about behaviour as we approach the boundary is given by:

Theorem (Osgood-Caratheodory). *Let U_1 and U_2 be the interiors of simple closed curves γ_1 and γ_2 and set $B_1 := U_1 \cup \gamma_1$ and $B_2 := U_2 \cup \gamma_2$. Let $f : U_1 \to U_2$ be conformal. (Such f exists by the Riemann Mapping Theorem.) Then f extends to a continuous bijection from B_1 to B_2 (which must, in particular, restrict to a continuous bijection from between the boundaries γ_1 and γ_2).*

The proof of this theorem is also difficult and we shall not go into it.

Fractional Linear Transformations.

To assist in finding conformal maps from a region B to another region B' we begin by looking at the special case where B and B' are the unit disk $D = \{z \mid |z| < 1\}$.

For any $p \in D$ and any ω with $|\omega| = 1$, define $\phi_{p,\omega} : \mathbb{C} - \{1/\bar{p}\} \to \mathbb{C}$ by

$$\phi_{p,\omega}(z) := \frac{\omega(z-p)}{1-\bar{p}z}.$$

Proposition.

1) *For any $p \in D$ and ω with $|\omega| = 1$, $\phi_{p,\omega}(D) \subset D$ and $\phi_{p,\omega}\Big|_D$ is a conformal self-map of D taking p to 0.*
2) *Any conformal self-map of D is $\phi_{p,\omega}$ for some $p \in D$ and ω with $|\omega| = 1$.*

Proof.

1) The domain of $\phi_{p,\omega}$ includes $\bar{D} = \{z \mid |z| \le 1\}$ since $p \in D$ implying that $|1/\bar{p}| > 1$. Let $C = \{z \mid |z| = 1\}$ be the boundary circle of $\bar{D}$. For $z \in C$,

$$|\phi_{p,\omega}(z)|^2 = \left|\frac{\omega(z-p)}{1-\bar{p}z}\right|^2 = \left|\frac{(z-p)(\bar{z}-\bar{p})}{(1-\bar{p}z)(1-p\bar{z})}\right| = \left|\frac{|z|^2 - p\bar{z} - \bar{p}z + |p|^2}{1 - p\bar{z} - \bar{p}z + |p|^2|z|^2}\right|$$
$$= \left|\frac{1 - p\bar{z} - \bar{p}z + |p|^2}{1 - p\bar{z} - \bar{p}z + |p|^2}\right| = 1$$

using $|z|^2 = 1$, since $z \in C$. Thus $\phi_{p,\omega}(C) \subset C$. Since $\phi_{p,\omega}$ is continuous, the image if each of the two connected components of $\mathbb{C} - C$ is connected so the either $\phi_{p,\omega}(D)$ is entirely contained in D or it is contained in the outside of C. Since $\phi(p) = 0$, we must have $\phi_{p,\omega}(D) \subset D$ and similarly the image of the outside of C is contained in the outside of C. It is clear that $\phi_{p,\omega}$ is differentiable and it is invertible with inverse given by $g(z) = \frac{z-p}{\omega(1-|p|^2)(z\bar{p}-1)}$.

2) Let $f : D \to D$ be conformal. Find p such that $f(p) = 0$. The derivative $\phi'_{p,1}(p)$ is given by

$$\phi'_{p,1}(p) = \frac{(1-\bar{p}z)1 - (z-p)(-\bar{p})}{(1-\bar{p}z)^2}\Big|_{z=p} = \frac{1-|p|^2}{(1-|p|^2)^2} = \frac{1}{1-|p|^2}$$

so it is a positive real number. Write $f'(p)/\phi'_{p,1}(p) = re^{i\theta}$, for some r and θ and set $\omega := e^{i\theta}$. The functions f/ω and $\phi_{p,1}$ are each conformal self-maps of D taking p to 0 with positive real derivatives at p. Therefore according to the Riemann Mapping Theorem, $f(z)/\omega = \phi_{p,1}(z)$ and so $f(z) = \omega\phi_{p,1}(z) = \phi_{p,\omega}(z)$. $\square$

Given a 2×2 matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of complex numbers with $\det(M) \neq 0$, define a function $\mathcal{F}(M)$ by

$$\mathcal{F}(M)(z) = \frac{az+b}{cz+d}.$$

Regard $\mathcal{F}(M)$ as a function on $\mathbb{C} \cup \{\infty\}$ with

$$\mathcal{F}(M)(-\frac{d}{c}) := \infty$$
$$\mathcal{F}(M)(\infty) := \lim_{z \to \infty} \frac{az+b}{cz+d} = \frac{a}{c}.$$

A function $f : \mathbb{C} \cup \infty \to \mathbb{C} \cup \infty$ of the form $f(z) = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$ is called a *fractional linear transformation*. We saw above that every conformal self-map of D is a fractional linear transformation.

Proposition. *Under the association $M \mapsto \mathcal{F}(M)$, matrix multiplication corresponds to composition of functions. i.e. $\mathcal{F}(MN) = \mathcal{F}(M) \circ \mathcal{F}(N)$.*

Proof. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $N = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. Hence $MN = \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' + dc' & cb' + dd' \end{pmatrix}$.

$$\mathcal{F}(M) \circ \mathcal{F}(N)(z) = \mathcal{F}(M) = \frac{a\left(\frac{a'z+b'}{c'z+d'}\right) + b}{c\left(\frac{a'z+b'}{c'z+d'}\right) + d} = \frac{a(a'z+b') + b(c'z+d')}{c(a'z+b') + d(c'z+d')}$$
$$= \frac{(aa'+bc')z + ab' + bd'}{(ca'+dc')z + cb' + dd'} = \mathcal{F}(MN)(z)$$

Corollary.
1) *The product of two fractional linear transformations is a fractional linear transformation.*
1) *The inverse of a fractional linear transformations is a fractional linear transformation.*

Proof. They correspond to the product and inverse of matrices. □

In the language of group theory (MATC01), $\mathcal{F} : \mathrm{GL}_2(\mathbb{C}) \to$ Self-maps of $(\mathbb{C} \cup \infty)$, is a group homomorphism where for a field F,

$$\mathrm{GL}_n(F) \equiv \{n \times n \text{ matrices with entries in } F \text{ and nonzero determininant}\}.$$

The homomorphism is not injective since $\mathcal{F}(kM) = \mathcal{F}(M)$.

The fractional linear transformation $I(z) := \frac{-1}{\bar{z}}$ is called *inversion* (or reflection) in the unit circle. In polar coordinates, $I(r, \theta) = (1/r, \theta)$.

I interchanges points P and Q on the same radial spoke if $|OP| = 1/|OQ|$ where O is the origin. Points on the unit circle $C = \{z \mid |z| = 1\}$ are fixed.

Consider the image of a line L under I.

Let A be the intersection of the perpenticular from 0 with L. Let $A' = I(A)$. Draw the circle D with diameter OA'.
Claim: $D = I(L)$.
Proof. Let P be any point on L. To show $I(P)$ lies on D:

Let Q be the intersection of D with OP. Since OA' is a diameter of D and Q is on the circumference, $\angle OQA' - \pi/2$. Therefore $\triangle OAP \approx \triangle OQA'$. Thus $\frac{|OA|}{|OQ|} = \frac{|OP|}{|OA'|}$ and so $|OP|\,|OQ| = |OA|\,|OA'| = 1$. Hence $Q = I(P)$. $\square$

We have just seen that for any line L, the image $I(L)$ is a circle passing through the origin. Conversely, I takes circles passing through the origin to lines.

What about circle not passing through the origin?

Let $\tilde{C}$ be a circle with centre $\tilde{O}$. As before C denotes the unit circle with centre $O =$ origin. Let A, B the intersections with $\tilde{C}$ of the line joining the centres O, $\tilde{O}$. Let M be some other line through O, intersection $\tilde{C}$ at points Q and P.

Set $A := |OA| \qquad b := |OB|$.

Let $\delta : \mathbb{C} \to \mathbb{C}$ be the dilation (expansion/contraction) $\delta(z) = z/ab$.

Set $A' := \delta(B) \qquad B' := \delta(A) \qquad P' := \delta(Q) \qquad Q' := \delta(P)$

$|OA'| = |OB|/(ab) = b/(ab) = 1/a$ so $A' = I(A)$. Similary $B' = I(B)$.

$\frac{|OA'|}{|OB|} = \frac{1}{ab} = \frac{|OP'}{|OQ|}$

Therefore $\triangle OA'P' \approx \triangle OBQ$.

Also $\angle ABQ = \angle APQ$ (they are angles at the circumference opposite the chord AQ).

Thus $\triangle OBQ \approx \triangle OPA$.

Therefore $\frac{|OA'|}{|OP|} = \frac{|OP'}{|OA|}$ which implies

$$|OP'|\,|OP| = |OA'|\,|OA| = (1/a)a = 1$$

and so $P' = I(P)$. Similarly $Q' = I(Q)$.

Since A, B, P, Q lie on circle $\tilde{C}$, their images B', A', Q', P' under I lie on the circle $\delta(\tilde{C})$. i.e. $I(\tilde{C})$ is the same as the circle $\delta(\tilde{C})$.

Thus, I takes circles not passing through the origin to circles not passing through the origin.

Theorem. *Let (z) be a fractional linear transformation. If S is either a line or circle then so is $f(S)$.*

Note: As in the special case of inversion, f(line) might be either a circle of a line. Similarly f(circle) might be either a circle of a line. We refer to a set which is either a circle or a line as a "circline".

Proof. Write $f(z) = \frac{az+b}{cz+d}$ with a, b, c, $d \in \mathbb{C}$, $ad - bc \neq 0$. Set $M := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Find $k \in \mathbb{C}$ such that $k^2 = \det M$. Then $\det M/k = 1$. Since multiplication by k has the desired property, it suffices to consider the special case where $\det M = 1$.

Set $\tau_r := \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$, $\sigma := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then $\mathcal{F}(\tau_r)(z) = z + r$ and $\mathcal{F}(\sigma) = \frac{-1}{z}$ so τ_r corresponds to translation by r and σ corresponds to the composition of inversion (reflection) in the unit circle followed by reflection about the y-axis. Therefore τ_r and σ each have the desired property.

$$\begin{aligned}
\sigma\tau_r &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & r \end{pmatrix} \\
\sigma\tau_s\sigma\tau_r &= \begin{pmatrix} 0 & -1 \\ 1 & s \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & r \end{pmatrix} = \begin{pmatrix} -1 & -r \\ s & rs-1 \end{pmatrix} \\
\sigma\tau_t\sigma\tau_s\sigma\tau_r &= \begin{pmatrix} 0 & -1 \\ 1 & t \end{pmatrix} \begin{pmatrix} -1 & -r \\ s & rs-1 \end{pmatrix} = \begin{pmatrix} -s & 1-rs \\ st-1 & rst-r-t \end{pmatrix}
\end{aligned}$$

If we can choose r, s, t so that $\sigma\tau_t\sigma\tau_s\sigma\tau_r = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then we will know that $\mathcal{F}(M)$ is the composite $\mathcal{F}(\sigma)\mathcal{F}(\tau_t)\mathcal{F}(\sigma)\mathcal{F}(\tau_s)\mathcal{F}(\sigma)\mathcal{F}(\tau_r)$ of functions each having the desired property which demonstates that $\mathcal{F}(M)$ has it as well.

If $a \neq 0$:

$$\text{Set} \qquad s := -a \qquad r := \frac{b-1}{a} \qquad t := \frac{-1-c}{a}$$

Then

$$\begin{aligned}
-s &= a \qquad \surd \\
1 - rs &= 1 - \left(\frac{b-a}{a}\right)(-a) = b \qquad \surd \\
st - 1 &= (-a)\left(\frac{-1-c}{a}\right) - 1 = c \qquad \surd \\
rst - r - t &= \left(\frac{b-1}{a}\right)(-a)\left(\frac{-1-c}{a}\right) - \frac{b-1}{a} - \frac{-1-c}{a} \\
&= \frac{b-1+bc-c-b+1+1+c}{a} = \frac{1+bc}{a} = \frac{ad}{a} = d \qquad \surd
\end{aligned}$$

using $ad - bc = 1$.

If $a = 0$ then $c \neq 0$ (since $ad - bc = 1$) so consider

$$\sigma M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}$$

By the above calculation, $\mathcal{F}(\sigma M)$ preserves circlines so $\mathcal{F}(M) = \mathcal{F}(\sigma)^{-1}\mathcal{F}(\sigma M)$ does as well. $\square$

Example. Find a conformal map from the unit disk D to the upper half plane H.

Solution.

Plan:

Step 1: Translate D so that its boundary passes through the origin.

Step 2: Apply $z \mapsto 1/z$ to convert the boundary circle to a line.

Step 3: Translate the line so that it passes through the origin
Step 4: Rotate so that the boundary line becomes the x-axis with the region on the correct side.

1) Let $g(z) = z - 1$. Then Im D is the disk D' of radius 1 centred at $(-1, 0)$.
2) Let $h(z) = 1/z$, which is the composition of inversion in the unit circle followed by reflection in the x-axis. Then $h(\partial D')$ is a line L. To determine L we need to know two points on it. Since points on unit circle are fixed by inversion, h reflects the unit circle in the x-axis. In particular, the two intersection points of ∂D and $\partial D'$ lie on the unit circle ∂D, so are reflected by $h(z)$, and lie on $\partial D'$ so their images lie on L. These two intersection points lie on the line $x = -1/2$ so L is its reflection in the x-axis, which is again the line $x = -1/2$.
3) Let $k(z) = z + 1/2$, which translates the line $x = 1/2$ to the y-axis. Let
$$q(z) = k \circ h \circ g(z) = \frac{1}{z-1} + \frac{1}{2} = \frac{2+z-1}{2(z-1)} \frac{1}{2}\left(\frac{z+1}{z-1}\right)$$
Then $q(\partial D)$ is the y-axis and since $q(0) = -1/2$ we see that $q(D)$ is the left half plane.
4) Rotate clockwise by $\pi/2$ (i.e. multiply by $-i$) to rotate the left half plane to H. We can also drop the factor of $1/2$ which is only a rescaling taking H to itself. Therefore a solution is given by $f(z) = -i\left(\frac{z+1}{z-1}\right)$. □

The solution found above is called the *Cayley transformation*.

Theorem (Schwarz Reflection Principle for a circle). *Let D be a domain which equals its inversion in the unit circle C. (i.e. $z \in D$ if and only if $-1/\bar{z} \in D$.) Let $I = D \cap C$. Let $U = \{z \in D \mid |z| < 1\}$ and let $\hat{U} = U \cup I$. Let $f : \hat{U} \to \mathbb{C}$ be a continuous function such that the restriction $f\big|_U$ is holomorphic and the restriction $f\big|_I$ is real-valued. Then f has a holomorphic extension to D.*

Proof. Let $C(z)$ be the Cayley transformation. Apply the Schwarz Reflection Principle to $(f \circ C^{-1})(z)$ to get an extension $g(z)$ of $(f \circ C^{-1})(z)$. Then $\bar{f} := (C \circ g)(z)$ is the desired extension of f. □

Example. Show that there does not exist a conformal map from the annulus $A = \{z \mid 1 < |z| \le 2\}$ to the punctured disk $D' = \{z \mid 0 < |z| \le 2\}$

Solution. Assume that there exists a conformal map f from A to D'. Let (w_n) be a sequence in A which converges to some point w on the circle $C = \{z \mid |z| = 1\}$. The set of images $\{f(w_n)\}$ is an infinite set so it has an accumulation point L (MATB43) lying in the closure $\bar{D} = \{z \mid 0 \le |z| \le 1\}$ of D'. If L lay in D' then $f^{-1}(L)$ would be an accumulation point of $\{w_n\}$ in A. But $\{w_n\}$ has no accumulation points in A since (w_n) converges to a point outside A. Therefore L does not lie in D' so $L = 0$. It follows that if we extend f to the closure $\bar{A} = \{z \mid 1 \le |z \le 2\}$ of A by setting $f(z) := 0$ for $z \in C$, the resulting extension is continuous. Therefore by the Schwartz Reflection for a circle, we can further extend f to a holomorphic function $\bar{f}$ on $\{z \mid 1/2 < |z| < 2\}$. But $\bar{f}$ is zero on C which contains a convergent sequence. By our first theorem on analytic continuation (uniqueness) this implies that $\bar{f}(z) = 0$ for all z, contradicting the fact that the origin f was a bijection from A to D'. Thus there is no conformal map from A to D'. □

Leaving aside now the difficult question of which regions are conformally equivalent, there remains the question of how to find a formula for a conformal map in cases where one exists. Fractional linear transformations are very useful for producing conformal maps (e.g. the Cayley transformation discussed earlier), so we consider their properties in greater detail.

Theorem. *Given any 3 distinct point p, q, $r \in \mathbb{C}\cup\{\infty\}$ and any other 3 distinct points p', q', $r' \in \mathbb{C}\cup\{\infty\}$, $\exists$ a fractional linear transformation $f(z)$ such that $f(p) = p'$, $f(q) = q'$, $f(r) = r'$.*

Proof. It suffices to consider the special case where $p = 0$, $q = 1$, $r = \infty$ since if we can solve this special case then there exists a fractional linear transformation $g(z)$ sending $(0, 1, \infty)$ to (p, q, r) and and a fractional linear transformation $h(z)$ sending $(0, 1, \infty)$ to (p', q', r') so we can set $f := h \circ g^{-1}$.

Want

$$p' = f(0) = b/d \tag{1}$$
$$q' = f(1) = (a + b)/(c + d) \tag{2}$$
$$p' = f(\infty) = a/c \tag{3}$$

$(1) \Rightarrow b = p'd$
$(3) \Rightarrow a = r'c$
Therefore $(2) \Rightarrow q' = \frac{r'c+p'd}{c+d} \Rightarrow q'c + q'd = r'c + p'd \Rightarrow (q' - r')c = (q' - p')d$

If $r' \neq \infty$,

$$\text{Choose} \qquad c := 1 \qquad d := \frac{q' - r'}{q' - p'} \qquad a := r' \qquad b := \frac{p'(q' - r')}{q' - p'}.$$

using that the denominators are nonzero since p', q', r' are distinct. Notice that

$$ad - bc = \frac{r'(q' - r')}{q' - r'} - \frac{p'(q' - r')}{p' - q'} = \frac{(r' - p')(q' - p')}{q' - p'} \neq 0$$

since $r' \neq p'$ and $r' \neq q'$.

Of $' = \infty$,

$$\text{Choose} \qquad c := 0 \qquad a := 1 \qquad d := \frac{1}{q' - p'} \qquad b := \frac{p'}{q' - p'}.$$

In this case $ad - bc = \frac{1}{q'-p'} \neq 0$. □

Lemma. *Let $f(z)$ be a fractional linear transformation. If $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$ then $f =$ identity.*

Proof. Write $f(z) = (az + b)/(cz + d)$.

$$0 = f(0) = b/d \tag{1}$$
$$1 = f(1) = (a + b)/(c + d) \tag{2}$$
$$\infty = f(\infty) = a/c \tag{3}$$

$(1) \Rightarrow b = 0$
$(3) \Rightarrow c = 0$
Therefore $(2) \Rightarrow 1 = a/d \Rightarrow d = a$. Thus $f(z) = az/a = z$. i.e. $f =$ identity. □

Corollary. *If a fractional linear transformation has 3 fixed points it is the identity.*

Proof. Suppose p, q, r are fixed points of f. By previous theorem, $\exists$ a fractional linear transformation g such that $g(0) = p$, $g(1) = q$, $g(\infty) = r$. Then $g^{-1}fg$ fixes 0, 1, and ∞. Therefore $g^{-1}fg =$ identity. Thus $f = gg^{-1}fgg^{-1} = g \circ \text{identity} \circ g^{-1} = \text{identity}$. $\square$

Corollary. *A fractional linear transformation is determined by its values on any 3 points.*

Proof. Suppose f and g are fractional linear transformations with $f(p) = q(p)$, $f(q = g(q)$, $f(r) = g(r)$. The p, q, r are fized points of $g^{-1} \circ f$. Thus $g^{-1} \circ f$ is the identity, so $f = g$. $\square$

Cross Ratio

We know that for any points p, q, r, p', q', r' there exists a unique fractional linear transformation f such that $f(p) = p'$, $f(q) = q'$, $f(r) = r'$. Would like a convenient way of finding f.

Notation: Given $a, b, c, d \in \mathbb{C} \cup \infty$ (with at least three distinct), set

$$(a, b; c, d) := \frac{(a-c)/(a-d)}{(b-c)/(b-d)} = \left(\frac{a-c}{b-c}\right)\left(\frac{b-d}{a-d}\right),$$

called the *cross-ratio* of a, b, c, d. If any of the elements is ∞, define it by $\lim_{t\to\infty}$. e.g. $(\infty, b; c, d) = \lim_{t\to\infty}\left(\frac{t-c}{b-c}\right)\left(\frac{b-d}{t-d}\right) = \lim_{t\to\infty}\frac{(1-c/t)(b-d)}{(1-d/t)(b-c)} = \frac{b-d}{b-c}$.

For a variable z,

$$T(z) = (z, a; b, c) = \frac{(z-b)/(z-c)}{(a-b)/(a-c)} = \frac{(z-b)(a-c)}{(z-c)(a-b)}$$

is a fractional linear transformation. Notice that $T(a) = \left(\frac{a-b}{a-c}\right)\left(\frac{a-c}{a-b}\right) = 1$, $T(b) = 0$, and $T(c) = \infty$.

Example. Find the fractional linear transformation f such that $f(1) = 1$, $f(-i) = 0$, $f(-1) = \infty$.

Solution. $f(z) = (z, 1; -i, -1) = \left(\frac{z+i}{z+1}\right)\left(\frac{1+1}{1+i}\right) = \frac{2z+2i}{(z+1)(1+i)}$. $\square$

Example. Find the fractional linear transformation f such that $f(i) = i$, $f(\infty) = 3$, $f(0) = -1/3$.

Solution1. $f = h^{-1} \circ g$ where $g(i) = 1$, $g(0) = 0$, $g(\infty) = \infty$ and $h(i) = i$, $h(-1/3) = 0$, $h(3) = \infty$.

$$g(z) = (z, i; 0, \infty) = \frac{z-0}{i-0} = \frac{z}{i} = -iz = \mathcal{F}\begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix}$$

$$h(z) = (z, i; -1/3, 3) = \left(\frac{z+1/3}{z-3}\right)\left(\frac{i+1/3}{i-3}\right) = \frac{(z+1/3)(i-3)}{(z-3)(i+1/3)} = \frac{(3z+1)(i-3)}{(z-3)(3i+1)}$$

$$= \frac{3(i-3)z + i - 3}{(3i+1)z - 3(3i+1)} = \mathcal{F}\begin{pmatrix} 3(i-3) & i-3 \\ 3i+1 & -3(3i+1) \end{pmatrix}$$

Inverting the matrix gives $h^{-1}(z) = \mathcal{F}\begin{pmatrix} -3(3i+1) & 3-i \\ -3i-1 & 3(i-3) \end{pmatrix}$. Therefore

$$\begin{aligned}
f(z) = (h^{-1} \circ g)(z) &= \mathcal{F}\left(\begin{pmatrix} -3(3i+1) & 3-i \\ -3i-1 & 3(i-3) \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}\right) \\
&= \mathcal{F}\begin{pmatrix} 3i(3i+1) & 3-i \\ i(3i+1) & 3(i-3) \end{pmatrix} = \mathcal{F}\begin{pmatrix} -3(3-i) & 3-i \\ -(3-i) & -3(3-i) \end{pmatrix} = \mathcal{F}\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} \\
&= \frac{3z-1}{z+3}
\end{aligned}$$

□

Proposition. *The fractional linear transformation $f(z)$ such that $f(a) = a'$, $f(b) = b'$, $f(c) = c'$ is given by the solution of $(z, a; b, c) = \big(f(z), a; b'c'\big)$. i.e.*

$$\frac{(z-b)/(z-c)}{(a-b)/(a-c)} = \frac{\big(f(z)-b'\big)/\big(f(z)-c'\big)}{(a'-b')/(a'-c')}$$

Proof. The fractional linear transformations on the left and right take on the same values at 0, 1, and ∞ if and only if f takes on the specified values. □

Solution2 to previous example. Set $w = f(z)$.

$(z, i; \infty, 0) = (w, i; 3, -1/3)$.

Therefore

$$\begin{aligned}
&\frac{i}{z} = \frac{(w-3)(i+1/3)}{(w+1/3)(i-3)} = \frac{(w-3)(3i+1)}{(3w+1)(i-3)} \\
&i(3wi - 3 + i - 9w) = z(3iw - pi - 3 + w) \\
&-3w - 3i - 1 = 9wi = 3izw - piz - 3z + zw \\
&-3w - 9wi - 3izw - zw = -9iz - 3 + 3i + 1 \\
&w = \frac{(-3-9i)z + (1+3i)}{(-1-3i)z - (3+9i)} = \frac{-3(1+3i)z + (1+3i)}{-(1+3i)z - 3(1+3i)} = \frac{3z-1}{z+3}
\end{aligned}$$

□

8. Infinite Product Decompositions

In this section, let $\log(z)$ denote a branch of the logarithm function defined on the domain $\mathbb{C} - \{(x,0) \mid x \leq 0\}$.

Suppose $|a_n| < 1$ for all n. We write $\prod_{k=1}^{\infty}(1+a_k)$ for $\lim_{n\to\infty}\prod_{k=1}^{n}(1+a_k)$ provided the limit exists and is greater than 0. The product is called divergent if either the limit does not exist or if it exists but equals 0. The motivation behind this convention is that we want $L = \prod_{k=1}^{\infty}(1+a_k)$ if and only if $\log L = \sum_{k=1}^{\infty}\log(1+a_k)$ and the latter would not converge if $L = 0$. The fact that this holds with our actual definition is an immediate consequence of the continuity of $\log(z)$.

Lemma.
1) *If $\prod_{k=1}^{\infty}(1+|a_k|)$ converges then $\prod_{k=1}^{\infty}(1+a_k)$ converges.*
2) *$\prod_{k=1}^{\infty}(1+|a_k|)$ converges if and only if $\sum_{k=1}^{\infty}|a_k|$ converges.*

Proof.
1) If $\prod_{k=1}^{\infty}(1+|a_k|)$ converges then $\sum_{k=1}^{\infty}\log(1+|a_k|)$ converges which implies by comparison that $\sum_{k=1}^{\infty}\log(1+a_k)$ converges and so $\prod_{k=1}^{\infty}(1+a_k)$ converges.
2) Unless $\lim_{n\to\infty}|a_n| = 0$ both sides diverge, so assume $\lim_{n\to\infty}|a_n| = 0$. Since $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$, for any constant $\epsilon > 0$, we have $1 < \frac{\log(1+x)}{x} < 1+\epsilon$ for all sufficiently small $x > 0$. Then $x < \log(1+x) < (1+\epsilon)x$ for all sufficiently small x, so by the Comparison Theorem, $\sum_{k=1}^{\infty}\log(1+|a_k|)$ converges if and only if $\sum_{k=1}^{\infty}|a_k|$ converges. $\square$

A holomorphic function whose domain is the entire complex plane is called *entire*. Clearly composition of entire functions gives an entire function. If $g(z)$ is an entire function, it is clear that $e^{g(z)}$ is an entire function with no zeros. Conversely

Theorem. *Let $f(z)$ be an entire function with no zeros. Then $f(z) = e^{g(z)}$ for some entire function $g(z)$.*

Proof. Since f has no zeros, f'/f is holomorphic. Let $\frac{f'(z)}{f(z)} = a_0 + a_1 z + a_2 z^2 + \ldots$ be the Taylor series of f'/f. Its radius of convergence is ∞ since f'/f is entire. Therefore (MATB43) the radius of the integrated series $\tilde{g}(z) := a_0 z + a_1 z^2/2 + a_2 z^3/3 + \ldots$ is also ∞. Thus $\tilde{g}(z)$ is an entire function with $\tilde{g}'(z) = \frac{f'(z)}{f(z)}$. Set $h(z) := e^{\tilde{g}(z)}$. Differentiating shows $\frac{h'(z)}{h(z)} = \tilde{g}'(z)$. Since f has no zeros, $h(z)/f(z)$ is entire and, since $\frac{h'(z)}{h(z)} = \frac{f'(z)}{f(z)}$, differentiating gives $(h/f)'(z) = 0$ for all z. Hence $kf(z) = h(z) = e^{\tilde{g}(z)}$ for some constant k. Setting $z = 0$ shows $k \neq 0$ so $k = e^c$ for some c. Therefore $f(z) = e^{g(z)}$ where $g(z) = \tilde{g} + c$. $\square$

Given a sequence of (a_n) of nonzero complex numbers, none occurring more than finitely many times, the Weierstrass Problem is to find an entire function (if one exists) whose zeros are pricely at the points $\{a_n\}$ with the order of the zero at a_n equal to the number of times a_n occurs in the sequence. One might think of $\prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right)$, but this will usually not converge so we modify it slightly. We will need a condition on the a_n's which will guarantee that their size increases quickly enough.

Lemma. *Suppose* $\sum_{n=1}^{\infty} \frac{1}{|a_n|^2} < \infty$. *Then* $f(z) = \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{a_n}\right) e^{z/a_n}\right]$ *solves the Weierstrass Problem for the sequence* (a_n).

Remark. In general, convergence of a series depends not only on its elements but also upon the order. In this case, since the convergence condition is given in terms of the absolute value, the order is irrelevant. Of course, since the function $f(z)$ determines only the set of its zeros and not the order in which we placed them in the sequence, for the answer to converge to $f(z)$ the right hand side must be independent of the numbering of the roots.

Remark. A function having the form of $f(z)$ is called a *canonical product.*

Proof. Set $f_n(z) = \prod_{k=1}^{n} \left(1 - \frac{z}{a_k}\right) e^{z/a_k}$. Since $f_n(z)$ is differentiable for all n, from the power series section, we know that to show that f is differentiable it suffices that each point has a neighbourhood in which $f_n(z)$ converges uniformly to $f(z)$. In particular, since every point in contained in a sufficiently large ball, it suffices to show that $f_n(z)$ converges uniformly to $f(z)$ on the closed ball $B_R[0]$ for every R.

SubLemma. *If* $|b| < 1$ *then* $|(1-b)e^b - 1| \leq \frac{|b|^2}{1-|b|}$.

Proof.

$$(1-b)e^b = (1 + b + \frac{b^2}{2!} + \frac{b^3}{3!} + \frac{b^4}{4! \ldots}) - (b + b^2 + \frac{b^3}{2!} + \frac{b^4}{3!} + \ldots)$$
$$= 1 + \sum_{n=2}^{\infty} \left(\frac{b^n}{n!} - \frac{b^n}{(n-1)!}\right) = 1 + \sum_{n=2}^{\infty} -b^n \left(\frac{1}{(n-1)!}\right)\left(1 - \frac{1}{n}\right)$$

Therefore

$$|(1-b)e^b - 1| = \left|\sum_{n=2}^{\infty} -b^n \left(\frac{1}{(n-1)!}\right)\left(1 - \frac{1}{n}\right)\right| \leq \sum_{n=2}^{\infty} |b|^n = |b|^2 \sum_{n=0}^{\infty} |b|^n = |b|^2 \frac{1}{1-|b|}$$

where the final step makes use of $|b| < 1$. ☐

Proof of Lemma (cont.). Since $\sum_{n=1}^{\infty} \frac{1}{|a_n|^2}$ converges, there exists N such that $|a_n| > 2R$ for $n \geq N$. If $|z| \in B_R[0]$ and $n \geq N$ then $|z/a_n| < 1$ and so from the SubLemma

$$\left|\left(1 - \frac{z}{a_n}\right) e^{z/a_n} - 1\right| \leq \frac{|z/a_n|^2}{1 - |z/a_n|} = \frac{|z|^2}{(1 - |z/a_n|)} \frac{1}{|a_n|^2} \leq \frac{R^2}{(1-1/2)} \frac{1}{|a_n|^2} = 2R^2 \frac{1}{|a_n|^2}$$

Therefore $\sum_{k=1}^{n} \left(1 - \frac{z}{a_k}\right) e^{z/a_k} - 1$ converges uniformly to $\sum_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right) e^{z/a_k} - 1$ and so it follows from the previous Lemma that $f_n(z)$ converges uniformly to $f(z)$. ☐

If $f_1(z)$ and $f_2(z)$ both solve the same Weierstrass Problem then f_1/f_2 has removable singularities, so extends to an entire function without zeros, and thus $f_1/f_2 = e^{g(z)}$ for some entire function $g(z)$. If we wish to include a zero of order k at the origin in our function we can just multiply the preceding function by z^k. Summing up, we have

Theorem. *Suppose* $\sum_{n=1}^{\infty} \frac{1}{|a_n|^2} < \infty$. *Then for any entire function* $g(z)$, *the function* $f(z) = e^{g(z)} z^k \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{z/a_n}$ *is an entire function which has a zero of order* k *at the origin and a zero at each* a_n *(where some* a_n*'s might be repeated resulting in a zero of higher order) but no other zeros. Conversely every function with these properties has this form for some entire function* $g(z)$.

Example. Show that $\sin(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right)$

Solution. The zeros of $\sin z$ are at πk where k is an integer and $\displaystyle\sum_{(k \neq 0) \in \mathbb{Z}} \frac{1}{\pi^2 k^2}$ converges so the theorem applies to give

$$\sin(z) = e^{g(z)} z \prod_{\text{nonzero roots } a_n \text{ of } \sin z} \left(1 - \frac{z}{a_n}\right) e^{z/a_n}$$

for some entire function $g(z)$. As noted earlier, the absolute convergence of the series means that the ordering of the roots does not affect the answer, so we will choose the order $(\pi, -\pi, 2\pi, -2\pi, 3\pi, -3\pi, \ldots)$. Therefore

$$\begin{aligned}
\sin(z) &= e^{g(z)} z \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{n\pi}\right) e^{z/(\pi n)} \left(1 - \frac{z}{-\pi n}\right) e^{z/(-\pi n)}\right] \\
&= e^{g(z)} z \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{\pi n}\right)\left(1 + \frac{z}{\pi n}\right)\right] = e^{g(z)} z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right)
\end{aligned}$$

so it remains to show that $g(z) = 1$. Logarithmic differentiation gives

$$\begin{aligned}
\cot(z) &= \frac{d \log\big(\sin(z)\big)}{dz} = \frac{d\left(g(z) + \log(z) + \sum_{n=1}^{\infty} \log(1 - \frac{z^2}{\pi^2 n^2})\right)}{dz} \\
&= g'(z) + \frac{1}{z} + \sum_{n=1}^{\infty} \frac{-2z/(\pi^2 n^2)}{1 - z^2/(\pi^2 n^2)} = g'(z) + \frac{1}{z} + \sum_{n=1}^{\infty} \frac{-2z}{\pi^2 n^2 - z^2} \\
&= g'(z) + \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - \pi^2 n^2}
\end{aligned}$$

(Note: There is no branch of the logarithm which is defined everywhere, but the method above can be used in the vicinity of any particular point z using a branch defined at $\cot z$. Alternatively, one could regard the preceding calculation as only providing intuition and verify the above formula for $\cot(z) = \sin(z)'/\sin(z)$ by the more laborious process of (non-logarithmic) differentiation followed by division.)

Comparing with our earlier formula for $\cot(z)$, we get $g'(z) = 0$ so $g = c$ for some constant c. Therefore $\frac{\sin(z)}{z} = C \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right)$ with $C = e^c$. Taking the limit $z \to 0$ gives $C = 1$. $\square$

9. Gamma Function

Since $\sum_{n=1}^{\infty} \frac{1}{|(-n)|^2}$ converges, the canonical $G(z) := \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$ defines a holomorphic function with zeros at the negative integers. Our earlier example shows that $\sin(z) = \pi z G(z) G(-z)$.

Set $H(z) := G(z-1)$. Then the zeros of $H(z)$ are the same as those of $zG(z)$ so we get $H(z) = e^{g(z)} z G(z)$ for some entire function $g(z)$. Therefore

$$\frac{d\log\big(H(z)\big)}{dz} = g'(z) + \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1/n}{1+z/n} - \frac{1}{n} \right) = g'(z) + \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{n+z} - \frac{1}{n} \right)$$

However the definition of $H(z)$ gives

$$\begin{aligned} \frac{d\log\big(H(z)\big)}{dz} = \frac{d\log\big(G(z-1)\big)}{dz} &= \sum_{n=1}^{\infty} \left(\frac{1/n}{1+(z-1)/n} - \frac{1}{n} \right) = \sum_{n=1}^{\infty} \left(\frac{1}{n+z-1} - \frac{1}{n} \right) \\ &= \frac{1}{z} - 1 + \sum_{n=2}^{\infty} \left(\frac{1}{n+z-1} - \frac{1}{n} \right) = \frac{1}{z} - 1 + \sum_{n=1}^{\infty} \left(\frac{1}{n+z} - \frac{1}{n+1} \right) \end{aligned}$$

Therefore comparing gives

$$g'(z) = -1 + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = -1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots = 0$$

Thus $g'(z)$ is a constant γ, known as "Euler's constant", and we get $G(z-1) = ze^{\gamma} G(z)$. To evaluate γ, set $z := 1$ yielding

$$\begin{aligned} e^{-\gamma} = G(1)/G(0) = G(1) &= \prod_{n=1}^{\infty} \left[\left(1 + \frac{1}{n}\right) e^{-1/n} \right] = \prod_{n=1}^{\infty} \left[\left(\frac{n+1}{n} \right) e^{-1/n} \right] \\ &= \lim_{k\to\infty} \left[\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)\left(\frac{4}{3}\right) \cdots \left(\frac{k+1}{k}\right) e^{-1-1/2-1/3-1/4-\ldots-1/k} \right] \\ &= \lim_{k\to\infty} \left[(k+1) e^{-1-1/2-1/3-1/4-\ldots-1/k} \right] \\ &= \lim_{k\to\infty} \left(k e^{-1-1/2-1/3-1/4-\ldots-1/k} \right) + \lim_{k\to\infty} \left(e^{-1-1/2-1/3-1/4-\ldots-1/k} \right) \\ &= \lim_{k\to\infty} \left(k e^{-1-1/2-1/3-1/4-\ldots-1/k} \right) \end{aligned}$$

since $\lim_{k\to\infty}\left(e^{-1-1/2-1/3-1/4-\ldots-1/k}\right) = 0$ due to the fact that $\sum_{k=1}^{\infty} \frac{-1}{k}$ diverges to $-\infty$. Therefore taking log( ) and solving for γ gives

$$\gamma = \lim_{k\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k} - \log(k) \right) \approx 0.577216$$

Define the Gamma function by

$$\Gamma(z) := \left(z e^{\gamma z} G(z) \right)^{-1} = \left(z e^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) e^{-z/n} \right] \right)^{-1}$$

It is a meromorphic function with simple poles at 0 and the negative integers.

Since $G(z-1) = ze^{\gamma}G(z)$,

$$\Gamma(z) = \frac{1}{ze^{\gamma z}G(z)} = \frac{ze^{\gamma}}{ze^{\gamma z}G(z-1)} = \frac{1}{e^{\gamma(z-1)}G(z-1)} = (z-1)\Gamma(z-1)$$

By definition of γ, $1 = G(0) = e^{\gamma}G(1) = \frac{1}{\Gamma(1)}$. Therefore $\Gamma(1) = 1$ and inductively $\Gamma(n) = (n-1)!$ for integer $n \geq 1$.

$$\Gamma(1-z)\Gamma(z) = (-z)\Gamma(-z)\Gamma(z) = \frac{1}{ze^{-\gamma z}G(-z)e^{\gamma z}G(z)} = \frac{1}{zG(-z)G(z)} = \frac{\pi}{\sin(\pi z)}$$

It follows that $\Gamma(z) \neq 0$ for any z in its domain.

It is clear from the definition of $\Gamma(z)$ that if x is real then $\Gamma(x)$ is real. Furthermore, since $\Gamma(x)$ has no zeros in its domain, which includes the positive x-axis, its restriction to the positive x-axis does not change sign and so $\Gamma(x) > 0$ for all $x \geq 0$.

The formula $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$ shows that $\Gamma(1/2)^2 = \pi$. Since $\Gamma(1/2) > 0$ this implies $\Gamma(1/2) = \sqrt{\pi}$.

Theorem (Euler).

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)\cdots(z+n)} = \frac{1}{z}\prod_{n=1}^{\infty}\left(\left(1+\frac{1}{n}\right)^z\left(1+\frac{z}{n}\right)^{-1}\right)$$

Proof.

$$\begin{aligned}
\frac{1}{\Gamma(z)} &= z\left(\lim_{n\to\infty} e^{(1+1/2+\ldots+1/n-\log n)z}\right)\left(\lim_{n\to\infty}\prod_{k=1}^{n}\left[\left(1+\frac{z}{k}\right)e^{-z/k}\right]\right) \\
&= z\lim_{n\to\infty}\left(e^{(1+1/2+\ldots+1/n-\log n)z}\left[\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)e^{-z/k}\right]\right) \\
&= z\lim_{n\to\infty}\left(\prod_{k=1}^{n} e^{z/k}e^{-(\log n)z}\left[\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)e^{-z/k}\right]\right) = z\lim_{n\to\infty}\left(n^{-z}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)\right) \\
&= z\lim_{n\to\infty}\left(\frac{1}{n^z n!}\prod_{k=1}^{n}(k+z)\right) = \lim_{n\to\infty}\left(\frac{z(1+z)\cdots(k+z)}{n^z n!}\right)
\end{aligned}$$

which gives the first identity. Also $n = \left(\frac{2}{1}\right)\left(\frac{3}{2}\right)\cdots\left(\frac{n}{n-1}\right) = \prod_{k=1}^{n-1}\left(\frac{k+1}{k}\right) = \prod_{k=1}^{n-1}\left(1+\frac{1}{k}\right)$ so

$$\begin{aligned}
\frac{1}{\Gamma(z)} &= z\lim_{n\to\infty}\left(n^{-z}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)\right) = z\lim_{n\to\infty}\left[\left(\prod_{k=1}^{n-1}\left(1+\frac{1}{k}\right)\right)^{-z}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)\right] \\
&= z\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{-z}\left(\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)\right)^{-z}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)\right] \\
&= z\lim_{n\to\infty}\left[\left(\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)\right)^{-z}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)\right]
\end{aligned}$$

since $\lim_{n\to\infty}(1+\frac{1}{n})^{-z}=1$. The second identity follows. $\square$

Theorem (Gauss). *For any integer $n \geq 2$,*

$$\Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\cdots\Gamma\left(z+\frac{n-1}{n}\right)=(2\pi)^{(n-1)/2}n^{(1/2)-nz}\Gamma(nz)$$

Proof. Set $f(z) := \frac{n^{nz}\Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\cdots\Gamma\left(z+\frac{n-1}{n}\right)}{n\Gamma(nz)}$. The theorem claims that $f(z)$ is the constant $(2\pi)^{(n-1)/2}n^{-(1/2)}$. We first show that $f(z)$ is constant.

$$\begin{aligned}\Gamma(z) &= \lim_{m\to\infty}\frac{m!m^z}{z(z+1)\cdots(z+m)} = \left(\lim_{m\to\infty}\frac{(m-1)!m^z}{z(z+1)\cdots(z+m-1)}\right)\left(\lim_{m\to\infty}\frac{m}{z+m}\right)\\ &= \lim_{m\to\infty}\frac{(m-1)!m^z}{z(z+1)\cdots(z+m-1)}\end{aligned}$$

When taking a limit as $m\to\infty$, we can if we wish, consider only the values of a subsequence. For example, we could choose some integer n and look only at terms indexed by multiples of n: in any convergent sequence $g(m)$, $\lim_{m\to\infty}g(m)=\lim_{m\to\infty}g(mn)$. Applying this gives

$$\Gamma(z)=\lim_{m\to\infty}\frac{(mn-1)!(mn)^z}{z(z+1)\cdots(z+mn-1)}$$

Then applying this rewritten form of Euler's Theorem to the denominator of $f(z)$ and the standard form to the numerator gives

$$f(z)=\frac{n^{nz-1}\prod_{k=0}^{n-1}\lim_{m\to\infty}\frac{(m-1)!m^{z+k/n}}{\left(z+\frac{k}{n}\right)\left(z+\frac{k}{n}+1\right)\cdots\left(z+\frac{k}{n}+m-1\right)}}{\lim_{m\to\infty}\frac{(mn-1)!(mn)^{nz}}{nz(nz+1)\cdots(nz+nm-1)}}$$

$$=\lim_{m\to\infty}\left[\frac{n^{nz-1}(m-1)!^n\prod_{k=0}^{n-1}m^{z+k/n}}{(mn-1)!(mn)^{nz}}\frac{nz(nz+1)\cdots(nz+nm-1)}{\prod_{k=0}^{n-1}\left(\left(z+\frac{k}{n}\right)\left(z+\frac{k}{n}+1\right)\cdots\left(z+\frac{k}{n}+m-1\right)\right)}\right]$$

$$=\lim_{m\to\infty}\left[\frac{n^{-1}(m-1)!^n\prod_{k=0}^{n-1}m^{k/n}}{(mn-1)!\,n^{-mn}}\frac{nz(nz+1)\cdots(nz+nm-1)}{\prod\limits_{k=0}^{n-1}\Big((nz+k)(nz+k+n)\cdots\big(nz+k+n(m-1)\big)\Big)}\right]$$

$$=\lim_{m\to\infty}\left[\frac{n^{-1}(m-1)!^n m^{(n-1)/2}}{(mn-1)!\,n^{-mn}}\frac{nz(nz+1)\cdots(nz+nm-1)}{\prod\limits_{k=0}^{n-1}\Big((nz+k)(nz+k+n)\cdots\big(nz+k+n(m-1)\big)\Big)}\right]$$

$$=\lim_{m\to\infty}\frac{n^{mn-1}(m-1)!^n m^{(n-1)/2}}{(mn-1)!}$$

since both the numerator and the denominator in the second factor consist of the product of the same nm numbers, beginning with nz and increasing by increments of 1 to $nz+nm-1$. Thus $f(z)$ is a constant. To evaluate the constant:

Notice that since $f(z)$ is a constant, $f(0) := \lim_{z\to 0} f(z)$ exists and by the definition of f is given by:

$$f(0) = \Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\cdots\Gamma\left(\frac{n-1}{n}\right)$$

Therefore, rearranging the order of multiplication,

$$\begin{aligned} f(0)^2 &= \Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{n-1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{n-2}{n}\right)\cdots\Gamma\left(\frac{n-1}{n}\right)\Gamma\left(\frac{1}{n}\right) \\ &= \prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}\right)\Gamma\left(1-\frac{k}{n}\right) = \prod_{k=1}^{n-1}\frac{\pi}{\sin(\pi k/n)} = \frac{\pi^{n-1}2^{n-1}}{n} = \frac{(2\pi)^{n-1}}{n} \end{aligned}$$

using our earlier formulas for $\Gamma(z)\Gamma(1-z)$ and for the product of the sin( ) terms. Therefore, since $f(0) > 0$ we get $f(0) = (2\pi)^{(n-1)/2}/\sqrt{n}$, as desired. □

The Gamma function is often defined instead by means of an improper integral.

Theorem (Euler). $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$

Proof. By uniqueness of analytic continuation, it suffices to check that the integral agrees with $\Gamma(z)$ when z is real and greater than 1. Set

$$f_n(x) := \begin{cases} t^{x-1}(1-t/n)^n & \text{if } 0 \le t \le n; \\ 0 & \text{if } n < t < \infty. \end{cases}$$

Then $f_1(x) \le f_2(x) \le \ldots \le f_n(x) \le t^{x-1}e^{-t}$ with $\lim_{n\to\infty} f_n(x) = t^{x-1}e^{-t}$. According to the Lebesgue Monotone Convergence Theorem (MATC37), these conditions imply

$$\begin{aligned} \int_0^\infty t^{x-1}e^{-t}\,dt &= \lim_{n\to\infty}\int_0^n f_n(t)\,dt \overset{(t:=n\tau)}{=} \lim_{n\to\infty}\int_0^1 (1-\tau)^n(n\tau)^{x-1}n\,d\tau \\ &= \lim_{n\to\infty} n^x\int_0^1 (1-\tau)^n\tau^{x-1}\,d\tau \\ &\overset{\text{(parts)}}{=} \lim_{n\to\infty} n^x\left[(1-\tau)^n\tau^x/x\Big|_0^1 + \frac{n}{x}\int_0^1 (1-\tau)^{n-1}\tau^x\,d\tau\right] \\ &= \lim_{n\to\infty} n^x\frac{n}{x}\int_0^1 (1-\tau)^{n-1}\tau^x\,d\tau \\ &\overset{\text{(parts)}}{=} \lim_{n\to\infty} n^x\frac{n(n-1)}{x(x+1)}\int_0^1 (1-\tau)^{n-2}\tau^{x+1}\,d\tau = \ldots \\ &= \lim_{n\to\infty} n^x\frac{n!}{x(x+1)\cdots(x+n-1)}\int_0^1 \tau^{x+n-1}\,d\tau \\ &= \lim_{n\to\infty} n^x\frac{n!}{x(x+1)\cdots(x+n)} = \Gamma(x) \end{aligned}$$

□

Example. Making the change of variable $u = x^2$ gives

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = 2\int_0^{\infty} e^{-x^2}\,dx = \int_0^{\infty} \frac{e^{-u}}{\sqrt{u}}\,du = \Gamma(1/2) = \sqrt{\pi}$$

$\square$

Remark. This integral, which figures prominently in STAB52, can also be calculated by the techniques of MATB42.

10. Laplace and Fourier Transforms

The Laplace Transform is studied in MATC46. We begin with a summary of its definition and basic properties.

Definition. Let $f : [0,\infty) \to \mathbb{C}$. The Laplace transform of f, denoted $\mathcal{L}(f)$ (or sometimes $\hat{f}$ when the context makes the meaning clear,) is the function

$$\mathcal{L}(f)\,(z) \equiv \int_0^{\infty} e^{-zt} f(t)\,dt.$$

The domain of $\mathcal{L}(f)$ is the set of z for which the integral converges.

This slightly generalizes the situation studied in MATC46 where only real values of z were considered.

Theorem. *If $\mathcal{L}(f)\,(z_0)$ converges, then $\mathcal{L}(f)\,(z)$ converges for all z such that $\operatorname{Re} z > \operatorname{Re} z_0$.* $\square$

Definition. Given α, a continuous function $f : [0,\infty) \to \mathbb{R}$ is said to have exponential order α if there exists a constant C such that $|f(x)| \le Ce^{\alpha x}$ for sufficiently large x. More precisely, f has exponential order if there exist constants C and b such that $|f(x)| \le Ce^{\alpha x}$ for $x > b$.

We write $f \in \xi_\alpha$ to mean that f is real-valued and has exponential order α.

Theorem. *If $f \in \xi_\alpha$, then $\mathcal{L}(f)\,(z)$ is defined for all z such that $\operatorname{Re} z > \alpha$.* $\square$

Theorem. *Properties of Laplace transforms:*

1. $\mathcal{L}(af + bg) = a\mathcal{L}(f) + b\mathcal{L}(g)$
2. $$\begin{aligned} \mathcal{L}(f')\,(z) &= z\mathcal{L}(f) - f(0), \\ \mathcal{L}(f'')\,(z) &= z\mathcal{L}(f') - f'(0) = z^2\mathcal{L}(f) - zf(0) - f'(0), \\ &\vdots \\ \mathcal{L}\big(f^{(n)}\big)\,(z) &= z^n\mathcal{L}(f) - z^{n-1}f(0) - z^{n-2}f'(0) - \cdots - f^{(n-1)}(0) \end{aligned}$$
3. If f and g are continuous on $[0,\infty)$ and $\mathcal{L}(f) = \mathcal{L}(g)$, then $f = g$.
4. $\mathcal{L}(e^{ax} f(x))\,(z) = \hat{f}(z-a)$

5. a) $\hat{f}$ is differentiable and $\mathcal{L}(x^n f(x))\,(z) = (-1)^n \hat{f}^{(n)}(z)$.
   b) $\hat{f}$ is integrable and $\mathcal{L}(f(x)/x)\,(z) = -\int_0^z \hat{f}(u)\,du$.
6. $\lim_{\mathrm{Re}\,z\to\infty} \hat{f}(z) = 0$
7. $\mathcal{L}(f * g) = \mathcal{L}(f)\,\mathcal{L}(g)$, where the convolution $f*g$ of f and g is defined by $(f*g)(x) := \int_0^x f(x-t)g(t)\,dt$. □

A few basic Laplace Transforms are given in the following table.

$f(x)$	$\mathcal{L}(f)\,(z)$
x^n	$\frac{\Gamma(n+1)}{z^{n+1}}$
e^{ax}	$\frac{1}{z-a}$
$\cos(ax)$	$\frac{z}{z^2+a^2}$, $\mathrm{Re}\,z > 0$
$\sin(ax)$	$\frac{a}{z^2+a^2}$
$x\cos(ax)$	$\frac{z^2-a^2}{(z^2+a^2)^2}$
$x\sin(ax)$	$\frac{2az}{(z^2+a^2)^2}$

Note: $z/\left(z^2+a^2\right)$ is, of course, defined for $z \neq \pm ia$, but it equals $\int_0^\infty e^{-zx}\cos(x)\,dx$ only when $\mathrm{Re}\,z > 0$. (The latter does not converge for $\mathrm{Re}\,z \le 0$.)

Example. Solve $y' - 4y = e^x$ with $y(0) = 1$.

Solution.

$$\begin{aligned}
\mathcal{L}(y' - 4y) &= \mathcal{L}(e^x) \\
\mathcal{L}(y') - 4\mathcal{L}(y) &= \frac{1}{z-1} \\
\frac{1}{z-1} &= z\mathcal{L}(y) - y(0) - 4\mathcal{L}(y) = (z-4)\,\mathcal{L}(y) - 1 \\
(z-4)\,\mathcal{L}(y) &= \frac{1}{z-1} + 1 = \frac{z}{z-1} \\
\mathcal{L}(y) &= \frac{z}{(z-1)\,(z-4)} = \frac{4}{3}\frac{1}{z-4} - \frac{1}{3}\frac{1}{z-1} \\
y &= \frac{4}{3}e^{4x} - \frac{1}{3}e^x.
\end{aligned}$$

□

A related transformation is the Fourier Transform.

Definition. Let $f : [0,\infty) \to \mathbb{C}$. The Fourier transform of f, denoted $\mathcal{F}(f)$ (or sometimes $\hat{f}$ when the context makes the meaning clear,) is the function

$$\mathcal{F}(f)\,(z) \equiv \int_{-\infty}^{\infty} e^{-izt} f(t)\,dt.$$

The domain of $\mathcal{F}(f)$ is the set of z for which the integral converges.

The domain of $\mathcal{F}(f)$ will, in general, be somewhat different from that of $\mathcal{L}(f)$. In particular, for real z the function e^{-izt} is periodic rather than approaching 0 as z increases. Hence for real-valued $f(t)$, unless $f(t) \to 0$ as $t \to 0$ the integral has no chance to converge. Like the Laplace Transform, it has the property of converting differentiation into multiplication.

Theorem. *Let $f(z)$ be meromorphic on $\mathbb{C}$ with no singularities on the x-axis. Suppose $|f(z)| \to 0$ as $|z| \to \infty$. Let $H := \{z \in \mathbb{C} \mid \operatorname{Im} z \geq 0\}$ be the upper half plane and $L := \{z \in \mathbb{C} \mid \operatorname{Im} z \leq 0\}$ be the lower half plane. Then for real $t < 0$*

$$\mathcal{F}(f)\,(t) = 2\pi i \sum \text{residues of } e^{-itz} f(z) \text{ at its singularities in } H$$

and for real $t > 0$

$$\mathcal{F}(f)\,(t) = -2\pi i \sum \text{residues of } e^{-itz} f(z) \text{ at its singularities in } L$$

Proof. Let $s = -t$. Consider case the case $t < 0$ so that $s > 0$. Let B be the rectangle $[-T, U] \times [0, S]$ where S, T, and U are chosen to be large enough so that B contains all the singularities of $f(z)$ in H. By the residue theorem

$$\int_{\partial B} e^{isz} f(z)\, dz = 2\pi i \sum \text{residues of } e^{isz} f(z) \text{ at its singularities in } H$$

$\int_{\partial B} e^{isz} f(z)\, dz = \int_{-T}^{U} e^{isz} f(z) + I_1 + I_2 + I_3$ where

$$\begin{aligned} I_1 &= \int_0^S e^{is(U+iy)} f(U+iy) i\, dy, \\ I_2 &= \int_U^{-T} e^{is(x+iS)} f(x+iS)\, dx, \\ \text{and} \qquad I_3 &= \int_S^0 e^{is(-T+iy)} f(-T+iy) i\, dy. \end{aligned}$$

□

Given $\epsilon > 0$ find R such that $|f(z)| < \epsilon$ for $|z| \geq R$. We may assume that S, T, and U, are chosen so that $S > T > R$ and $S > U > R$ and $e^{-sS}(T + U) < \epsilon$. Let

$$\begin{aligned} M_1 &= \max\{|f(U+iy)| \mid 0 \leq y \leq S\} < \epsilon \\ M_2 &= \max\{|f(x+iS)| \mid -T \leq x \leq U\} < \epsilon \\ M_3 &= \max\{|f(-T+iy)| \mid 0 \leq y \leq S\} < \epsilon \end{aligned}$$

using the choices of T, U, S, and R. Then

$$|I_1| \leq \int_0^S e^{-sy} |f(U+iy)|\, dy \leq M_1 \int_0^S e^{-sy}\, dy = M_1 \frac{e^{-sy}}{s}\Big|_0^S = \frac{M_1}{s}(1 - e^{-sS}) \leq \frac{M_1}{s} < \frac{\epsilon}{s}$$

and similarly $|I_3| \le \frac{\epsilon}{s}$. Also $|I_2| \le \int_{-T}^{U} e^{-sS}|f(x+iS)|\,dx \le M_2 e^{-sS}(T+U) < \epsilon^2$. Since this is true for all $\epsilon > 0$, taking the limit as $T \to \infty$ and $U \to \infty$ gives

$$2\pi i \sum \text{residues of } e^{-izt} f(z) \text{ at its singularities in } H = \mathcal{F}(f)\,(z).$$

If $t > 0$ then we consider instead a rectangle in L. The argument is the same, but the counter-clockwise orientation of the boundary curve results in the portion along the x-axis running backwards resulting in the minus sign. □

Example. Let $f(z) = \frac{1}{z^2+a^2}$, where $a > 0$. Then

$$\begin{aligned}
\int_0^\infty \frac{\cos x}{x^2+a^2}\,dx &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,dx = \frac{1}{2}\operatorname{Re}\big(\mathcal{F}(f)\,(-1)\big) \\
&= \operatorname{Re}\left(\pi i \sum \text{residues of } \tfrac{e^{iz}}{z^2+a^2} \text{ at its singularities in } H\right) \\
&= \operatorname{Re}\left(\pi i \operatorname{Res}_{ia} \frac{e^{iz}}{z^2+a^2}\right) = \operatorname{Re}\left(\pi i \frac{e^{iz}|_{z=ia}}{2z|_{z=ia}}\right) = \operatorname{Re}\left(\pi i \frac{e^{-a}}{2ia}\right) = \frac{\pi e^{-a}}{2a}
\end{aligned}$$

using $\operatorname{Res}_p \frac{g(z)}{h(z)} = \frac{g(p)}{h'(z)}$ when $g(z)$ is holomorphic at p and $h(z)$ has a simple pole at p. □

We return now to the Laplace transform. While property (3) of the earlier theorem says that a function is uniquely determined by its Laplace Transform, there remains the question of how to compute inverse Laplace Transforms. In simple cases one can find the inverse Laplace Transform by inspecting a table a table of Laplace Transforms. Our goal is to derive a general formula for the inverse Laplace Transform.

Given $F(z)$ we wish to find $f(x)$ such that $\mathcal{L}(f) = F$. According to property (6) of the theorem, there is no chance unless $\lim_{\operatorname{Re} z \to \infty} F(z) = 0$. We will need to assume the stronger condition that there exist positive constants M, β, and R such that $|F(z)| \le M/|z|^\beta$ whenever $|z| > R$. For example, if $F(z) = P(z)/Q(z)$ where $P(z)$ and $Q(z)$ are polynomials with $\deg Q > \deg P$, then the condition is satisfied.

Theorem. *Let $F(z)$ be a meromorphic function such that there exist positive constants M, β, and R such that $|F(z)| \le M/|z|^\beta$ whenever $|z| > R$. Let σ be the maximum of the real parts of the singularities of $F(z)$. Then the inverse Laplace transform of $F(z)$ on the domain $\operatorname{Re} z > \sigma$ is given by*

$$\mathcal{L}^{-1}(F)\,(t) = \sum \text{residues of } e^{zt}F(z) \text{ at its singularities} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\alpha+iy)t} F(\alpha+iy)\,dy$$

for any real $\alpha > \sigma$.

Proof. Suppose $\alpha > \sigma$ and consider the domain $\operatorname{Re} z > \alpha$. The bound on $|F(z)|$ implies, in particular, that $\lim_{|z|\to\infty} |F(z)| = 0$, so given $\epsilon > 0$ by choosing a larger R, we may assume that $|F(z)| < \epsilon$ whenever $|z| \ge R$. Let $\rho > \max\{R, |z|\}$ be arbitrary. Let B_1 and B_2 be the rectangles $B_1 := [\alpha, S] \times [T, U]$, $B_2 := [-S, \alpha] \times [T, U]$ where S, T, and U are chosen to be large enough so that B_2 contains all the singularities of $F(z)$ and $B_1 \cup B_2$ contains the ball $B_\rho[0]$. Set $\gamma_1 := \partial B_1$ and $\gamma_2 := \partial B_2$ be the boundaries with the counterclockwise orientation.

Set

$$f(t) := \sum \text{residues of } e^{zt}F(z) \text{ at its singularities.}$$

We wish to show that $\mathcal{L}(f) = F$.

By the residue theorem, $2\pi i f(t) = \int_{\gamma_2} e^{zt}F(z)\,dz$. Therefore by definition

$$\begin{aligned}
2\pi i \mathcal{L}(f)\,(z) &= \lim_{r\to\infty}\int_0^r e^{-zt}2\pi i f(t)\,dt = \lim_{r\to\infty}\int_0^r e^{-zt}\int_{\gamma_2} e^{wt}F(w)\,dw\,dt \\
&= \lim_{r\to\infty}\int_0^r\int_{\gamma_2} e^{(w-z)t}F(w)\,dw\,dt
\end{aligned}$$

This is a surface integral over the sides of the cube of height r over the base γ_2. Changing the order of integration gives

$$\begin{aligned}
2\pi i \mathcal{L}(f)\,(z) &= \lim_{r\to\infty}\int_{\gamma_2}\int_0^r e^{(w-z)t}F(w)\,dt\,dw = \lim_{r\to\infty}\int_{\gamma_2}\frac{e^{(w-z)t}F(w)}{w-z}\Big|_{t=0}^{t=r}\,dw \\
&= \lim_{r\to\infty}\int_{\gamma_2}\frac{e^{(w-z)r}F(w)}{w-z}\,dw - \int_{\gamma_2}\frac{F(w)}{w-z}\,dw \qquad , \\
&= \int_{\gamma_2}\lim_{r\to\infty}\frac{e^{(w-z)r}F(w)}{w-z}\,dw - \int_{\gamma_2}\frac{F(w)}{w-z}\,dw
\end{aligned}$$

where we used that it is a proper integral to move the limit inside. Since $\operatorname{Re} z > \alpha$ and $\operatorname{Re} w \le \alpha$ for w in the region of integration, $e^{(w-z)r} \to 0$ as $r \to \infty$, so the first term is 0 and we get

$$2\pi i \mathcal{L}(f)\,(z) = -\int_{\gamma_2}\frac{F(w)}{w-z}\,dw = \int_{\gamma_1}\frac{F(w)}{w-z}\,dw - \int_{\partial(B_1\cup B_2)}\frac{F(w)}{w-z}\,dw.$$

Since all the singularties of $F(z)$ lie to the left of $\operatorname{Re} z = \alpha$, $F(z)$ is holomorphic throughout B_1 and so $\int_{\gamma_1}\frac{F(w)}{w-z}\,dw = 2\pi F(z)$. Using the fact that $F(z)$ is holomorphic outside of $B_1 \cup B_2$ gives $\int_{\partial(B_1\cup B_2)}\frac{F(w)}{w-z}\,dw = \int_{\partial B_\rho[0]}\frac{F(w)}{w-z}\,dw$. However on $B_\rho[0]$, $|F(z)| \le M/|z|^\beta$, and so

$$\begin{aligned}
\left|\int_{\partial B_\rho[0]}\frac{F(w)}{w-z}\,dw\right| &\le \int_{\partial B_\rho[0]}\left|\frac{F(w)}{w-z}\right|\,dw \le \int_{\partial B_\rho[0]}\frac{M}{|z|^\beta\,|w-z|}\,dw \\
&\le \int_{\partial B_\rho[0]}\frac{M}{\rho^\beta(\rho-|z|)}\,dw \le \frac{2\pi M\rho}{\rho^\beta(\rho-|z|)}.
\end{aligned}$$

Since ρ is arbitrary, this implies the second term is 0 and so $\mathcal{L}(f)\,(z) = F(z)$, which is the first equality in the theorem. $\square$

For the restatement given by the second equality

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\alpha+iy)t}F(\alpha+iy)\,dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iyt}e^{\alpha t}F(\alpha+iy)\,dy = \frac{1}{2\pi}\mathcal{F}(G)\,(-t)$$

where $G(z) = e^{\alpha t}F(\alpha+iz)$. Since $t > 0$, the earlier theorem gives

$$\begin{aligned}\frac{1}{2\pi}\mathcal{F}(G)\,(-t) &= \frac{1}{2\pi}2\pi i\sum \text{residues of } e^{izt}G(z) \text{ at its singularities in } H \\ &= i\sum \text{residues of } e^{izt}G(z) \text{ at its singularities in } H\end{aligned}$$

p is a singularity of $e^{zt}F(z)$ if and only if $q := \frac{p-\alpha}{i} = i(\alpha - p)$ is a singularity of $e^{izt}G(z)$. The condition $i(\alpha-p) \in H$ is equivalent to the condition $\operatorname{Re} p < \alpha$ which is satisfied by all singularities of $e^{zt}F(z)$. Thus all singularities of $e^{izt}G(z)$ lie in H. Also, if $a = \operatorname{Res}_p e^{zt}F(z)$ then the Laurent expansion of $e^{zt}F(z)$ at p contains the term $\frac{a}{z-p}$. Making the change of variable $z := \alpha + iw$ gives that the Laurent expansion of $e^{\alpha+iw}F(\alpha+iw) = e^{wt}G(w)$ at the corresponding point $q = i(\alpha-p)$ contains the term $\frac{a}{\alpha+iw-p} = \frac{-ia}{w-q}$. Therefore

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\alpha+iy)t}F(\alpha+iy)\,dy &= i\sum \text{residues of } e^{izt}G(z) \text{ at its singularities in } H \\ &= i(-i)\sum \text{residues of } e^{zt}F(z) \text{ at its singularities} \\ &= \sum \text{residues of } e^{zt}F(z) \text{ at its singularities} = f(t)\end{aligned}$$

Therefore the equalities stated in the theorem hold in the domain $\operatorname{Re} z > \alpha$. Since α was arbitrary (aside from the condition $\alpha > \sigma$), they hold in the domain $\operatorname{Re} z > \sigma$. $\square$

Example. Let $F(z) = \frac{1}{3z-6}$. The conditions of the preceding theorem are satisfied so there is a function $f(t)$ such that $\mathcal{L}(f)\,(z) = F(z)$ and it is given by

$$f(t) = \sum \text{residues of } \tfrac{e^{zt}}{3z-6} \text{ at its singularities} = \operatorname{Res}_2 \frac{e^{zt}}{3z-6} = \frac{e^{zt}|_{z=2}}{(3z-6)'|_{z=2}} = \frac{e^{2t}}{3}.$$

In this simple case, obviously one could just look up the answer in the table instead.

11. Stirling's Formula

Letting $\tau := zt$ in the integral $\Gamma(z+1) = \int_0^\infty \tau^z e^{-\tau}\, d\tau$ gives

$$\Gamma(z+1) = \int_0^\infty (zt)^z e^{-zt} z\, dt = z^{z+1} \int_0^\infty e^{z\log(t)} e^{-zt} dt = z^{z+1} \int_0^\infty e^{zh(t)}\, dt$$

where $h(t) = \log(t) - t$.

We consider the case where z is real. Notice that $h'(t) = 1/t - 1$, so $h(t)$ has a global maximum at $t = 1$, with $h(1) = -1$. We will show that the portion of the integral around $t = 1$ dominates the rest of the integral. Some approximate values of $h(t)$ are:

$$\begin{aligned} h(1/2) &\approx -1.19 \\ h(1) &= -1 \\ h(2) &= -1.31 \end{aligned}$$

Choose constants $c \in h(1/2, -1)$ and $\epsilon > 0$ such that $h(t) < c - \epsilon$ for all $t \notin [1/2, 2]$. For example, let $c = -1.1$ and $\epsilon = 0.05$. Since $c < -1$, $h(t) = c$ has two solution, say a and b, with $1/2 < a < 1 < b < 2$. Then $h(t) \geq c$ for $t \in [a, b]$.

$$\Gamma(x+1)/x^{x+1} = \int_0^\infty e^{xh(t)}\, dt = \int_0^{1/2} e^{xh(t)}\, dt + \int_{1/2}^2 e^{xh(t)}\, dt + \int_2^\infty e^{xh(t)}\, dt.$$

Using the definitions of c and ϵ:

$$\begin{aligned} \int_0^{1/2} e^{xh(t)}\, dt + \int_2^\infty e^{xh(t)}\, dt &= \int_0^{1/2} e^{h(t)} e^{(x-1)h(t)}\, dt + \int_2^\infty e^{h(t)} e^{(x-1)h(t)}\, dt \\ &\leq e^{|x-1|(c-\epsilon)} \left(\int_0^{1/2} e^{h(t)}\, dt + \int_2^\infty e^{h(t)}\, dt \right) = Me^{|x-1|(c-\epsilon)} \end{aligned}$$

where $M = \left(\int_0^{1/2} e^{h(t)}\, dt + \int_2^\infty e^{h(t)}\, dt \right)$ is a constant, noting that the integrals defining M converge since they are bounded by $\Gamma(2)$. Similarly

$$\int_{1/2}^2 e^{xh(t)}\, dt \geq \int_a^b e^{xh(t)}\, dt = \int_a^b e^{h(t)} e^{(x-1)h(t)}\, dt \geq e^{|x-1|c} \int_a^b e^{h(t)}\, dt = M' e^{|x-1|c}$$

where M' is a constant. Therefore

$$\int_0^{1/2} e^{xh(t)}\, dt + \int_2^\infty e^{xh(t)}\, dt \leq e^{-\epsilon|x-1|} \frac{M}{M'} \int_{1/2}^2 e^{xh(t)}\, dt$$

We now consider the term $\int_{1/2}^2 e^{xh(t)}\, dt$ in detail.

The function $-\big(h(t)+1\big) = t-1-\log(t)$ is always non-negative, with a global minimum of 0 at $t=1$ where its tangent line is the x-axis. Define

$$w(t) := \begin{cases} -\sqrt{-h(t)-1} & \text{if } t \le 1 \\ \sqrt{-h(t)-1} & \text{if } t > 1. \end{cases}$$

Then $w(t)$ is monotonically increasing so it is invertible. Let $t = v(w)$ be the inverse function to w and let $v = 1 + a_1 w + a_2 w^2 + \dots$ be its Taylor expansion about 0, noting that the constant term is 1 since $w(1) = 0$ and thus $v(0) = 1$.

$$\int_{1/2}^{2} e^{xh(t)}\,dt = \int_{1/2}^{2} e^{-x} e^{x\left(h(t)+1\right)}\,dt = e^{-x}\int_{1/2}^{2} e^{-xw(t)^2}\,dt$$

Make change of variable $y := \sqrt{x}w(t)$, $t = v(y/\sqrt{x})$. Then $dt/dy = v'(w)/\sqrt{x}$ and so

$$e^{-x}\int_{1/2}^{2} e^{-xw(t)^2}\,dt = \frac{e^{-x}}{\sqrt{x}}\int_{p\sqrt{x}}^{q\sqrt{x}} e^{-y^2} v'(w)\,dy$$

where $p = w(1/2)$ and $q = w(2)$. Notice that the definition of w implies that $p < 0$ and $q > 0$.

$$\begin{aligned}
\int_{p\sqrt{x}}^{q\sqrt{x}} e^{-y^2} v'(w)\,dy &= \int_{p\sqrt{x}}^{q\sqrt{x}} e^{-y^2} \sum_{k=0}^{\infty}(k+1)a_{k+1}w^k\,dy \\
&= \int_{p\sqrt{x}}^{q\sqrt{x}} e^{-y^2} \sum_{k=0}^{\infty}(k+1)a_{k+1}\left(\frac{y}{\sqrt{x}}\right)^k dy \\
&= \int_{p\sqrt{x}}^{q\sqrt{x}} a_1 e^{-y^2}\,dy + \int_{p\sqrt{x}}^{q\sqrt{x}} e^{-y^2} \sum_{k=1}^{\infty}(k+1)a_{k+1}\left(\frac{y}{\sqrt{x}}\right)^k dy
\end{aligned}$$

For the second term, write $\sum_{k=1}^{\infty}(k+1)a_{k+1}\left(\frac{y}{\sqrt{x}}\right)^k = \left(\frac{y}{\sqrt{x}}\right)B(x,y)$ where

$$B(x,y) = 2a_2 + 3a_3\left(\frac{y}{\sqrt{x}}\right) + \dots = \frac{v'(w)-v'(0)}{w}\Big|_{w=y/\sqrt{x}}$$

Since $\frac{v'(w)-v'(0)}{w}$ is continuous (singularity at 0 is removable) there is a bound B for its values over the finite interval $[p,q]$ which is thus a uniform bound on $B(x,y)$ in the range of integration. Hence

$$\int_{p\sqrt{x}}^{q\sqrt{x}} e^{-y^2} \sum_{k=1}^{\infty}(k+1)a_{k+1}\left(\frac{y}{\sqrt{x}}\right)^k dy \le \int_{p\sqrt{x}}^{q\sqrt{x}} B\frac{|y|}{\sqrt{x}}e^{-y2}\,dy \le \frac{B}{\sqrt{x}}\int_{-\infty}^{\infty}|y|e^{-y2}\,dy = \frac{2B}{\sqrt{x}}.$$

$$\int_{p\sqrt{x}}^{q\sqrt{x}} e^{-y^2}\,dy = \int_{-\infty}^{\infty} e^{-y^2}\,dy - \int_{-\infty}^{p\sqrt{x}} e^{-y^2}\,dy - \int_{q\sqrt{x}}^{\infty} e^{-y^2}\,dy$$

Lemma. *There exists a constant K such that $\int_r^\infty e^{-y^2} \le \frac{K}{r}$ for all sufficiently large r. (In fact, the statement is true for every $K > 0$.)*

Proof. Choose $K > 0$. Since $\lim_{y\to\infty} y^2 e^{-y^2} = 0$, $\exists Y$ such that $e^{-y^2} < K/y^2$ for all $y \ge Y$. Then for $r > Y$,

$$\int_r^\infty e^{-y^2} \le \int_r^\infty \frac{K}{y^2}\, dy = K \frac{-1}{y}\Big|_r^\infty = \frac{K}{r}$$

□

The Lemma shows that $\int_{q\sqrt{x}}^\infty e^{-y^2}\, dy < \frac{K}{q\sqrt{x}}$ for sufficiently large x and similarly $\int_{-\infty}^{p\sqrt{x}} e^{-y^2}\, dy < \frac{K}{p\sqrt{x}}$ for sufficiently large x.

The Lemma's show that

$$\left| \int_{p\sqrt{x}}^{q\sqrt{x}} e^{-y^2}\, dy - \int_{-\infty}^{\infty} e^{-y^2}\, dy \right| < \frac{L}{\sqrt{x}}$$

for all sufficiently large x, where $L = K/q + K/p$.

Putting it all together, we showed

$$\left| \int_{1/2}^{2} e^{xh(t)}\, dt - \frac{e^{-x}}{\sqrt{x}} \int_{-\infty}^{\infty} a_1 e^{-y^2}\, dy \right| \le \frac{e^{-x}}{\sqrt{x}} \left(\frac{2B}{\sqrt{x}} + \frac{a_1 L}{\sqrt{x}} \right) \tag{1}$$

and

$$\left| \frac{\Gamma(x+1)}{x^{x+1}} - \frac{e^{-x}}{\sqrt{x}} \int_{-\infty}^{\infty} a_1 e^{-y^2}\, dy \right| \le e^{-\epsilon|x-1|} \frac{M}{M'} \int_{1/2}^{2} e^{xh(t)}\, dt + \frac{e^{-x}}{\sqrt{x}} \left(\frac{2B}{\sqrt{x}} + \frac{a_1 L}{\sqrt{x}} \right) \tag{2}$$

Since (1) shows that for large x, $\int_{1/2}^2 e^{xh(t)}\, dt \le C\frac{e^{-x}}{\sqrt{x}}$ for some constant C, and $e^{-\epsilon(x-1)} < 1/\sqrt{x}$ for large x, equation (2) can be rewritten as

$$\left| \frac{\Gamma(x+1)}{x^{x+1}} - \frac{e^{-x}}{\sqrt{x}} \int_{-\infty}^{\infty} a_1 e^{-y^2}\, dy \right| \le \frac{e^{-x}}{\sqrt{x}} \left(\frac{CM/M'}{\sqrt{x}} + \frac{2B}{\sqrt{x}} + \frac{a_1 L}{\sqrt{x}} \right)$$

In other words, for large x, $\frac{\Gamma(x+1)}{x^{x+1}}$ is approximately equal to $\frac{e^{-x}}{\sqrt{x}} \int_{-\infty}^\infty a_1 e^{-y^2}\, dy$ and the error in the approximation is sufficiently small that even when multiplied by $\sqrt{x}e^x$ it still approaches 0 as $x \to \infty$.

To complete the calculation, we need to find a_1 and $\frac{e^{-x}}{\sqrt{x}} \int_{-\infty}^\infty a_1 e^{-y^2}\, dy$.

In an earlier example, we showed:

Lemma. $\int_{-\infty}^\infty e^{-y^2}\, dy = \sqrt{\pi}$. □

$a_1 = v'(0)$. Since $w(1) = 0$, $v'(0) = 1/w'(1)$. By definition $w^2(t) = -h(t) - 1$. Therefore $2w(t)w'(t) = -h'(t)$. Substituting $t = 1$ at this point does not help, since $w(1) = 0$, so differentiate again to get $2\big(w'(t)\big)^2 + w(t)w''(t) = -h''(t)$. Thus

$$2\big(w'(1)\big)^2 + 0 = -h''(1) = \big(t - \log(t)\big)''|_{t=1} = 1$$

which gives $w'(1) = 1/\sqrt{2}$, taking the positive square root since $w(t)$ is monotonically increasing. Hence $a_1 = \sqrt{2}$.

$$\text{Stirling's Formula :} \qquad \Gamma(x+1) \approx x^{x+1}e^{-x}\sqrt{2}\sqrt{\pi}/\sqrt{x} = \sqrt{2\pi}x^{x+1/2}e^{-x}$$

Still better approximations could be obtained by taking more terms in the Taylor expansion of $v(w)$.

The formula is still valid when z is not real, but in the proof, instead of looking at the fixed interval $[1/2, 2]$ one concentrates on a small interval about 1 determined by $\operatorname{Im} z$ where the interval is chosen so that the deviation of $e^{i \operatorname{Im} z h(t)}$ from $e^{i \operatorname{Im} z h(1)}$ is small throughout the interval. Details omitted.

12. Riemann Zeta Function

In this section, the symbol p will indicate a prime.

The Riemann-zeta function is defined for $\operatorname{Re} z > 1$ by the convergent series $\zeta(z) := \sum_{n=1}^{\infty} n^z$.

Theorem. *For* $\operatorname{Re} z > 1$, *the infinite product* $\prod_p \left(1 - \frac{1}{p^z}\right)$ *converges and equals* $\frac{1}{\zeta(z)}$.

Proof. Since $\sum_p \frac{1}{p^z} \le \sum_{n=1}^{\infty} \frac{1}{n^z}$ converges for $\operatorname{Re} z > 1$ so does $\prod_p \left(1 - \frac{1}{p^z}\right)$.

Given z with $\operatorname{Re} z > 1$, for $\epsilon > 0$, $\exists N$ such that $\sum_{n=N+1}^{\infty} \frac{1}{n^z} < \epsilon$.

Collecting the terms with denominators divisible by 2 we get.

$$\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \frac{1}{5^z} + \ldots = \frac{1}{2^z}\left(1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \frac{1}{5^z} + \ldots\right) + 1 + \frac{1}{3^z} + \frac{1}{5^z} + \ldots$$

so $\left(1 - \frac{1}{2^z}\right)\zeta(z) = 1 + \frac{1}{3^z} + \frac{1}{5^z} + \ldots$. Repeating the procedure with the prime 3 gives $\left(1 - \frac{1}{3^z}\right)\left(1 - \frac{1}{2^z}\right)\zeta(z) = 1 + \frac{1}{5^z} + \frac{1}{7^z} + \ldots$. Continuing, if p_k denotes the kth prime, we get

$$\left(1 - \frac{1}{(p_N)^z}\right)\cdots\left(1 - \frac{1}{3^z}\right)\left(1 - \frac{1}{2^z}\right)\zeta(z) = 1 + \frac{1}{(p_{N+1})^z} + \ldots.$$

Therefore, by choice N,

$$\left|\left(1 - \frac{1}{(p_N)^z}\right)\cdots\left(1 - \frac{1}{3^z}\right)\left(1 - \frac{1}{2^z}\right)\zeta(z) - 1\right| < \epsilon$$

. Since this is true for all ϵ,

$$\left(1 - \frac{1}{(p_N)^z}\right)\cdots\left(1 - \frac{1}{3^z}\right)\left(1 - \frac{1}{2^z}\right)\zeta(z) = 1.$$

□

Theorem.
$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}e^{-t}}{1-e^{-t}}\, dt$$

Proof. For any positive integer n,
$$n^{-z} \int_0^\infty \tau^{z-1} e^{-\tau}\, d\tau \overset{(\tau = nt)}{=} \int_0^\infty t^{z-1} e^{-nt}\, dt$$
and so $n^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} e^{-nt}\, dt$. Therefore
$$\sum_{n=1}^\infty n^{-z} = \frac{1}{\Gamma(z)} \sum_{n=1}^\infty \int_0^\infty t^{z-1} e^{-nt}\, dt = \frac{1}{\Gamma(z)} \int_0^\infty \sum_{n=1}^\infty t^{z-1} (e^{-t})^n\, dt = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}e^{-t}}{1-e^{-t}}\, dt$$
where the interchange of the integration and summation is justified by the Lebesgue Monotone Convergence Theorem (MATC37). ☐

For ϵ and δ, let $C_{\epsilon,\delta}$ be the boundary (with counter-clockwise orientation) of the thermometer-shaped region consisting of the union of the ball $B_\epsilon[0]$ and the strip $[0,\infty] \times [-\delta,\delta]$. $C_{\epsilon,\delta}$ is called a "Hankel contour". The Hankel functions are defined by $H(z) := \int_{C_{\epsilon,\delta}} u(w,z)\, dw$ where $u(w,z) = \frac{(-w)^{z-1}e^{-w}}{1-e^{-w}}$. Here we are using the branch of the logarithm defined except on the portion $(0,\infty)$ of the x-axis to define $(-w)^{z-1} := e^{(z-1)\log(-w)}$. The answer is independent of ϵ and δ since, for $\epsilon' < \epsilon$ and $\delta' < \delta$, the region between $C_{\epsilon',\delta'}$ and $C_{\epsilon,\delta}$ contains no poles of u, all whose poles (as a function of w) lie on the x-axis.

Theorem.
$$\zeta(z) = \frac{-H(z)}{2i \sin(\pi z)\Gamma(z)}$$
if $\mathrm{Re}(z) > 1$.

Proof. Let $(\tilde{\epsilon},\delta)$ and $(\tilde{\epsilon},-\delta)$ be the intersection points of the boundary of the strip with the boundary circle of the disk. In polar coodinates these points become $(\epsilon,\tilde{\delta})$ and $(\epsilon, 2\pi-\tilde{\delta})$ for some angle $\tilde{\delta}$. (Explicitly, $\tilde{\epsilon} = \sqrt{\epsilon^2-\delta^2}$ and $\tilde{\delta} = \sin^{-1}(\frac{\delta}{\epsilon})$.)

$H_\epsilon(z) = I_1 + I_2 + I_3$ where
$$I_1 = \int_\infty^{\tilde{\epsilon}} \frac{e^{(z-1)\log(-(t+i\delta))} e^{-(t+i\delta)}}{1-e^{-(t+i\delta)}}\, dt,$$
$$I_2 = \int_{\tilde{\epsilon}}^\infty \frac{e^{(z-1)\log(-(t-i\delta))} e^{-(t-i\delta)}}{1-e^{-(t-i\delta)}}\, dt,$$
$$I_3 = \int_{\tilde{\delta}}^{2\pi-\tilde{\delta}} \frac{(-\epsilon e^{i\theta})^{z-1} e^{-\epsilon e^{i\theta}}}{1-e^{-\epsilon}e^{i\theta}} i\epsilon e^{i\theta}\, d\theta$$

For small ϵ, $|1-e^{\epsilon e^{i\theta}}| \geq |1-e^{-\epsilon}| \geq \epsilon/2$ and so
$$|I_3| \leq 2\pi\epsilon \max_\theta \frac{|(-\epsilon e^{i\theta})^{z-1}|\,|e^{-\epsilon e^{i\theta}}|}{\epsilon/2} \leq 4\pi\epsilon^{\mathrm{Re}\, z-1} e^{2\pi|\operatorname{Im} z|} e^\epsilon$$

which implies that $I_3 \to 0$ as $\epsilon \to 0$.

$I_1 + I_2 = \int_\infty^{\tilde\epsilon} \frac{e^{(z-1)\left[\log(\sqrt{t^2+\delta^2})+i(-\pi+\delta')\right]}e^{-t-i\delta}}{1-e^{-t-i\delta}}\,dt + \int_{\tilde\epsilon}^\infty \frac{e^{(z-1)\left[\log(\sqrt{t^2+\delta^2})+i(\pi-\delta')\right]}e^{-t+i\delta}}{1-e^{-t-i\delta}}\,dt$
where δ'_t is chosen so that $-\pi + \delta'_t = \operatorname{Im}\log\bigl(-(t+i\delta)\bigr)$ and δ''_t similarly. Letting $\delta \to 0$ gives

$$\begin{aligned} H(z) &= \int_\infty^{\tilde\epsilon} \frac{e^{(z-1)[\log(t)-i\pi]}e^{-t}}{1-e^{-t}}\,dt + \int_{\tilde\epsilon}^\infty \frac{e^{(z-1)[\log(t)+i\pi]}e^{-t}}{1-e^{-t}}\,dt \\ &= -(e^{i\pi z} - e^{-i\pi z}) \int_{\tilde\epsilon}^\infty \frac{t^{z-1}e^{-t}}{1-e^{-t}}\,dt = -2i\sin(\pi z) \int_{\tilde\epsilon}^\infty \frac{t^{z-1}e^{-t}}{1-e^{-t}}\,dt \end{aligned}$$

so taking the limit as $\epsilon \to 0$ and applying the previous theorem gives the result. ☐

Corollary. *$\zeta(z)$ has an analytic continuation to $\mathbb{C} - \{1\}$.*

For $b > 1$ let $C_{\epsilon,\delta,b}$ be the portion of $C_{\epsilon,\delta}$ for which $\delta \le b$. Since for proper integrals we can interchange the order of differentiation and integration, we can deduce that for each $b > 0$ the integral $H_b(z) := \int_{C_{\epsilon,\delta,b}} u(w,z)\,dw$ defines an entire function. Using that $b > 1$ implies, $|u(b,z)| \le 2b^{\operatorname{Re} z - 1}e^{-b}$ we can deduce that in the neighbourhood of any point z, these functions converge uniformly to $H(z) := \lim_{b\to 1} H_b(z)$, so we conclude that $H(z)$ is an entire function. Thus, since $\Gamma(z)$ has no zeros, $\frac{-H(z)}{2i\sin(\pi z)\Gamma(z)}$ is an analytic continuation of $\zeta(z)$ to $\mathbb{C} - \{\text{zeros of } \sin(\pi z)\} = \mathbb{C} - \mathbb{Z}$. We already know that $\zeta(z)$ is holomorphic at $n > 1$, and the simple poles of $\Gamma(z)$ at the integers $n \le 0$ cancel those zeros of $\sin(\pi z)$, meaning that those singularities are removable. Therefore we have produced an analytic continuation of $\zeta(z)$ to $\mathbb{C} - \{1\}$. ☐

Theorem. *$\zeta(z)$ has a simple pole at $z = 1$ with $\operatorname{Res}_1 \zeta(z) = 1$.*

Proof. Let I_1, I_2, and I_3 be as in the proof of the previous theorem. As before, $I_1 + I_2 = -2i\sin(\pi z)$, so $I_1 + I_2 = 0$ when $z = 1$. Our previous proof that $I_3 = 0$ depended upon $\operatorname{Re} z > 1$. For $z = 1$ we get instead

$$H(z) = I_3 = \int_0^{2\pi} \frac{e^{-\epsilon e^{i\theta}}}{1 - e^{-\epsilon e^{i\theta}}} i\epsilon e^{i\theta}\,d\theta = \int_0^{2\pi} \frac{i\epsilon e^{i\theta}}{e^{\epsilon e^{i\theta}} - 1}\,d\theta$$

Letting $v := \epsilon e^{i\theta}$, we have $v \to 0$ as $\epsilon \to 0$. Therefore

$$I_3 = \lim_{v\to 0} \int_0^{2\pi} \frac{iv}{e^v - 1}\,d\theta = \int_0^{2\pi} \lim_{v\to 0} \frac{iv}{e^v - 1}\,d\theta = \int_0^{2\pi} i\,d\theta = 2\pi i$$

Therefore

$$\lim_{z\to 1}(z-1)\zeta(z) = \lim_{z\to 1} \frac{-H(z)}{\Gamma(z)} \frac{(z-1)}{2i\sin(\pi z)} = \frac{-2\pi i}{1} \frac{1}{-2\pi i} = 1$$

and so $\zeta(z)$ has a simple pole with residue 1 at $z = 1$. ☐

13. Prime Number Theorem

In this section, the symbol p will indicate a prime.

Let $\pi(n) =$ number of primes which are less than or equal to n. We wish to show that $\pi(n) \approx n/\log(n)$. More precisely, the claim is that $\lim_{n\to\infty} \frac{\pi(n)}{n/\log(n)} = 1$.

Set $V(x) := \sum_{p\leq x} \log(p)$.

Lemma. *$V(x) \approx x$ implies $\pi(x) \approx x/\log(x)$.*

Proof. Suppose that we know $V(x) \approx x$.

$V(x) = \sum_{p\leq x} \log(p) \leq \sum_{p\leq x} \log(x) = \pi(x)\log(x)$. Therefore $\frac{\pi(x)\log(x)}{x} \geq \frac{V(x)}{x}$ so $\lim_{x\to\infty} \frac{\pi(x)\log(x)}{x} \geq 1$, giving a lower bound for the limit.

However for all $\epsilon > 0$,

$$\begin{aligned} V(x) &= \sum_{p\leq x} \log(p) \geq \sum_{x^{1-\epsilon}\leq p\leq x} \log(p) \geq \sum_{x^{1-\epsilon}\leq p\leq x} \log(x^{1-\epsilon}) = \sum_{x^{1-\epsilon}\leq p\leq x} (1-\epsilon)\log(x) \\ &= (1-\epsilon)\log(x) \sum_{x^{1-\epsilon}\leq p\leq x} 1 \\ &= (1-\epsilon)\log(x)\big(\pi(x) - \pi(x^{1-\epsilon})\big) \end{aligned}$$

Therefore, noting that $\pi(y) \leq y$ for any y we have

$$\begin{aligned} \frac{(1-\epsilon)\log(x)(\pi(x))}{x} &\leq \frac{V(x)}{x} + \frac{(1-\epsilon)\log(x)\pi(x^{1-\epsilon})}{x} \leq \frac{V(x)}{x} + \frac{(1-\epsilon)\log(x)x^{1-\epsilon}}{x} \\ &= \frac{V(x)}{x} + \frac{(1-\epsilon)\log(x)}{x^{\epsilon}} \end{aligned}$$

Since $\lim_{x\to\infty} \frac{\log(x)}{x^{\epsilon}} = 0$, this gives

$$(1-\epsilon)\lim_{x\to\infty} \frac{\log(x)(\pi(x))}{x} \leq \lim_{x\to\infty} \frac{V(x)}{x} = 1.$$

Since this is true for every $\epsilon > 0$, $\lim_{x\to\infty} \frac{\log(x)(\pi(x))}{x} \leq 1$. □

Lemma. *If $\int_1^\infty \frac{V(t)-t}{t^2}\,dt$ converges then $V(x) \approx x$.*

Proof. Suppose $\int_1^\infty \frac{V(t)-t}{t^2}\,dt$ converges to L. Then for any λ,

$$\lim_{x\to\infty} \int_x^{\lambda x} \frac{V(t)-t}{t^2}\,dt = \lim_{x\to\infty} \int_1^{\lambda x} \frac{V(t)-t}{t^2}\,dt - \lim_{x\to\infty} \int_1^{x} \frac{V(t)-t}{t^2}\,dt = L - L = 0.$$

If $V(x) \not\approx x$ then either there exists $\lambda > 1$ such that $\frac{V(x)}{x} > \lambda$ for infinitely many x, or there exists $\lambda < 1$ such that $\frac{V(x)}{x} < \lambda$ for infinitely many x.

Assume there exists $\lambda > 1$ such that $\frac{V(x)}{x} > \lambda$ for infinitely many x. For any such x,

$$\int_x^{\lambda x} \frac{V(t)-t}{t^2}\,dt \geq \int_x^{\lambda x} \frac{\lambda x - t}{t^2}\,dt = \int_1^{\lambda} \frac{x-t}{t^2}\,dt$$

so $\lim_{x\to\infty}\int_x^\infty \frac{V(t)-t}{t^2}\,dt \geq \int_1^\lambda \frac{x-t}{t^2}\,dt > 0$, a contradiction.

Assume there exists $\lambda > 1$ such that $\frac{V(x)}{x} < \lambda$ for infinitely many x. For any such x,

$$\int_{\lambda x}^{x} \frac{V(t)-t}{t^2}\,dt \leq \int_{\lambda x}^{x} \frac{\lambda x - t}{t^2}\,dt = -\int_1^\lambda \frac{x-t}{t^2}\,dt$$

so $\lim_{x\to\infty}\int_x^\infty \frac{V(t)-t}{t^2}\,dt \leq \int_1^\lambda \frac{x-t}{t^2}\,dt < 0$, a contradiction.

Since $V(x) \not\approx x$, leads to a contradiction in both cases, we conclude $V(x) \approx x$. $\square$

Lemma. *There exists a constant K such that $V(x) \leq Kx$ for all $x \geq 1$.*

Proof. Let N be a positive integer. If $N < p \leq 2N$ then p divides $\frac{(2N)!}{(N!)^2} = \binom{2N}{N}$ and thus

$$V(2N) - V(N) = \sum_{N<p\leq 2N} \log(p) = \log\left[\prod_{N<p\leq 2N} p\right] \leq \log\left[\binom{2N}{N}\right].$$

However

$$2^{2N} = (1+1)^{2N} = \sum_{k=0}^{2N}\binom{2N}{k} \geq \binom{2N}{N}$$

and so $V(2N) - V(N) \leq \log(2^{2N}) = 2N\log 2$. Thus for any $k \geq 2$,

$$\begin{aligned} V(2^k) &= (V^{2^k} - V^{2^{k-1}}) + (V^{2^{k-1}} - V^{2^{k-2}}) + \ldots + \big(V(4) - V(2)\big) + \big(V(2) - V(1)\big) \\ &\leq (2 + 4 + 8 + \ldots + 2^{k-1})\log(2) < 2^k \log(2) \end{aligned}$$

Given $x \geq 2$, find k such that $2^{k-1} \leq x \leq 2^k$. Then $V(x) \leq V(2^k) \leq \log(2)2^k \leq \log(2)2x$ so let $K = 2\log(2)$. $\square$

Lemma. *The function defined for* $\operatorname{Re} z > 1$ *by $\zeta(z) - \frac{1}{z-1}$ has an analytic continuation to the domain* $\operatorname{Re} z > 0$.

Proof. Set $q(z) := \zeta(z) - \frac{1}{z-1}$.

$$\zeta(z) - \frac{1}{z-1} = \sum_{n=1}^\infty \frac{1}{n^z} - \int_1^\infty \frac{1}{t^z}\,dt = \sum_{n=1}^\infty \int_n^{n+1}\left(\frac{1}{n^z} - \frac{1}{t^z}\right)dt$$

For any z,

$$\left|\int_n^{n+1}\left(\frac{1}{n^z} - \frac{1}{t^z}\right)dt\right| = \left|\int_n^{n+1}\int_n^t \frac{z}{u^{z+1}}\,du\,dt\right| \leq M(\text{Area of } B)$$

where the region of integration B is the triangle forming half of the square $[n, n+1] \times [n, n+1]$ and M is the maximum value of $|\frac{z}{u^{z+1}}|$ on B. The area of B is $1/2$ and in the region of integration $u \geq n$, so $M = |\frac{z}{n^{z+1}}|$. For any real a,

$$|a^z| = |e^{z\log(a)}| = |e^{x\log(a)}\,|e^{iy\log(a)}| = |e^{x\log(a)}| = a^{\operatorname{Re} z}$$

so we conclude

$$\left|\int_n^{n+1}\left(\frac{1}{n^z}-\frac{1}{t^z}\right)dt\right| \le \frac{|z|}{2n^{\operatorname{Re} z+1}}.$$

Thus for every $c > 0$, the partial sums of the series $\sum_{n=1}^{\infty}\int_n^{n+1}\left(\frac{1}{n^z}-\frac{1}{t^z}\right)dt$ converge uniformly in a ball around z providing an analytic continuation of $q(z)$ to the domain $\operatorname{Re} z > c$. Since this is true for every $c > 0$, $q(z)$ has an analytic continuation to the domain $\operatorname{Re} z > 0$. $\square$

Set $\Phi(z) := \sum_p p^{-z}\log(p)$. For any $c > 0$, $\log(p) < p^c$ for sufficiently large p, so $|p^{-z}\log(p)| \le |p^{-z+c}| = |p^{-\operatorname{Re} z+c}|$ and hence $\sum_p p^{-z}\log(p)$ converges and defines a holomorphic function in the domain $\operatorname{Re} z > 1+c$. This is true for all $c > 0$ and so $\Phi(z)$ is a holomorphic function in the domain $\operatorname{Re} z > 1$.

$\zeta(z) = \prod_p \frac{1}{1-\frac{1}{p^z}}$ so logarithmic differentiation shows

$$\begin{aligned}\frac{\zeta'(z)}{\zeta(z)} &= -\sum_p \frac{d(1-p^{-z})/dz}{1-p^{-z}} = -\sum_p \frac{p^{-z}\log(p)}{1-p^{-z}} = -\sum_p \frac{\log(p)}{p^z-1} = -\sum_p \frac{\log(p)p^z}{p^z(p^z-1)} \\ &= -\sum_p \frac{\log(p)(1+p^z-1)}{p^z(p^z-1)} = -\sum_p \frac{\log(p)}{p^z(p^z-1)} - \sum_p \frac{\log(p)}{p^z} \\ &= -\sum_p \frac{\log(p)}{p^z(p^z-1)} - \Phi(z)\end{aligned}$$

Since for any z with $\operatorname{Re} z > 1/2$, $-\sum_p \frac{\log(p)}{p^z(p^z-1)}$ converges uniformly, it defines a holomorphic function in the domain $\operatorname{Re} z > 1/2$, and therefore $\Phi(z)$ extends to a meromorphic function in the domain $\operatorname{Re} z > 1/2$ with simple poles at the poles of $\frac{\zeta'(z)}{\zeta(z)}$, which consist of $z = 1$ together with the zeros of $\zeta(z)$.

Suppose $\operatorname{Re} a > 1/2$. Then $\sum_p \frac{\log(p)}{p^z(p^z-1)}$ is holomorphic in the neighbourhood of a, so $\operatorname{Res}_a \Phi(z) = \operatorname{Res}_a\left(-\frac{\zeta'(z)}{\zeta(z)}\right)$. In particular, $\operatorname{Res}_1 \Phi(z) = -(-1) = 1$, since $\zeta(z)$ has a pole of order 1 at 1. (Recall, if f has a pole of order k at a then $\operatorname{Res}_a\big(f'(z)/f(z)\big) = -k$.) Equivalently, $\lim_{z\to 1}(z-1)\Phi(z) = 1$ so $\Phi(z) - \frac{1}{z-1}$ is holomorphic in the neighbourhood of 1.

Theorem. *$\zeta(z) \neq 0$ if $z = 1 + i\alpha$ with $\alpha \neq 0 \in \mathbb{R}$.*

Proof. Let the order of the zeros of $\zeta(z)$ at $1+i\alpha$ and $1+2i\alpha$ be μ and ν respectively, where we use the convention that a zero of order 0 means a place where the function is nonzero. Notice that $\zeta(\bar{z}) = \overline{\zeta(z)}$, so the order of the zeros of η at $1-i\alpha$ and $1-2i\alpha$ are also μ and ν.

As above, for $\operatorname{Re} a > 1/2$, $\operatorname{Res}_a \Phi(z) = \operatorname{Res}_a -\frac{\zeta'(z)}{\zeta(z)}$ and so

$$\lim_{z\to 1\pm\alpha}(z\pm\alpha)\Phi(z) = -\mu \qquad \text{and} \qquad \lim_{z\to 2\pm\alpha}(z\pm\alpha)\Phi(z) = -\nu.$$

Equivalent, letting $\epsilon := z - a$,

$$\lim_{\epsilon \to 0} \epsilon \Phi(1 + \epsilon \pm i\alpha) = -\mu \qquad \text{and} \qquad \lim_{\epsilon \to 0} \epsilon \Phi(1 + \epsilon \pm 2i\alpha) = -\nu.$$

$p^{i\alpha/2} + p^{-i\alpha/2}$ is real and by definition

$$\sum_p \frac{\log(p)}{p^{1+\epsilon}} \left(p^{i\alpha/2} + p^{-i\alpha/2}\right)^4$$
$$= \Phi(1 + \epsilon - 2i\alpha) + 4\Phi(1 + \epsilon - i\alpha) + 6\Phi(1 + \epsilon) + 4\Phi(1 + \epsilon + i\alpha) + \Phi(1 + \epsilon + 2i\alpha)$$

Since the left hand side is always positive, multiplying by $\epsilon > 0$ and taking the limit at $\epsilon \to 0$ gives

$$0 \leq -\nu - 4\mu + 6 - 4\mu - \nu = 6 - 8\mu - 2\nu.$$

Since μ is a non-negative integer, this implies $\mu = 0$. $\square$

Since we showed earlier that on $\operatorname{Re} z > 1/2$ the poles of $\Phi(z)$ occur at the zeros of $\zeta(z)$, we get

Corollary. $\Phi(z) - \frac{1}{1-z}$ *is holomorphic on a domain containing* $\operatorname{Re} z \geq 1$. $\square$

Lemma.

$$\Phi(z) = z \int_0^\infty e^{-zt} V(e^t)\, dt$$

Proof. Letting $u := e^t$,

$$z \int_0^\infty e^{-zt} V(e^t)\, dt = z \int_1^\infty \frac{V(u)}{u^z} \frac{1}{u}\, du = z \int_1^\infty \frac{\sum_{p \leq u} \log(p)}{u^{z+1}}\, du = \sum_p z \int_p^\infty \frac{\log(p)}{u^{z+1}}\, du$$
$$= \sum_p \frac{\log(p)}{p^z} = \Phi(z)$$

$\square$

Recall that to prove the prime number theorem, it suffices to show that $\int_1^\infty \frac{V(u)-u}{u^2}\, du$ converges. Making the change of variable $u := e^t$,

$$\int_1^\infty \frac{V(u) - u}{u^2}\, du = \int_0^\infty \frac{V(e^t) - e^t}{e^{2t}} e^t\, dt = \int_0^\infty \left(e^{-t} V(e^t) - 1\right) dt$$

Set $f(t) := e^{-t} V(t) - 1$. Then the prime number theorem follows from the following Lemma.

Lemma. *$\int_0^\infty f(t)\,dt$ converges.*

Proof. Let $g(z) = \int_0^\infty e^{-zt} f(t)\,dt$.

We want to show that $g(0)$ converges. From the previous Lemma,

$$\begin{aligned}\frac{1}{z+1}\Phi(z+1) &= \int_0^\infty e^{-(z+1)t} V(e^t)\,dt = \int_0^\infty e^{-zt} e^{-t} V(e^t)\,dt \\ &= \int_0^\infty e^{-zt} f(t)\,dt + \int_0^\infty e^{-zt} = \int_0^\infty e^{-zt} f(t)\,dt + \frac{1}{z}\end{aligned}$$

and so $g(z) = \frac{1}{z+1}\Phi(z+1) - \frac{1}{z}$ is holomorphic on a domain D containing $\operatorname{Re} z \geq 0$, since we showed earlier that $\Phi(z) - \frac{1}{z-1}$ is holomorphic on a domain containing $\operatorname{Re} z \geq 1$.

For $T > 0$, let $g_T(z) = \int_0^T e^{-zt} f(t)\,dt$. Since the interval is bounded, we can differentiate under the integral sign to conclude that $g_T(z)$ is holomorphic for all z. To prove the Lemma, we show that $\lim_{T\to\infty} g_T(0) = g(0)$.

Pick a large R. The open set D contains the closed interval $0 \times [-R, R]$ along the y-axis so we can choose a small $\delta > 0$ such that all points whose distance to $0 \times [-R, R]$ is less than δ lie in D. Let C be the boundary of the region $B := \{z \in \mathbb{C} \mid |z| \leq R \text{ and } \operatorname{Re} z \geq -\delta\}$. Then $g(z)$ is holomorphic on B by choice of δ.

Set $h_T(z) := \big(g(z) - g_T(z)\big) e^{zT}\left(1 + \frac{z^2}{R^2}\right)$. Since $g(z)$ and $g_T(z)$ are holomorphic on B, so is $h(z)$. Notice that $h(0) = g(0) - g_T(0)$. Applying the Cauchy integral theorem to $h(z)$ and the curve C gives

$$g(0) - g_T(0) = \frac{1}{2\pi i}\int_C \frac{h(z)}{z}\,dz.$$

We showed earlier that there exists a postive constant K such $V(x) \leq Kx$ for all $x \geq 1$. Therefore there exists $M > 0$ such that $|f(t)| \leq M$ for all $t \geq 0$.

On the semicircle $C_+ := C \cap \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$,

$$|g(z) - g_T(0)| = \left|\int_T^\infty f(t) e^{-zt}\,dt\right| \leq M \int_T^\infty |e^{-zt}|\,dt = \frac{M e^{-(\operatorname{Re} z)T}}{\operatorname{Re} z}$$

and

$$\begin{aligned}\left|e^{zT}\left(1 + \frac{z^2}{R^2}\right)\frac{1}{z}\right| &= e^{T\operatorname{Re} z}\left|\left(1 + \frac{z^2}{R^2}\right)\frac{1}{z}\right| = e^{T\operatorname{Re} z}\left|\frac{R^2 + z^2}{R^2}\frac{1}{z}\right| \\ &= e^{T\operatorname{Re} z}\frac{1}{R}\left|\frac{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 + z^2}{R^2}\right| \\ &= e^{T\operatorname{Re} z}\frac{1}{R}\left|\frac{(\operatorname{Re} z)^2 + (\operatorname{Re} z)^2 + 2(\operatorname{Re} z)(\operatorname{Im} z)}{R^2}\right| \\ &\leq e^{T\operatorname{Re} z}\frac{1}{R}\left|\frac{(\operatorname{Re} z)R + (\operatorname{Re} z)R + 2(\operatorname{Re} z)R}{R^2}\right| \\ &\leq e^{T\operatorname{Re} z}\frac{4\operatorname{Re} z}{R^2}.\end{aligned}$$

Therefore

$$\left|\int_{C_+} \frac{h(z)}{z}\,dz\right| \leq \int_{C_+} \frac{4M}{R^2}\,dz = \frac{4\pi M}{R}.$$

Next look at $C_- := C \cap \{z \in \mathbb{C} \mid \operatorname{Re} z < 0\}$ where we will consider $g(z)$ and $g_T(z)$ separately.

First consider $g_T(z)$. Let C'_- be the semicircle $C'_- = \{z \in \mathbb{C} \mid |z| = R \text{ and} \operatorname{Re} z < 0\}$ Since $g_T(z)$ is entire, $\int_{C-} g_T e^{zT}\left(1+\frac{z^2}{R^2}\right)\frac{1}{z}\,dz = \int_{C'_-} g_T e^{zT}\left(1+\frac{z^2}{R^2}\right)\frac{1}{z}\,dz$.

$$|g_T(z)| = \left|\int_0^T f(t)e^{-zt}\,dt\right| \leq M\int_{-\infty}^T |e^{-zt}|\,dt = \frac{Me^{-T\operatorname{Re} z}}{|\operatorname{Re} z|}$$

and so $\left|\int_{C'_-} g_T e^{zT}\left(1+\frac{z^2}{R^2}\right)\frac{1}{z}\,dz\right| \leq \frac{4\pi M}{R}$ as before.

Now consider $g(z)$. Since $\operatorname{Re} z < 0$ on C_-,

For every $\epsilon > 0$, in the region $|z| \leq -\epsilon$, e^{zT} converges uniformly to 0 as $T \to \infty$, while the other factor is independent of T. Letting $C_{\epsilon,-} = C \cap \{z \in \mathbb{C} \mid \operatorname{Re} z \leq -\epsilon\}$, this shows that $\left|\int_{C_{\epsilon,-}} g(z)e^{zT}\left(1+\frac{z^2}{R^2}\right)\frac{1}{z}\,dz\right|$ goes to 0 as $T \to \infty$. Since $C_- = \cup_\epsilon C_{\epsilon,-}$, this implies that $\lim_{T\to\infty}\int_{C_-} g(z)e^{zT}\left(1+\frac{z^2}{R^2}\right)\frac{1}{z}\,dz = 0$.

Combining the previous estimates shows that

$$\lim_{T\to\infty}|g(0) - g_T(0)| = \lim_{T\to\infty}\left|\frac{1}{2\pi i}\int_C \frac{h(z)}{z}\,dz\right| \leq \frac{1}{2\pi}\left(\frac{4\pi M}{R} + \frac{4\pi M}{R} + 0\right) = \frac{4M}{R}.$$

Since this is true for all R, $\int_0^\infty f(t)\,dt = g(0) = \lim_{T\to\infty} g_T$ converges. □

This concludes the proof of

Prime Number Theorem.

$$\pi(n) \approx \frac{n}{\log(n)}$$

□