Differential Equations II

MATC46H3S

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CHAPTER 1

Laplace Transforms

1. Definitions

Definition 1.1 (Laplace transform). Let $f : [0, \infty) \to \mathbb{R}$. The *Laplace transform* of f, denoted $\mathcal{L}(f)$ or $\hat{f}$, is the function

$$\hat{f}(s) := \int_0^\infty e^{-st} f(t)\, dt.$$

The domain of $\hat{f}$ is the set of s for which the integral converges. $\Diamond$

Theorem 1.2. *If $\hat{f}(s_0)$ converges, then $\hat{f}(s)$ converges for all $s > s_0$.*

Proof. Suppose that $s > s_0$. We wish to show that the "tail" of the integral is small, i.e., show that, given an $\epsilon > 0$, there exists an a such that

$$\left| \int_a^b e^{-st} f(t)\, dt \right| < \epsilon k$$

for all $b \geq a$, where k is a constant, i.e., independent of a and b. Let

$$\beta(x) = - \int_x^\infty e^{-s_0 t} f(t)\, dt,$$

where $x \geq 0$. The integral exists since $\hat{f}(s_0)$ converges. Note that $\beta(x)$ is also differentiable (fundamental theorem of calculus) with $\beta'(x) = e^{-s_0 x} f(x)$. Therefore, $\lim_{x\to\infty} \beta(x) = 0$.

Choose an a such that $x \geq a \Rightarrow \left|\beta(x)\right| < \epsilon$. Then

$$\begin{aligned}
\int_a^b e^{-st} f(t)\, dt &= \int_a^b e^{-st} e^{s_0 t} e^{-s_0 t} f(t)\, dt \\
&= \int_a^b e^{-(s-s_0)t} \beta'(t)\, dt \\
&= e^{-(s-s_0)t\beta(t)}\Big|_a^b + \int_a^b (s - s_0)\, e^{-(s-s_0)t} \beta(t)\, dt.
\end{aligned}$$

Choosing

$$\begin{aligned}
u &= e^{-(s-s_0)t}, & dv &= \beta'(t)\, dt, \\
du &= -(s - s_0)\, e^{-(s-s_0)t}\, dt, & v &= \beta(t),
\end{aligned}$$

we have

$$\int_a^b e^{-st} f(t)\, dt = e^{-(s-s_0)b} \beta(b) - e^{-(s-s_0)a} \beta(a) + (s - s_0) \int_a^b e^{-(s-s_0)t} \beta(t)\, dt.$$

Since $s \geq s_0$ and $b \geq a \geq 0$, we have $e^{-(s-s_0)b} \leq 1$ and $e^{-(s-s_0)a} \leq 1$. Therefore,

$$\left| \int_a^b e^{-st} f(t)\, dt \right| \leq \left|\beta(b)\right| + \left|\beta(a)\right| + (s - s_0) \left| \int_a^b e^{-(s-s_0)t} \beta(t)\, dt \right|$$

$$
\begin{aligned}
&\le \epsilon + \epsilon + (s - s_0) \int_a^b e^{-(s-s_0)t} \epsilon \, dt \\
&= 2\epsilon + \epsilon\,(s - s_0)^2 \left(\, (s - s_0)^2 \Big|_a^b \right) \\
&= 2\epsilon + \epsilon\,(s - s_0)^2 \left(e^{-(s-s_0)a} - e^{-(s-s_0)b} \right) \\
&\le 2\epsilon + \epsilon\,(s - s_0)^2 \, e^{-(s-s_0)a} \\
&\le 2\epsilon + \epsilon\,(s - s_0)^2 \\
&= \epsilon \left(2 + (s - s_0)^2 \right).
\end{aligned}
$$

Therefore, letting $k = 2 + (s - s_0)^2$, we have

$$
\left| \int_a^b e^{-st} f(t) \, dt \right| < \epsilon k
$$

for all $b \ge a$. □

Recall from `MATA30/36/37` that

(1) if $0 \le g(x) \le h(x)$, then $\int_0^\infty h(x)\,dx$ converges implies that $\int_0^\infty g(x)\,dx$ converges.
(2) $\int_0^\infty \left|g(x)\right| dx$ converges implies that $\int_0^\infty g(x)\,dx$ converges.

Also note that $s > s_0 \Rightarrow e^{-st} \le e^{-s_0 t}$. Therefore, if $\int_0^\infty e^{-s_0 t}\left|f(t)\right| dt$ converges, then Theorem 1.2 follows immediately. The general case requires more careful analysis.

Example 1.3. Compute $\mathcal{L}(f)$ for $f(x) = e^{ax}$. ◇

Solution. We have

$$
\begin{aligned}
\hat{f}(s) &= \int_0^\infty e^{-st} e^{at} \, dt = \int_0^\infty e^{(a-s)t} \, dt = \lim_{b \to \infty} \left. \frac{e^{(a-s)t}}{a - s} \right|_0^b \\
&= \lim_{b \to \infty} \left(\frac{e^{(a-s)b}}{a - s} - \frac{1}{a - s} \right) = 0 - \frac{1}{a - s} = \frac{1}{s - a},
\end{aligned}
$$

where the domain is $s > a$, i.e., (a, ∞). ◇

Definition 1.4 (Exponential order). Given a constant α, a continuous function $f : [0, \infty) \to \mathbb{R}$ is said to have *exponential order* α if there exists a constant C such that $\left|f(x)\right| \le Ce^{\alpha x}$ for sufficiently large x. More precisely, f has exponential order if there exist constants C and b such that $\left|f(x)\right| \le Ce^{\alpha x}$ for $x > b$. We write $f \in \xi_\alpha$ to mean that f has exponential order α. ♢

Theorem 1.5 (Comparison theorem). *If $f \in \xi_\alpha$, then $\hat{f}(s)$ is defined for all $s > \alpha$.*

From now on, we will assume that $f \in \xi_\alpha$ for some α.

2. Laplace Transforms of Derivatives

To take into account the Laplace transform of derivatives, note that, from the definition of the Laplace transform, we have

$$
\mathcal{L}(f') = \int_0^\infty e^{-st} f'(t) \, dt = \lim_{b \to \infty} \int_0^b e^{-st} f'(t) \, dt.
$$

Letting

$$u = e^{-st}, \qquad dv = f'(t)\,dt,$$
$$du = -se^{-st}\,dt, \qquad v = f(t),$$

we have

$$\mathcal{L}(f') = \lim_{b\to\infty}\left(e^{-st}f(t)\Big|_0^b + s\int_0^b e^{-st}f(t)\,dt \right) = \lim_{b\to\infty}\left(e^{-sb}f(b)\right) - f(0) + s\mathcal{L}(f).$$

Our interest now lies in $\lim_{b\to\infty}\left(e^{-sb}f(b)\right)$. Note that

$$0 \le \left|e^{-sb}f(b)\right| \le \underbrace{Ce^{-sb}e^{\alpha b}}_{\star}.$$

But $\star \to 0$ as $b\to\infty$. So by squeezing, we have $\lim_{b\to\infty}\left|e^{-st}f(b)\right| = 0$. But then $\lim_{b\to\infty}\left(-\left|e^{-sb}f(b)\right|\right) = 0$ while

$$-\left|e^{-sb}f(b)\right| \le e^{-sb}f(b) \le \left|e^{-sb}f(b)\right|,$$

so we conclude that $\lim_{b\to\infty} e^{-sb}f(b) = 0$. Hence,

$$\mathcal{L}(f') = 0 - f(0) + s\mathcal{L}(f) = s\mathcal{L}(f) - f(0).$$

Example 1.6. Solve $y' - 4y = e^x$ with $y(0) = 1$. ◇

Solution. Taking the Laplace transform of the entire equation, we have

$$\begin{aligned}
\mathcal{L}(y' - 4y) &= \mathcal{L}(e^x)\,,\\
\mathcal{L}(y') - 4\mathcal{L}(y) &= \frac{1}{s-1},\\
s\mathcal{L}(y) - y(0) - 4\mathcal{L}(y) &= \frac{1}{s-1},\\
(s-4)\,\mathcal{L}(y) - 1 &= \frac{1}{s-1},\\
(s-4)\,\mathcal{L}(y) &= \frac{1}{s-1} + 1,\\
(s-4)\,\mathcal{L}(y) &= \frac{s}{s-1},\\
\mathcal{L}(y) &= \frac{s}{(s-1)\,(s-4)}.
\end{aligned}$$

Rearranging the expression to reveal terms with easily identifiable inverse Laplace transforms, we have

$$\mathcal{L}(y) = \frac{4}{3}\frac{1}{s-4} - \frac{1}{3}\frac{1}{s-1}.$$

Taking the inverse Laplace transform gives

$$y = \frac{4}{3}e^{4x} - \frac{1}{3}e^x.$$

◇

Property 1.7 (Laplace transform).

(1) $\mathcal{L}(af + bg) = a\mathcal{L}(f) + b\mathcal{L}(g)$

(2)

$$\begin{aligned}
\mathcal{L}(f') &= s\mathcal{L}(f) - f(0),\\
\mathcal{L}(f'') &= s\mathcal{L}(f') - f'(0) = s^2\mathcal{L}(f) - sf(0) - f'(0),\\
&\vdots\\
\mathcal{L}\Big(f^{(n)}\Big) &= s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)
\end{aligned}$$

(3) If f and g are continuous on $[0,\infty)$ and $\mathcal{L}(f) = \mathcal{L}(g)$, then $f = g$.

(4) $\mathcal{L}\big(e^{ax}f(x)\big) = \hat{f}(s-a)$

(5) (a) $\hat{f}$ is differentiable and $\mathcal{L}\big(x^n f(x)\big) = (-1)^n \hat{f}^{(n)}(s)$.

(b) $\hat{f}$ is integrable and $\mathcal{L}\big(f(x)/x\big) = -\int_0^s \hat{f}(u)\,du$.

(6) $\lim_{s\to\infty}\hat{f}(s) = 0$

Proof of (1). This is trivial. □

Proof of (2). Note that

$$\mathcal{L}\big(f'(x)\big) = \int_0^\infty e^{-sx}f'(x)\,dx.$$

Letting

$$\begin{aligned}
u &= e^{-sx}, & dv &= f'(x)\,dx,\\
du &= -se^{-sx}\,dx, & v &= f(x),
\end{aligned}$$

we have

$$\begin{aligned}
\mathcal{L}\big(f'(x)\big) &= e^{-sx}f(x)\Big|_0^\infty + s\int_0^\infty e^{-sx}f(x)\,dx\\
&= 0 - 1f(x) + s\mathcal{L}(f)\\
&= s\mathcal{L}(f) - f(0).
\end{aligned}$$

□

Proof of (3). Suppose that $\mathcal{L}(f) = \mathcal{L}(g)$ and let $h = f - g$. Then $\mathcal{L}(h) = 0$. To show that $h = 0$ as well, we use the corollary to the Weierstrass Approximation Theorem (`MATC37`). If f is continuous on $[0,1]$ and $\int_0^1 t^n f(t)\,dt = 0$ for all $n = 0,1,2,\ldots$, then $f \equiv 0$. Suppose that $\hat{f}(s) = 0$ for all $s \geq s_0$. Consider $s = s_0 + n$. Then

$$\begin{aligned}
\hat{h}(s_0+n) &= \int_0^\infty e^{-(s_0+n)t}h(t)\,dt\\
&= \int_0^\infty e^{-nt}e^{-s_0t}h(t)\,dt\\
&= \int_0^\infty e^{-nt}v'(t)\,dt.
\end{aligned}$$

Let $v(x) = \int_0^x e^{-s_0t}h(t)\,dt$ so that $v'(x) = e^{-s_0x}h(x)$. Letting

$$\begin{aligned}
u &= e^{-nt}, & dv &= v'(t)\,dt,\\
du &= -ne^{-nx}\,dx, & v &= v,
\end{aligned}$$

we have

$$\begin{aligned}
\hat{h}(s_0+n) &= e^{-nt}v(t)\Big|_0^\infty + n\int_0^\infty e^{-nt}v(t)\,dt \\
&= \lim_{b\to\infty} e^{-nb}\int_0^b e^{-s_0t}h(t) - v(0) + n\int_0^\infty e^{-nt}v(t)\,dt \\
&= 0\hat{h}(s_0) - v(0) + n\int_0^\infty e^{-nt}v(t)\,dt \\
&= 0\cdot 0 - 0 + n\int_0^\infty e^{-nt}v(t)\,dt \\
&= n\int_0^\infty e^{-nt}v(t)\,dt.
\end{aligned}$$

Therefore, $\int_0^\infty e^{-nt}v(t)\,dt = 0$ for all n. Now let $z = e^t$ so that $t = \ln(1/z) = -\ln(z)$ and $dt = -(1/z)\,dz$. Then $t = 0 \Rightarrow z = 0$ and $\lim_{t\to\infty} e^{-t} = 0$. Therefore,

$$\begin{aligned}
0 &= \int_0^\infty e^{-nt}v(t)\,dt \\
&= \int_0^1 z^n v\big(-\ln(z)\big)\left(-\frac{1}{z}\right)dz \\
&= \int_0^1 z^{n-1}v\big(-\ln(z)\big)dz
\end{aligned}$$

for all n. Therefore, $v\big(-\ln(z)\big) = 0$. As z runs through $[0,1]$, $-\ln(z)$ runs through $[0,\infty)$, i.e., $v(t) = 0$ for all $t \in [0,\infty)$. Therefore, $v(t) = 0$ and $v'(t) = 0 = e^{-s_0t}h(t)$ for all $t \in [0,\infty)$. Therefore, $h(t) = 0$ as well since $e^{-s_0t} \neq 0$. □

Proof of (4). We have

$$\begin{aligned}
\mathcal{L}\big(e^{ax}f(x)\big) &= \int_0^\infty e^{-sx}e^{ax}f(x)\,dx \\
&= \int_0^\infty e^{-(s-a)x}f(x)\,dx \\
&= \hat{f}(s-a).
\end{aligned}$$

□

Proof of (5).

We have

$$\hat{f}(s) = \lim_{h\to 0}\frac{\hat{f}(s+h)-\hat{f}(s)}{h}$$

and

$$\mathcal{L}\big(-xf(x)\big) = \int_0^\infty -e^{-st}tf(t)\,dt.$$

We must show that for all $\epsilon > 0$, we have

$$\left|\frac{\hat{f}(s+h)-\hat{f}(s)}{h} - I\right| < \epsilon$$

for sufficiently small h. We have

$$\frac{\hat{f}(s+h)-\hat{f}(s)}{h} - I = \frac{\int_0^\infty e^{-(s+h)t}f(t)\,dt - \int_0^\infty e^{-st}f(t)\,dt}{h} - \int_0^\infty -e^{-st}tf(t)\,dt$$

$$= \int_0^\infty e^{-st}\frac{e^{-ht}-1}{h} f(t)\,dt + \int_0^\infty e^{-st} t f(t)\,dt$$
$$= \int_0^\infty e^{-st}\left(\frac{e^{-ht}-1}{h}+t\right) f(t)\,dt.$$

Note that

$$\begin{aligned}
\frac{e^{-ht}-1}{h}+t &= \frac{1-ht+\frac{h^2t^2}{2!}+\cdots-1}{h}+t \\
&= \frac{ht^2}{2}+\frac{h^2t^3}{6}+\cdots \\
&= ht^2\left(\frac{1}{2}-\frac{th}{3!}+\frac{t^2h^2}{4!}+\cdots\right) \\
&< ht^2\left(1+\frac{th}{2!}+\frac{t^2h^2}{3!}+\cdots\right) \\
&= ht^2e^{th}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|\frac{\hat{f}(s+h)-\hat{f}(s)}{h}-I\right| &\le \int_0^\infty ht^2e^{th}e^{-st}\big|f(t)\big|\,dt \\
&\le h\int_0^\infty t^2e^{(h-s-\alpha)t}\,dt.
\end{aligned}$$

If h is sufficiently small, then $h-s-\alpha<0$. So

$$\int_0^\infty t^2e^{(h-s-\alpha)t}\,dt<\infty$$

for h sufficiently small. So we can make

$$h\int_0^\infty t^2e^{(h-s-\alpha)t}\,dt<\epsilon$$

for h sufficiently small. Therefore,

$$\hat{f}'(s)=\lim_{h\to 0}\frac{\hat{f}(s+h)-\hat{f}(s)}{h}=I=\mathcal{L}\big(-xf(x)\big).$$

We can differentiate again by the same procedure. □

Proof of (6). For large x, we have $\big|f(x)\big|\le Ce^{\alpha x}$ for some C and α. So

$$\underbrace{\big|\hat{f}(s)\big|}_{\left|\int_0^\infty e^{-sx}f(x)\,dx\right|} \le \begin{array}{c}\int_b^\infty e^{-sx}\big|f(x)\big|\,dx \\ +\int_0^b e^{-sx}\big|f(x)\big|\,dx\end{array} \le \underbrace{\begin{array}{c}C\int_b^\infty e^{-(s-\alpha)x}\,dx \\ +\int_0^b e^{-sx}\big|f(x)\big|\,dx.\end{array}}_{\frac{C}{s-\alpha}+\int_0^b e^{-sx}\left|f(x)\right|\,dx}$$

Therefore, by the Squeeze Theorem, we have

$$\lim_{s\to\infty}\hat{f}(s)=0,$$
$$\lim_{s\to\infty}\int_0^b e^{-sx}\big|f(x)\big|\,dx=0,$$

$$\int_0^b \left(\lim_{s\to\infty} e^{-sx}\right) \left|f(x)\right| dx = 0. \qquad \square$$

Table 1.1 shows a partial list of Laplace transforms.* Note that $s/\left(s^2+1\right)$ is defined for all s, but it

Table 1.1: Laplace transforms

f	$\mathcal{L}(f)$
x^n	$\dfrac{\Gamma(n+1)}{s^{n+1}}$
$\cos(ax)$	$\dfrac{s}{s^2+a^2}$, $s>0$
$\sin(ax)$	$\dfrac{a}{s^2+a^2}$
$x\cos(ax)$	$\dfrac{s^2-a^2}{\left(s^2+a^2\right)^2}$
$x\sin(ax)$	$\dfrac{2as}{\left(s^2+a^2\right)^2}$

equals $\int_0^\infty e^{-sx}\cos(x)\,dx$ only when $s>0$ (it does not converge for $s\le 0$).

3. The Gamma Function

Definition 1.8 (Gamma function). The *Gamma function* is defined as

$$\Gamma(n) := \int_0^\infty x^{n-1}e^{-x}\,dx, \quad n>0.$$

$\lozenge$

Property 1.9 (Gamma function).

(1) $\Gamma(n) = (n-1)\,\Gamma(n-1)$
(2) $\Gamma(n) = (n-1)!$ *if n is a positive integer*
(3) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof of (1). From the definition of the Gamma function, we have

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\,dx.$$

Considering integration by parts, we have

$$\begin{aligned} u &= x^{n-1}, & dv &= e^{-x}\,dx, \\ du &= (n-1)\,x^{n-2}\,dx, & v &= -e^{-x}. \end{aligned}$$

Therefore,

$$\begin{aligned} \Gamma(n) &= \left. -x^{n-1}e^{-x}\right|_0^\infty + (n-1)\int_0^\infty x^{n-2}e^{-x}\,dx \\ &= 0 + 0 + (n-1)\,\Gamma(n-1) \end{aligned}$$

*See [?, p. 304] for a more comprehensive list.

$$= (n-1)\,\Gamma(n-1).$$ □

PROOF OF (2). First note that

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = \left. -e^{-x}\right|_0^\infty = 1.$$

So by induction in part (1), $\Gamma(n) = (n-1)!$ for any positive integer n. □

PROOF OF (3). From the definition of the Gamma function, we have

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx.$$

Considering the substitution $u = \sqrt{x}$, it follows that $x^2 = u$ and $dx = 2u\,du$. Therefore,

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{e^{-u^2}}{u} 2u\,du = 2\int_0^\infty e^{-u^2}\,du.$$

Let $I = \int_0^\infty e^{-u^2}\,du$. Then

$$\begin{aligned} I^2 &= \int_0^\infty \int_0^\infty e^{-\left(u^2+v^2\right)}\,dv\,du \\ &= \int_0^{\pi/2} \int_0^\infty e^{-r^2} r\,dr\,d\theta \\ &= \left.\frac{\pi}{2}\frac{e^{-r^2}}{2}\right|_0^\infty = \frac{\pi}{4}. \end{aligned}$$

Therefore, $I = \sqrt{\pi}/2$. So $\Gamma(1/2) = 2I = \sqrt{\pi}$. □

With the Gamma function and its properties established, we can now prove the entries in Table 1.1.

PROOF OF TABLE 1.1.

(1) We have

$$\mathcal{L}(x^n) = \int_0^\infty e^{-sx} x^n\,dx.$$

If n is an integer, then

$$\begin{aligned} \mathcal{L}(1) = \mathcal{L}\left(e^{0\cdot x}\right) &= \frac{1}{s-0} = \frac{1}{s}, \\ \mathcal{L}(x) &= -\frac{d}{ds}\left(\frac{1}{s}\right) = \frac{1}{s^2}, \\ \mathcal{L}\left(x^2\right) &= -\frac{d}{ds}\left(\frac{1}{s^2}\right) = \frac{2!}{s^3}, \dots \end{aligned}$$

and so on. Let $t = sx$. Then

$$\mathcal{L}(x^n) = \int_0^\infty e^{-t}\frac{t^n}{s^n}\frac{dt}{s} = \frac{1}{s^{n+1}}\int_0^\infty e^{-t}t^n\,dt = \frac{\Gamma(n+1)}{s^{n+1}}.$$

(2) We have

$$\mathcal{L}\left(e^{bx}\right) = \frac{1}{s-b}.$$

Let $b = ia$. Then

$$\mathcal{L}\left(e^{iax}\right) = \frac{1}{s-ia} = \frac{s+ia}{s^2+a^2}.$$

Therefore,

$$\mathcal{L}\big(\cos(ax)\big) = \mathrm{Re}\Big(\mathcal{L}\big(e^{iax}\big)\Big) = \frac{s}{s^2+a^2}.$$

(3) It follows immediately from (2) that

$$\mathcal{L}\big(\sin(ax)\big) = \mathrm{Im}\Big(\mathcal{L}\big(e^{iax}\big)\Big) = \frac{a}{s^2+a^2}.$$

(4) We have

$$\mathcal{L}\big(xe^{bx}\big) = \frac{1}{(s-b)^2}.$$

Therefore,

$$\mathcal{L}\big(xe^{iax}\big) = \frac{1}{(s-ia)^2} = \frac{(s+ia)^2}{(s^2+a^2)^2} = \frac{s^2-a^2+2ias}{(s^2+a^2)^2}.$$

So it follows that

$$\mathcal{L}\big(x\cos(ax)\big) = \frac{s^2-a^2}{(s^2+a^2)^2}.$$

(5) It follows immediately from (4) that

$$\mathcal{L}\big(x\sin(ax)\big) = \frac{2as}{(s^2+a^2)^2}.$$

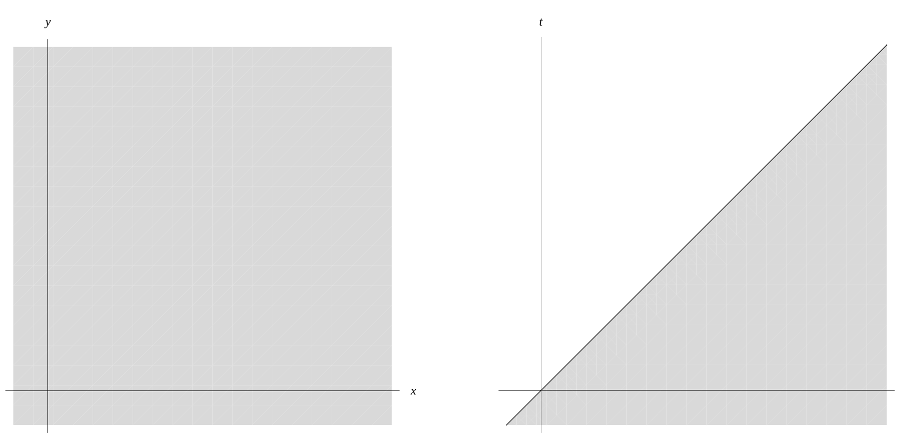
□

4. Convolutions

Let $\mathcal{L}(f) = \hat{f}$ and $\mathcal{L}(g) = \hat{g}$. Then we wish to find an h such that $\mathcal{L}(h) = \hat{f}\hat{g}$. To do so, we have

$$\begin{aligned}\mathcal{L}(h)(s) &= \hat{f}(s)\hat{g}(s)\\ &= \left(\int_0^\infty e^{-sx} f(x)\,dx\right)\left(\int_0^\infty e^{-sy} g(y)\,dy\right)\\ &= \int_0^\infty\int_0^\infty e^{-s(x+y)} f(x)g(y)\,dx\,dy.\end{aligned}$$

Let $u = x+y$ and $t = y$. Figure 1.1 shows graphically this substitution. Then



(a) The xy-plane.

(b) The half-plane $u > t$.

Figure 1.1: The xy-plane and half-plane below $t = u$.

$$\mathbf{J} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and $|\mathbf{J}| = 1$. Thus,

$$\begin{aligned}
\mathcal{L}(h)(s) &= \int_0^\infty \int_0^u e^{-su} f(u-t)g(t)\,dt\,du \\
&= \int_0^\infty \left(e^{-su} \int_0^u f(u-t)g(t)\,dt \right) du \\
&= \int_0^\infty \left(e^{-sx} \int_0^x f(x-t)g(t)\,dt \right) dx \\
&= \mathcal{L}\left(\int_0^x f(x-t)g(t)\,dt \right).
\end{aligned}$$

So

$$h(x) = \int_0^x f(x-t)g(t)\,dt,$$

called the *convolution* of f and g, written $f * g$.

Property 1.10 (Convolution).

(1) $f * g = g * f$
(2) $(f * g) * h = f * (g * h)$
(3) $f * (g + h) = f * g + f * h$
(4) $(\lambda f) * g = \lambda (f * g)$, *where* λ *is a constant*

Example 1.11. Solve $y'' + y = f(x)$, where $y(0) = 0$ and $y'(0) = 0$. In addition, consider the case where $f(x) = \tan(x)$. $\diamond$

Solution. Taking the Laplace transform of every term, we have

$$\begin{aligned}
\mathcal{L}(y'') &= s\mathcal{L}(y') - y'(0) = s^2 \hat{y}, \\
\mathcal{L}(y') &= s\hat{y} - y(0) = s\hat{y}.
\end{aligned}$$

Therefore,

$$\left(s^2 + 1\right) \hat{y} = \hat{f} \Longrightarrow \hat{y} = \frac{1}{s^2+1} \hat{f},$$

so

$$\begin{aligned}
y &= \mathcal{L}^{-1}\left(\frac{1}{s^2+1} \hat{f} \right) = \mathcal{L}^{-1}\left(\frac{1}{s^2+1} \right) * \mathcal{L}^{-1}\left(\hat{f} \right) \\
&= \sin(x) * f = \int_0^x f(t) \sin(x-t)\,dt.
\end{aligned}$$

With $f(x) = \tan(x)$, we have

$$\begin{aligned}
y &= \int_0^x \tan(t) \sin(x-t)\,dt \\
&= \int_0^x \tan(t) \big(\sin(x)\cos(-t) + \cos(x)\sin(-t) \big)\,dt \\
&= \int_0^x \tan(t) \big(\sin(x)\cos(t) - \cos(x)\sin(t) \big)\,dt
\end{aligned}$$

$$\begin{aligned}
&= \int_0^x \left(\sin(x)\sin(t) - \cos(x)\frac{\sin^2(t)}{\cos(t)}\right) dt \\
&= \int_0^x \sin(x)\sin(t)\, dt - \int_0^x \cos(x)\left(\frac{1-\cos^2(t)}{\cos(t)}\right) dt \\
&= \sin(x)\int_0^x \sin(t)\, dt - \cos(x)\int_0^x \sec(t)\, dt + \cos(x)\int_0^x \cos(t)\, dt \\
&= \sin(x)\left(-\cos(t)\Big|_0^x\right) - \cos(x)\left(\ln\Big(\big|\sec(t)+\tan(t)\big|\Big)\Big|_0^x\right) + \cos(x)\left(\sin(t)\Big|_0^x\right) \\
&= \sin(x)\big(-\cos(x)+1\big) - \cos(x)\Big(\ln\Big(\big|\sec(x)+\tan(x)\big|\Big) - \ln\big(|1+0|\big)\Big) + \cos(x)\sin(x) \\
&= \cancel{-\sin(x)\cos(x)} + \sin(x) - \cos(x)\ln\Big(\big|\sec(x)+\tan(x)\big|\Big) + \cancel{\cos(x)\sin(x)} \\
&= \sin(x) - \cos(x)\ln\Big(\big|\sec(x)+\tan(x)\big|\Big).
\end{aligned}$$

◇

5. Laplace Transforms of Some Discontinuous Functions

5.1. Step Functions. Let

$$u(x) := \begin{cases} 1, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

called the *step function*. Suppose $x_0 \geq 0$ and let $g(x) := u(x - x_0)$. Then

$$g(x) = \begin{cases} 1, & x \geq x_0, \\ 0, & x < x_0. \end{cases}$$

We have

$$\mathcal{L}(\hat{g})\,(s) = \int_0^\infty e^{-sx} g(x)\, dx = \int_{x_0}^\infty e^{-sx}\, dx = \frac{e^{-sx_0}}{s}.$$

Therefore,

$$\mathcal{L}^{-1}\left(\frac{e^{-sx_0}}{s}\right) = u(x - x_0).$$

Recall that $\mathcal{L}(1) = 1/s$. This is the case where $x_0 = 0$. More generally, we have the following theorem.

Theorem 1.12. *We have*

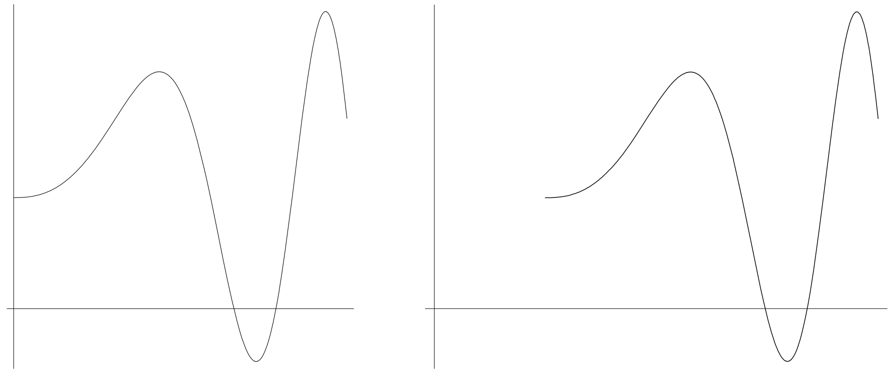
$$\mathcal{L}\big(u(x-x_0)f(x-x_0)\big) = e^{-sx_0}\hat{f}(s).$$

In particular, f shifted to the right by x_0.

Figure 1.2 illustrates the idea of these shifts.

Proof 1. We have

$$\begin{aligned}
\mathcal{L}^{-1}\Big(e^{-sx_0}\hat{f}(s)\Big) &= \mathcal{L}^{-1}\Big(e^{-sx_0}\hat{f}(s)\Big) \\
&= \mathcal{L}^{-1}\left(e^{-sx_0}\frac{\mathcal{L}(f') + f(0)}{s}\right) \\
&= \mathcal{L}^{-1}\left(\frac{e^{-sx_0}}{s}\mathcal{L}^{-1}(f')\right) + \mathcal{L}^{-1}\left(\frac{e^{-sx_0}f(0)}{s}\right) \\
&= u(x-x_0) * f'(x) + f(0)u(x-x_0)
\end{aligned}$$



(a) $f(x)$ (b) $u(x-x_0)f(x-x_0)$

Figure 1.2: A plot showing $f(x)$ and $f(x)$ shifted to the right by x_0.

$$
\begin{aligned}
&= \int_0^x f'(x-t)u(t-x_0)\,dt + f(0)u(x-x_0) \\
&= \left(\begin{cases} \int_{x_0}^x f'(x-t)\,dt, & x \geq x_0, \\ 0, & x < x_0 \end{cases}\right) + f(0)u(x-x_0) \\
&= u(x-x_0)\left(\int_{x_0}^x f'(x-t)\,dt + f(0)\right) \\
&= u(x-x_0)\left(-f(x-t)\Big|_{x_0}^x + f(0)\right) \\
&= u(x-x_0)\big(-f(0) + f(x-x_0) + f(0)\big) \\
&= u(x-x_0)f(x-x_0).
\end{aligned}
$$

□

PROOF 2. We have

$$\mathcal{L}\big(u(x-x_0)f(x-x_0)\big) = \int_{x_0}^\infty e^{-st}f(t-x_0)\,dt.$$

Considering a substitution, let $v = t - x_0$ so that $dv = dt$. Then

$$
\begin{aligned}
\mathcal{L}\big(u(x-x_0)f(x-x_0)\big) &= \int_0^\infty e^{-s(v-x_0)}f(v)\,dv \\
&= e^{-sx_0}\int_0^\infty e^{-sv}f(v)\,dv \\
&= e^{-sx_0}\hat{f}(s).
\end{aligned}
$$

□

Also note that

$$
\begin{aligned}
\mathcal{L}\big(u(x-x_0)f(x)\big) &= \mathcal{L}\big(u(x-x_0)g(x-x_0)\big) \\
&= e^{-sx_0}\mathcal{L}(g) \\
&= e^{-sx_0}\mathcal{L}\big(f(x-x_0)\big),
\end{aligned}
$$

where $g(x-x_0) = f(x)$ and $g(x) = f(x+x_0)$.

EXAMPLE 1.13. Solve $3y'' + 7y' + 2y = f(x)$ with $y(0) = 0$ and $y'(0) = 0$, where

$$f(x) = \begin{cases} x, & x \geq 2, \\ -1, & x < 2. \end{cases}$$

$\diamond$

SOLUTION. Taking the Laplace transform, we have

$$\begin{aligned} \mathcal{L}(y') &= s\hat{y} - y(0) = s\hat{y}, \\ \mathcal{L}(y'') &= s\,(s\hat{y}) - y'(0) = s^2\hat{y}. \end{aligned}$$

Therefore, our equation becomes

$$\begin{aligned} \left(3s^2 + 7s + 2\right)\hat{y} &= \mathcal{L}\big(u(x-2)(x+1)\big) - \mathcal{L}(1), \\ (s+2)\,(3s+1)\,\hat{y} &= e^{-2s}\mathcal{L}(x+3) - \mathcal{L}(1) \\ &= e^{-2s}\left(\frac{1}{s^2} + \frac{3}{s}\right) - \frac{1}{s} \\ &= e^{-2s}\left(\frac{3s+1}{s^2}\right) - \frac{1}{s}. \end{aligned}$$

Solving for $\hat{y}$ gives

$$\hat{y} = e^{-2s}\underbrace{\frac{1}{s^2\,(s+2)}}_{\star} - \frac{1}{s\,(s+2)\,(3s+1)}.$$

We now must consider partial fractions. First considering $\star$, note that

$$\frac{1}{s^2\,(s+2)} = \frac{As+B}{s^2} + \frac{C}{s+2} = \frac{(As+B)\,(s+2)\,Cs^2}{s^2\,(s+2)}.$$

Comparing coefficients, we see that

$$\begin{aligned} s = -2 &\Longrightarrow 1 = 4C \Longrightarrow C = \frac{1}{4}, \\ s = 0 &\Longrightarrow 2B = 1 \Longrightarrow B = \frac{1}{2}, \end{aligned}$$

and

$$\begin{aligned} (A+B)\,3 + C &= 1, \\ 3\left(A + \frac{1}{2}\right) + \frac{1}{4} &= 1, \\ 12A + 6 + 1 &= 4, \\ 12A &= -3, \\ A &= -\frac{1}{4}. \end{aligned}$$

Therefore,

$$\frac{1}{s^2\,(s+2)} = -\frac{1/4}{s} + \frac{1/2}{s^2} + \frac{1/4}{s+2}$$

and we tentatively have

$$\hat{y} = e^{-2s}\left(-\frac{1/4}{s} + \frac{1/2}{s^2} + \frac{1/4}{s+2}\right) - \underbrace{\frac{1}{s\,(s+2)\,(3s+1)}}_{\star\star}.$$

Now considering $\star\star$, note that

$$\begin{aligned}\frac{1}{s\,(s+2)\,(3s+1)} &= \frac{A}{s} + \frac{B}{s+2} + \frac{C}{3s+1} \\ &= \frac{A\,(s+2)\,(3s+1) + Bs\,(3s+1) + Cs\,(s+2)}{s\,(s+2)\,(3s+1)}.\end{aligned}$$

Comparing coefficients gives us

$$\begin{aligned}s = 0 &\Longrightarrow 1 = 2A \Longrightarrow A = \frac{1}{2}, \\ s = -2 &\Longrightarrow 1 = 10 \Longrightarrow B = \frac{1}{10}, \\ s = -\frac{1}{3} &\Longrightarrow 1 = -\frac{5}{9} \Longrightarrow C = -\frac{9}{5}.\end{aligned}$$

Therefore,

$$\frac{1}{s\,(s+2)\,(3s+1)} = \frac{1/2}{s} + \frac{1/10}{s+2} - \frac{9/5}{3s+1}$$

and we finally have

$$\hat{y} = e^{-2s}\left(-\frac{1/4}{s} + \frac{1/2}{s^2} + \frac{1/4}{s+2}\right) - \frac{1/2}{s} - \frac{1/10}{s+2} + \frac{3/5}{s+1/3}.$$

Therefore,

$$y = u(x-2)\left(-\frac{1}{4} + \frac{1}{2}\,(x-2) + \frac{1}{4}e^{-2(x-2)}\right) - \frac{1}{2} - \frac{1}{10}e^{-2x} + \frac{3}{5}e^{-x/3}.$$

◇

Example 1.14 (Trick). Solve $xy'' + (2x+3)\,y' + (x+3)\,y = 3e^{-x}$ with $y(0) = 0$. ◇

Solution. First note that the initial condition $y(0) = 0$ alone specifies the solution as it implies the value of $y'(0)$. More precisely,

$$0 + 3y'(0) + 3y(0) = 3 \Longrightarrow y'(0) = 1.$$

Taking the Laplace transform, we have

$$\begin{aligned}\mathcal{L}(xy'') + 2\mathcal{L}(xy') + 3\mathcal{L}(y') + \mathcal{L}(xy) + 3\mathcal{L}(y) &= 3\mathcal{L}\big(e^{-x}\big)\,, \\ -\frac{d}{ds}\big(s^2\hat{y} - sy(0) - y'(0)\big) - 2\frac{d}{ds}\big(s\hat{y} - y(0)\big) + 3\big(s\hat{y} - y(0)\big) - \frac{d}{ds}\hat{y} + 3\hat{y} &= \frac{3}{s+1}, \\ -2s\hat{y} - s^2\frac{d\hat{y}}{ds} - 2\hat{y} - 2s\frac{d\hat{y}}{ds} + 3s\hat{y} - \frac{d\hat{y}}{ds} + 3\hat{y} &= \frac{3}{s+1}, \\ -\left(s^2 + 2s + 1\right)\frac{d\hat{y}}{ds} + s\hat{y} + \hat{y} &= \frac{3}{s+1}, \\ (s+1)^2\frac{d\hat{y}}{ds} - (s+1)\,\hat{y} &= -\frac{3}{s+1}, \\ \frac{d\hat{y}}{ds} - \frac{1}{s+1}\hat{y} &= -\frac{3}{(s+1)^3}.\end{aligned}$$

At this point it is appropriate to introduce the integrating factor

$$I = e^{-\int \frac{ds}{s+1}} = e^{-\ln(s+1)} = \frac{1}{s+1}.$$

Multiplying both sides by $1/(s+1)$ gives us

$$\begin{aligned}\frac{1}{s+1}\frac{d\hat{y}}{ds} - \frac{1}{(s+1)^2}\hat{y} &= -\frac{3}{(s+1)^4},\\ \frac{\hat{y}}{s+1} &= \frac{1}{(s+1)^3} + C,\\ \hat{y} &= \frac{1}{(s+1)^2} + C(s+1).\end{aligned}$$

Note that $\lim_{s\to\infty} \hat{y} = 0 \Rightarrow C = 0$. Therefore,

$$\hat{y}(s) = \frac{1}{(s+1)^2}$$

and

$$y = e^{-x}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = xe^{-x}.$$

Note that if any x^2 had appeared in the original equation, the resulting differential equation for $\hat{y}$ would have had order 2. So this trick has limited applicability. $\diamond$

5.2. Impulse Functions. A force $F(t)$ acting between $t = a$ and $t = b$ produces momentum $\rho = \int_a^b F\,dt$. An "instantaneous" transfer of momentum ρ at time a can be thought of as the limit as $\epsilon \to 0$ of the result of a force of size ρ/ϵ acting over time ϵ (from a to $a+\epsilon$). Consider the step function shown in Figure 1.3. We have

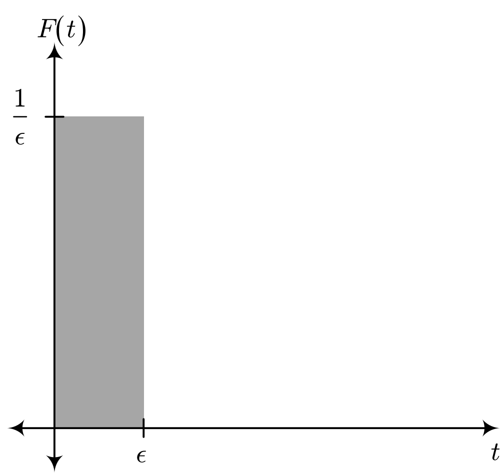


Figure 1.3: The step function with width ϵ and height $1/\epsilon$, bounded by the y-axis, encloses a region with area 1.

$$\int_0^\infty f_\epsilon(x)\,dx = \epsilon \cdot \frac{1}{\epsilon} = 1,$$

where

$$f_\epsilon = \frac{1}{\epsilon}\big(1 - u(x-\epsilon)\big).$$

Thus,

$$\mathcal{L}(f_\epsilon) = \frac{1}{\epsilon}\left(\frac{1}{s} - \frac{e^{-\epsilon s}}{s}\right) = \frac{1-e^{-\epsilon s}}{\epsilon s}.$$

Taking the limit, we have

$$\lim_{\epsilon\to 0}\mathcal{L}(f_\epsilon) = \underbrace{\lim_{\epsilon\to 0}\frac{1-e^{-\epsilon s}}{\epsilon s}}_{\text{l'Hôpital's Rule}} = \lim_{\epsilon\to 0}\frac{se^{-\epsilon s}}{s} = 1.$$

EXAMPLE 1.15. A block of wood of mass 80 g is motionless at the end of a spring with spring constant 10 g/sec². At time $t = 0$, it is hit by a bullet weighing 1 g and traveling upward at 100 m/sec. Find the equation of motion of the block of wood (assuming that there is no resistance). ◇

SOLUTION. Let $x(t)$ be the distance above the starting position at time t. Then we have

$$90\frac{d^2x}{dt^2} + 10x = F(t)$$

with initial conditions $x(0) = 0$ and $x'(0) = 0$, where $F(t)$ is the "impulse" function with momentum

$$1\,\mathrm{g} \times 100\,\mathrm{m/sec} = 100\,\mathrm{g{\cdot}m/sec}.$$

Taking the Laplace transform, we have

$$\begin{aligned} s^2\hat{x} + 10\hat{x} &= 100, \\ \hat{x} &= \frac{100}{90s^2+10} = \frac{10}{9s^2+1} \\ &= \frac{10}{9}\frac{1}{s^2+1/9} = \frac{10}{3}\frac{1/3}{s^2+1/9}. \end{aligned}$$

Therefore,

$$x = \frac{10}{3}\sin\left(\frac{t}{3}\right).$$

Figure 1.4 shows the plot of the equation. ◇

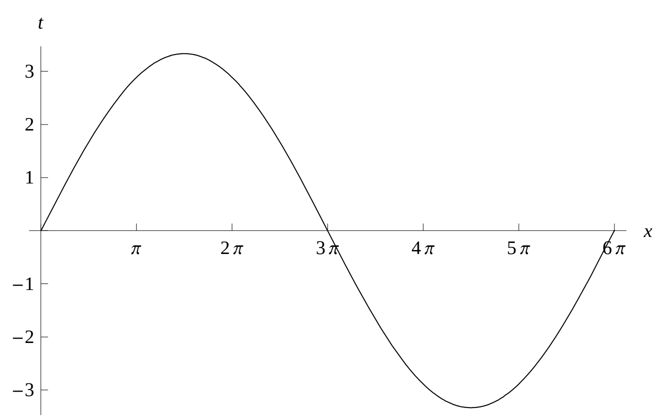


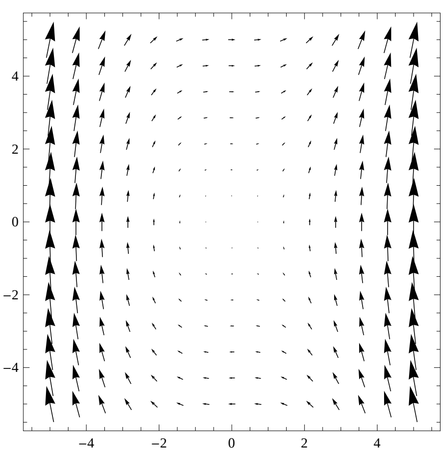
Figure 1.4: The plot of the equation of the motion of the block of wood in Example 1.15.

CHAPTER 2

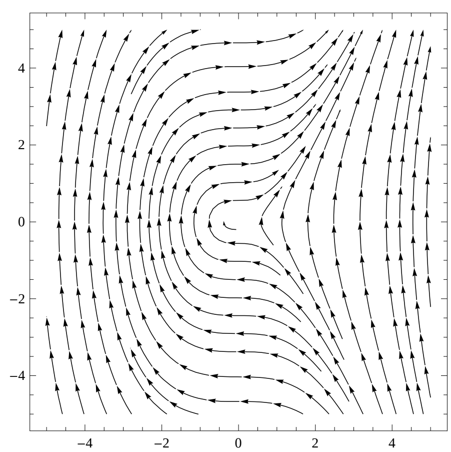
Phase Portraits: Qualitative and Pictorial Descriptions of Solutions of Two-Dimensional Systems

1. Introduction

Let $\mathbf{V} : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field. Imagine, for example, that $\mathbf{V}(x, y)$ represents the velocity of a river at the point (x, y).* We wish to get the description of the path that a leaf dropped in the river at the point (x_0, y_0) will follow. For example, Figure 2.1 shows the vector field of $\mathbf{V}(x, y) = \big(y, x^2\big)$. Let



(a) The vector field plot of $\big(y, x^2\big)$.



(b) The trajectories of $\big(y, x^2\big)$

Figure 2.1: The vector field plot of $\big(y, x^2\big)$ and its trajectories.

$\gamma(t) = \big(x(t), y(t)\big)$ be such a path. At any time, the leaf will go in the direction that the river is flowing at the point at which it is presently located, i.e., for all t, we have $\gamma'(t) = \mathbf{V}\big(x(t), y(t)\big)$. If $\mathbf{V} = (F, G)$, then

$$\frac{dx}{dt} = \underbrace{F(x, y)}_{y}, \qquad \frac{dy}{dt} = \underbrace{G(x, y)}_{x^2}.$$

In general, it will be impossible to solve this system exactly, but we want to be able to get the overall shape of the solution curves, e.g., we can see that in Figure 2.1, no matter where the leaf is dropped, it will head towards (∞, ∞) as $t \to \infty$.

*We are assuming here that $\mathbf{V}$ depends only on the position (x, y) and not also on time t.

2. Phase Portraits of Linear Systems

Before considering the general case, let us look at the linear case where we can solve it exactly, i.e., $\mathbf{V} = (ax+by, cx+dy)$ with

$$\frac{dx}{dt} = ax + by, \qquad \frac{dy}{dt} = cx + dy,$$

or $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Recall the existence and uniqueness theorem for ODE's from `MATB44`: if all the entries of $\mathbf{A}$ are continuous, then for any point (x_0, y_0), there is a unique solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ satisfying $x(t_0) = x_0$ and $y(t_0) = y_0$. In other words, there exists a unique solution through each point; in particular, the solution curves do not cross.

The above case can be solved explicitly, where

$$\mathbf{x} = e^{\mathbf{A}t} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

is a solution passing through (x_0, y_0) at time $t = 0$. We will consider only cases where $\det(\mathbf{A}) \neq 0$.

2.1. Real Distinct Eigenvalues. Let λ_1 and λ_2 be distinct eigenvalues of $\mathbf{A}$ and let $\mathbf{v}$ and $\mathbf{w}$ be their corresponding eigenvectors. Let $\mathbf{P} = [\mathbf{v}, \mathbf{w}]$. Then

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \underbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}_{\mathbf{D}}.$$

Therefore, $\mathbf{A}t = \mathbf{P}\,(\mathbf{D}t)\,\mathbf{P}^{-1}$ and we have

$$\begin{aligned} \mathbf{x} = e^{\mathbf{A}t} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} &= \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = [\mathbf{v}, \mathbf{w}] \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \\ &= [\mathbf{v}, \mathbf{w}] \begin{bmatrix} C_1 e^{\lambda_1 t} \\ C_2 e^{\lambda_2 t} \end{bmatrix} = C_1 e^{\lambda_1 t}\mathbf{v} + C_2 e^{\lambda_2 t}\mathbf{w}. \end{aligned}$$

Different C_1 and C_2 values give various solution curves.

Note that $C_1 = 1$ and $C_2 = 0$ implies that $\mathbf{x} = e^{\lambda_1 t}\mathbf{v}$. If $\lambda_1 < 0$, then the arrows point toward the origin, as shown in Figure 2.2a which contains a *stable node*. Note that

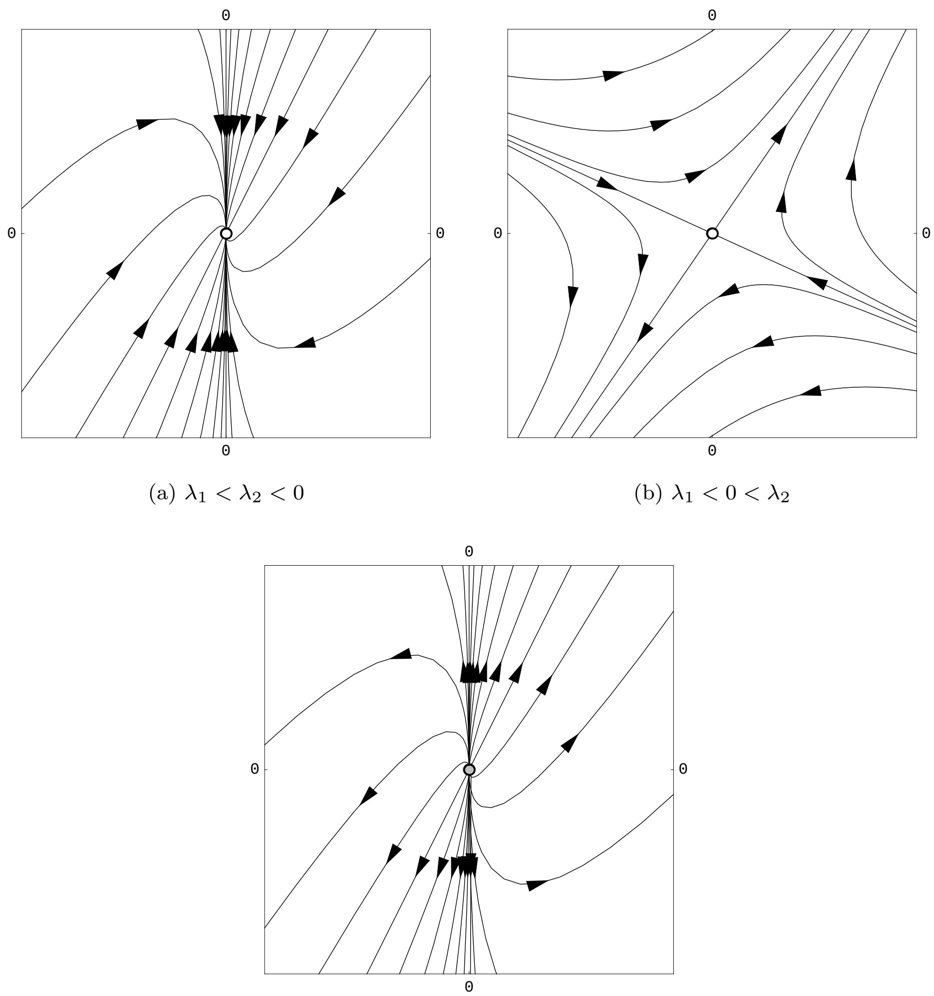
$$\mathbf{x} = C_1 e^{\lambda_1 t}\mathbf{v} + C_2 e^{\lambda_2 t}\mathbf{w} = e^{\lambda_2 t}\left(C_1 e^{(\lambda_1 - \lambda_2)t}\mathbf{v} + C_2\mathbf{w}\right).$$

The coefficient of $\mathbf{v}$ goes to 0 as $t \to \infty$, i.e., as $t \to \infty$, $\mathbf{x} \to (0,0)$, approaching along a curve whose tangent is $\mathbf{w}$. As $t \to -\infty$, $\mathbf{x} = e^{\lambda_1 t}\left(C_1\mathbf{v} + C_2 e^{(\lambda_2 - \lambda_1)t}\mathbf{w}\right)$, i.e., the curves get closer and closer to being parallel to $\mathbf{v}$ as $t \to -\infty$.

We have the degenerate case when $\lambda_1 < \lambda_2 = 0$, in which case $\mathbf{x} = C_1 e^{\lambda_1 t}\mathbf{v} + C_2\mathbf{w}$.

The case when $\lambda_1 < 0 < \lambda_2$ gives us the phase portrait shown in Figure 2.2b which contains a *saddle point*. This occurs when $\det(\mathbf{A}) < 0$. The case when $0 < \lambda_1 < \lambda_2$ gives us the phase portrait shown in Figure 2.2c which contains an *unstable node*. We have

$$\mathbf{x} = C_1 e^{\lambda_1 t}\mathbf{v} + C_2 e^{\lambda_2 t}\mathbf{w} = e^{\lambda_1 t}\left(C_1\mathbf{v} + C_2 e^{(\lambda_2 - \lambda_1)t}\mathbf{w}\right).$$



(a) $\lambda_1 < \lambda_2 < 0$

(b) $\lambda_1 < 0 < \lambda_2$

(c) $0 < \lambda_1 < \lambda_2$

Figure 2.2: The cases for λ_1 and λ_2.

Therefore, as $t \to -\infty$, $\mathbf{x} \to [0,0]$, approaching $\mathbf{v}$ as tangent; as $t \to \infty$, $\mathbf{x}$ approaches parallel to $\mathbf{w}$ asymptotically.

Note that in all cases, the origin itself is a fixed point, i.e., at the origin, $x' = 0$ and $y' = 0$, so anything dropped at the origin stays there. Such a point is called an equilibrium point; in a stable node, if it is disturbed, it will come back; in an unstable node, if perturbed slightly, it will leave the vicinity of the origin.

2.2. Complex Eigenvalues. Complex eigenvalues come in the form $\lambda = \alpha \pm \beta i$, where $\beta \neq 0$. In such a case, we have

$$\mathbf{x} = C_1 \operatorname{Re}\big(e^{\lambda t}\mathbf{v}\big) + C_2 \operatorname{Im}\big(e^{\lambda t}\mathbf{v}\big)\,,$$

where $\mathbf{v} = \mathbf{p} + i\mathbf{q}$ is an eigenvector for λ. Then

$$\begin{aligned} e^{\lambda t}\mathbf{v} &= e^{\alpha t} e^{\beta i t}\,(\mathbf{p} + i\mathbf{q}) \\ &= e^{\alpha t}\big(\cos(\beta t) + i\sin(\beta t)\big)\,(\mathbf{p} + i\mathbf{q}) \end{aligned}$$

$$= e^{\alpha t}\big(\cos(\beta t)\mathbf{p} - \sin(\beta t)\mathbf{q} + i\cos(\beta t)\mathbf{q} + i\sin(\beta t)\mathbf{p}\big).$$

Therefore,

$$\begin{aligned}
\mathbf{x} &= e^{\alpha t}\Big[\big(C_1\cos(\beta t)\mathbf{p} - C_1\sin(\beta t)\mathbf{q}\big) + C_1\cos(\beta t)\mathbf{q} + C_2\sin(\beta t)\mathbf{p}\Big] \\
&= e^{\alpha t}\begin{bmatrix} k_1\cos(\beta t) + k_2\sin(\beta t) \\ k_3\cos(\beta t) + k_4\sin(\beta t)\end{bmatrix} \\
&= e^{\alpha t}\left(\cos(\beta t)\begin{bmatrix} k_1 \\ k_3 \end{bmatrix} + \sin(\beta t)\begin{bmatrix} k_2 \\ k_4\end{bmatrix}\right).
\end{aligned}$$

Note that $\operatorname{tr}(\mathbf{A}) = 2\alpha$.* So

$$\begin{aligned}
\alpha = 0 &\Longrightarrow \operatorname{tr}(\mathbf{A}) = 0, \\
\alpha > 0 &\Longrightarrow \operatorname{tr}(\mathbf{A}) < 0, \\
\alpha < 0 &\Longrightarrow \operatorname{tr}(\mathbf{A}) > 0.
\end{aligned}$$

Consider first $\alpha = 0$. To consider the axes of the ellipses, first note that

$$\mathbf{x} = \underbrace{\begin{bmatrix} p_1 & q_1 \\ p_2 & q_2 \end{bmatrix}}_{\mathbf{P}}\underbrace{\begin{bmatrix} C_1 & C_2 \\ C_2 & -C_1 \end{bmatrix}}_{\mathbf{C}}\begin{bmatrix}\cos(\beta t) \\ \sin(\beta t)\end{bmatrix}.$$

Except in the degenerate case, where $\mathbf{p}$ and $\mathbf{q}$ are linearly dependent, we have

$$\begin{bmatrix}\cos(\beta t) \\ \sin(\beta t)\end{bmatrix} = \mathbf{C}^{-1}\mathbf{P}^{-1}\mathbf{x}.$$

Therefore,

$$\begin{aligned}
\cos(\beta t) + \sin(\beta t)\begin{bmatrix}\cos(\beta t) \\ \sin(\beta t)\end{bmatrix} &= \mathbf{x}^t\left(\mathbf{P}^{-1}\right)^t\left(\mathbf{C}^{-1}\right)^t\mathbf{C}^{-1}\mathbf{P}^{-1}\mathbf{x}, \\
\cos^2(\beta t) + \sin^2(\beta t) &= \mathbf{x}^t\left(\mathbf{P}^{-1}\right)^t\left(\mathbf{C}^{-1}\right)^t\mathbf{C}^{-1}\mathbf{P}^{-1}\mathbf{x}, \\
1 &= \mathbf{x}^t\left(\mathbf{P}^{-1}\right)^t\left(\mathbf{C}^{-1}\right)^t\mathbf{C}^{-1}\mathbf{P}^{-1}\mathbf{x}.
\end{aligned}$$

Note that $\mathbf{C} = \mathbf{C}^t$, so

$$\begin{aligned}
\mathbf{C}\mathbf{C}^t &= \begin{bmatrix} C_1 & C_2 \\ C_2 & -C_1 \end{bmatrix}\begin{bmatrix} C_1 & C_2 \\ C_2 & -C_1 \end{bmatrix}, \\
\mathbf{C}^2 &= \begin{bmatrix} C_1^2 + C_2^2 & 0 \\ 0 & C_1^2 + C_2^2 \end{bmatrix} = \left(C_1^2 + C_2^2\right)\mathbf{I}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{C}^{-1} &= \frac{\mathbf{C}}{C_1^2 + C_2^2}, \\
\left(\mathbf{C}^{-1}\right)^t = \mathbf{C}^{-1} &= \frac{\mathbf{C}}{C_1^2 + C_2^2},
\end{aligned}$$

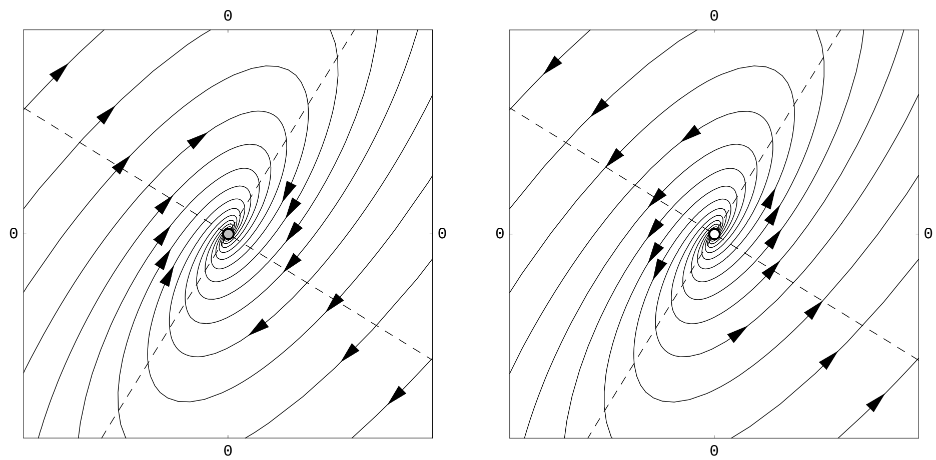
*Recall that the *trace* of a matrix is the sum of the elements on the main diagonal.

$$\left(\mathbf{C}^{-1}\right)^t \mathbf{C}^{-1} = \left(\mathbf{C}^{-1}\right)^2 = \frac{\left(C_1^2 + C_2^2\right)\mathbf{I}}{\left(C_1^2 + C_2^2\right)^2} = \frac{\mathbf{I}}{C_1^2 + C_2^2}.$$

Therefore,

$$1 = \mathbf{x}^t \left(\mathbf{P}^{-1}\right)^t \left(\mathbf{C}^{-1}\right)^t \mathbf{C}^{-1}\mathbf{P}^{-1}\mathbf{x} = \frac{1}{C_1^2 + C_2^2}\mathbf{x}^t \left(\mathbf{P}^{-1}\right)^t \mathbf{P}^{-1}\mathbf{x}.$$

Let $\mathbf{T} = \left(\mathbf{P}^{-1}\right)^t \mathbf{P}^{-1}$. Then $\mathbf{x}^t\mathbf{T}\mathbf{x} = C_1^2 + C_2^2$ and $\mathbf{T} = \mathbf{T}^t$ ($\mathbf{T}$ is symmetric). Therefore, the eigenvectors of $\mathbf{T}$ are mutually orthogonal and form the axes of the ellipses. Figure 2.3 shows a stable spiral and an unstable spiral.



(a) $\alpha < 0$ (b) $\alpha > 0$

Figure 2.3: The cases for α, where we have a stable spiral when $\alpha > 0$ and an unstable spiral when $\alpha > 0$.

2.3. Repeated Real Roots. We have $\mathbf{N} = \mathbf{A} - \lambda\mathbf{I}$, where $\mathbf{N}^2 = \mathbf{0}$ and $\mathbf{A} = \mathbf{N} + \lambda\mathbf{I}$. So

$$e^{\mathbf{A}t} = e^{\mathbf{N}t + \lambda t\mathbf{I}} = e^{\mathbf{N}t}e^{\lambda t\mathbf{I}} = (\mathbf{I} + \mathbf{N}t)\begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{bmatrix}.$$

Therefore,

$$\mathbf{x} = e^{\mathbf{A}t}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = (\mathbf{I} + \mathbf{N}t)\begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = e^{\lambda t}(\mathbf{I} + \mathbf{N}t)\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$

Note that $\mathbf{N}^2 = \mathbf{0} \Rightarrow \det(\mathbf{N})^2 = 0 \Rightarrow \det(\mathbf{N}) = 0$. Therefore,

$$\mathbf{N} = \begin{bmatrix} n_1 & n_2 \\ \alpha n_1 & \alpha n_2 \end{bmatrix}.$$

Also, $\mathbf{N}^2 = \mathbf{0} \Rightarrow \operatorname{tr}(\mathbf{N}) = 0 \Rightarrow n_1 + \alpha n_2 = 0$. Let

$$\mathbf{v} = \begin{bmatrix} 1 \\ \alpha \end{bmatrix}.$$

Then

$$\mathbf{N}\mathbf{v} = \begin{bmatrix} n_1 + \alpha n_2 \\ \alpha\left(n_1 + \alpha n_2\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So $\mathbf{A}\mathbf{v} = (\mathbf{N} + \lambda\mathbf{I})\,\mathbf{v} = \lambda\mathbf{v}$, i.e., $\mathbf{v}$ is an eigenvector for λ. Therefore,

$$\begin{aligned}
\mathbf{x} = e^{\lambda t}(\mathbf{I} + \mathbf{N}t)\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} &= e^{\lambda t}\begin{bmatrix} C_1 + (n_1 C_1 + n_2 C_2)\,t \\ C_2 + \alpha\,(n_1 + n_2 C_2)\,t \end{bmatrix} \\
&= e^{\lambda t}\left(\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + (n_1 C_1 + n_2 C_2)\,t \begin{bmatrix} 1 \\ \alpha \end{bmatrix}\right) \\
&= e^{\lambda t}\left(\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + (n_1 C_1 + n_2 C_2)\,t\mathbf{v}\right).
\end{aligned}$$

If $\lambda < 0$, we have

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

as $t \to \infty$. If $\lambda > 0$, then

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

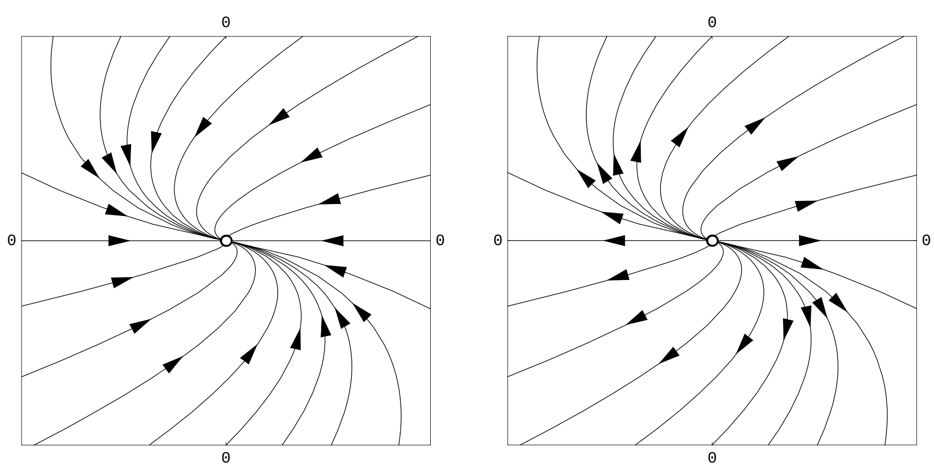
as $t \to -\infty$. What is the limit of the slope? In other words, what line is approached asymptotically? We have

$$\lim_{t\to\infty} \frac{y}{x} = \lim_{t\to\infty} \frac{C_2 + (n_1 C_1 + n_2 C_2)\,t\mathbf{v}_2}{C_1 + (n_1 C_1 + n_2 C_2)\,t\mathbf{v}_1} = \frac{\mathbf{v}_2}{\mathbf{v}_1},$$

i.e., it approaches $\mathbf{v}$. Similarly,

$$\lim_{t\to-\infty} \frac{y}{x} = \frac{\mathbf{v}_2}{\mathbf{v}_1},$$

i.e., it also approaches $\mathbf{v}$ as $t \to -\infty$. Figure 2.4 illustrates the situation.



(a) $\lambda < 0$ (b) $\lambda > 0$

Figure 2.4: The cases for λ, where we have a stable node when $\lambda < 0$ and an unstable node when $\lambda > 0$.

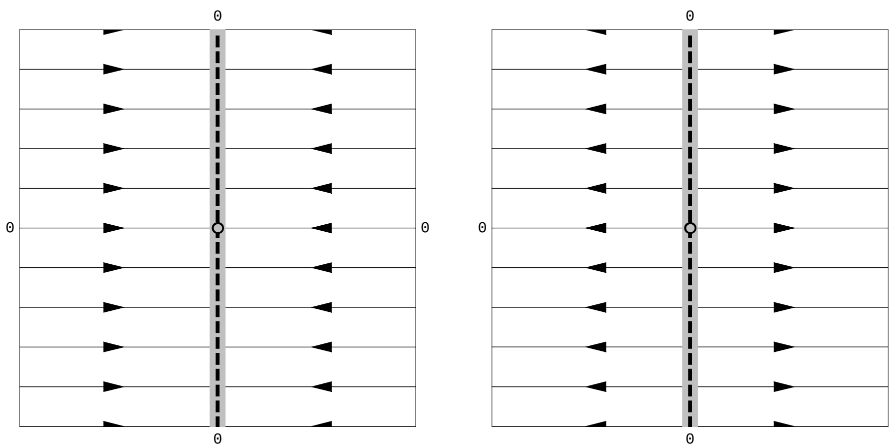
We encounter the degenerate case when $\mathbf{N} = \mathbf{0}$. This does not work, but then $\mathbf{A} = \lambda\mathbf{I}$, so

$$\mathbf{x} = e^{\mathbf{A}t}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = e^{\lambda t}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix},$$

which is just a straight line through

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$

Figure 2.5 illustrates this situation.



(a) $\lambda < 0$ (b) $\lambda > 0$

Figure 2.5: The degenerate cases for λ when $\mathbf{N} = \mathbf{0}$, where we have a stable node when $\lambda < 0$ and an unstable node when $\lambda > 0$.

3. Phase Portraits of Non-Linear Systems

Returning to the general case, we have

$$\begin{cases} \dfrac{dx}{dt} = F(x, y), \\ \dfrac{dy}{dt} = G(x, y). \end{cases}$$

Definition 2.1 (Equilibrium point). A point where $dx/dy = 0$ and $dy/dt = 0$ is called an *equilibrium point* (or *singular point* or *critical point*). ♢

We can get an approximation to the behaviour in the vicinity of each equilibrium point by determining the behaviour of the linear approximation. Let (p, q) be an equilibrium point. Since $F(p, q) = 0$ and $G(p, q) = 0$, the Taylor expansion of $F(x, y)$ and $G(x, y)$ around (p, q) are

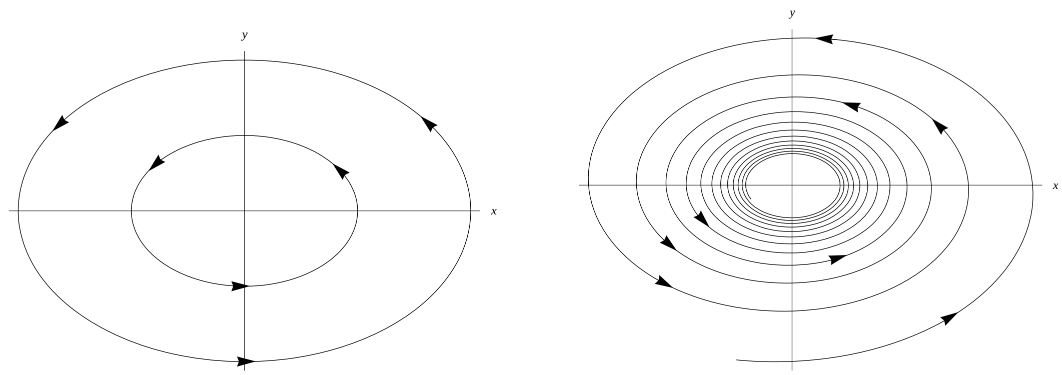
$$\begin{aligned} F(x, y) &= \left.\frac{\partial F}{\partial x}\right|_{(p,q)} (x - p) + \left.\frac{\partial F}{\partial y}\right|_{(p,q)} (y - p) + \cdots, \\ G(x, y) &= \left.\frac{\partial G}{\partial x}\right|_{(p,q)} (x - p) + \left.\frac{\partial G}{\partial y}\right|_{(p,q)} (y - p) + \cdots. \end{aligned}$$

Let $\tilde{x} = x - p$ and $\tilde{y} = y - q$. So the behaviour near (p, q) is approximated by that of $d\mathbf{x}/dt = \mathbf{A}\mathbf{x}$, where

$$\mathbf{x} = \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} x - p \\ y - q \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} \left.\frac{\partial F}{\partial x}\right|_{(p,q)} & \left.\frac{\partial F}{\partial y}\right|_{(p,q)} \\ \left.\frac{\partial G}{\partial x}\right|_{(p,q)} & \left.\frac{\partial G}{\partial y}\right|_{(p,q)} \end{bmatrix}.$$

Definition 2.2 (Stable equilibrium point). An equilibrium point p is called *stable* if for all $\epsilon > 0$, there exists a $\delta > 0$ such that any solution which comes within δ of p never gets farther than ϵ from p at any later time. $\Diamond$

Definition 2.3 (Asymptotically stable equilibrium point). A stable equilibrium point p is called *asymptotically stable* if, in addition to the properties of a stable equilibrium point, there exists an r such that every solution which comes within r of p approaches p as $t \to \infty$. Figure 2.6 illustrates this situation. $\Diamond$



(a) Stable but not asymptotically stable. (b) Asymptotically stable.

Figure 2.6: Stability and asymptotic stability.

From §2.1, linear systems in which both eigenvalues have negative real parts are stable, while, if at least one eigenvalue has a positive real part, it is unstable. The follow theorem ties these ideas together.*

Theorem 2.4. *An equilibrium point is stable if the real parts of both eigenvalues of the corresponding linear system are negative. It is unstable if the real part of at least one eigenvalue is positive.*

In these cases, stability is determined by behaviour of the corresponding linear system. In other words, (e.g., no eigenvalue with positive real part but at least one eigenvalue with no real part) we would require the need to analyze higher order terms (not just linear terms) in the Taylor expansion to determine its behaviour.

Example 2.5. Find and classify the equilibrium points of

$$\begin{cases} F(x,y) = 3x - 3y - x^2 + xy, \\ G(x,y) = 3y + x^2 - 4xy. \end{cases}$$

$\diamond$

Solution. To find the equilibrium points, we set

$$3x - 3y - x^2 + xy = 0, \tag{$*$}$$

$$3y + x^2 - 4xy = 0. \tag{$**$}$$

*See §5, p. 32.

Equation $(*)$ implies that

$$\begin{aligned} 3\left(x-y\right)-x\left(x-y\right)=0 &\Longrightarrow \left(3-x\right)\left(x-y\right)=0 \\ &\Longrightarrow x=3 \text{ or } x=y. \end{aligned}$$

If $x=3$, then

$$\begin{aligned} 3y+9-12y&=0, \\ 9-9y&=0, \end{aligned}$$

so $y=1$ and $(3,1)$ is an equilibrium point.

If $x=y$, then

$$\begin{aligned} 3y+y^2-4y^2&=0, \\ 3y-3y^2&=0, \end{aligned}$$

and $y=y^2$ implies that $y=0$ or $y=1$. Therefore, two more equilibrium points are $(0,0)$ and $(1,1)$. So in summary, the equilibrium points are

$$(0,0), \quad (1,1), \quad (3,1).$$

Note that

$$\mathbf{A}=\begin{bmatrix} \left.\frac{\partial F}{\partial x}\right|_{(p,q)} & \left.\frac{\partial F}{\partial y}\right|_{(p,q)} \\ \left.\frac{\partial G}{\partial x}\right|_{(p,q)} & \left.\frac{\partial G}{\partial y}\right|_{(p,q)} \end{bmatrix}=\begin{bmatrix} 3-2p+q & -3+p \\ 2p-4q & 3-4p \end{bmatrix},$$

where (p,q) is an equilibrium point, i.e., the matrix $\mathbf{A}$ is obtained by evaluating its entries at the equilibrium points.

At $(0,0)$, we have

$$\mathbf{A}=\begin{bmatrix} 3-2(0)+0 & -3+0 \\ 2(0)-4(0) & 3-4(0) \end{bmatrix}=\begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix},$$

which gives us a double root $\lambda=3$, which is indicative of an unstable equilibrium point. Note that

$$\begin{bmatrix} 0 & -3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}=\begin{bmatrix} 0 \\ -3b \end{bmatrix}\Longrightarrow b=0.$$

The eigenvector is $[1,0]$.

At $(1,1)$, we have

$$\mathbf{A}=\begin{bmatrix} 3-2(1)+1 & -3+1 \\ 2(1)-4(1) & 3-4(1) \end{bmatrix}=\begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}.$$

Finding eigenvalues, we have

$$\begin{aligned} \left|\begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}-\lambda\mathbf{I}\right|&=0, \\ \left(2-\lambda\right)\left(-1-\lambda\right)-4&=0, \\ -2-\lambda+\lambda^2-4&=0, \\ \lambda^2-\lambda-6&=0, \\ \left(\lambda-3\right)\left(\lambda+2\right)&=0. \end{aligned}$$

Therefore, $\lambda \in \{-2, 3\}$, which is indicative of an unstable equilibrium point. For $\lambda = 3$, we have

$$\begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a - 2b \\ -2a - 4b \end{bmatrix} \Longrightarrow a = -2b,$$

which gives us the eigenvector $[-2, 1]$. For $\lambda = -2$, we have

$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 4a - 2b \\ -2a + b \end{bmatrix} \Longrightarrow b = 2a,$$

which gives us the eigenvector $[1, 2]$.

At $(3, 1)$, we have

$$\mathbf{A} = \begin{bmatrix} 3 - 2(3) + 1 & -3 + 3 \\ 2(3) - 4(1) & 3 - 4(3) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 2 & -9 \end{bmatrix}.$$

Finding eigenvalues, we have

$$\begin{aligned} \left| \begin{bmatrix} -2 & 0 \\ 2 & -9 \end{bmatrix} - \lambda \mathbf{I} \right| &= 0, \\ \lambda^2 + 11\lambda + 18 &= 0, \\ (\lambda + 9)(\lambda + 2) &= 0. \end{aligned}$$

Therefore, $\lambda \in \{-9, -2\}$, which is indicative of a stable equilibrium point.

For $\lambda = -2$, we have

$$\begin{bmatrix} 0 & 0 \\ 2 & -7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 2a - 7b \end{bmatrix},$$

so $2a = 7b$ and the eigenvector is $[7, 2]$.

For $\lambda - 9$, we have

$$\begin{bmatrix} 7 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 7a \\ 2a \end{bmatrix},$$

so $a = 0$ and the eigenvector is $[0, 1]$. Figure 2.7 shows the phase portrait. ◇

4. Applications

4.1. The Pendulum. Consider the pendulum in Figure 2.8. Let $\mathbf{a} = (a_x, a_y)$ denote the acceleration and let $\mathbf{a}_{\text{normal}}$ denote its component in the normal direction. Then we know that

$$g \sin(\theta) = \mathbf{a}_{\text{normal}} = -a_x \cos(\theta) - a_y \sin(\theta).$$

Since $x = \ell \sin(\theta)$, we have

$$\frac{dx}{dt} = \ell \cos(\theta) \frac{d\theta}{dt}.$$

Therefore,

$$a_x = \frac{d^2x}{dt^2} = -\ell \sin(\theta) \left(\frac{d\theta}{dt} \right)^2 + \ell \cos(\theta) \frac{d^2\theta}{dt^2}.$$

Similarly, since $y = -\ell \cos(\theta)$, we have

$$\frac{dy}{dt} = \ell \sin(\theta) \frac{d\theta}{dt},$$

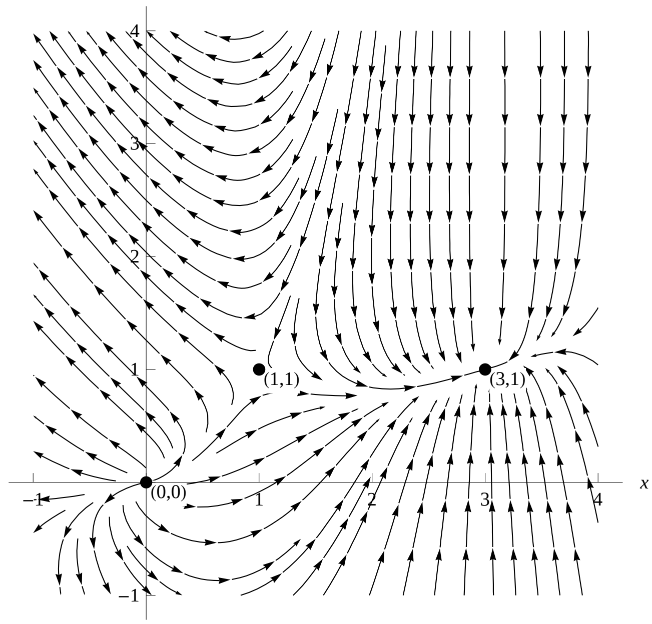


Figure 2.7: The phase portrait of (F, G).

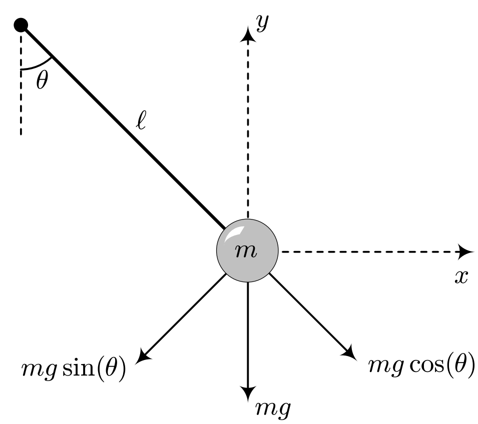


Figure 2.8: A pendulum with a length ℓ and a mass m.

and it follows that

$$a_y = \frac{d^2y}{dt^2} = \ell\cos(\theta)\left(\frac{d\theta}{dt}\right)^2 + \ell\sin(\theta)\frac{d^2\theta}{dt^2}.$$

Therefore, we have

$$\begin{aligned} g\sin(\theta) &= \cancel{\ell\sin(\theta)\cos(\theta)\left(\frac{d\theta}{dt}\right)^2} - \ell\cos^2(\theta)\frac{d^2\theta}{dt^2} - \cancel{\ell\cos(\theta)\sin(\theta)\left(\frac{d\theta}{dt}\right)^2} - \ell\cos^2(\theta)\frac{d^2\theta}{dt^2} \\ &= -\ell\frac{d^2\theta}{dt^2}, \end{aligned}$$

which finally results in

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin(\theta) = 0, \tag{2.1}$$

where θ is the angle, t is time, g is the acceleration due to gravity, and ℓ is the length of the pendulum.

Change the notation: use x to represent the angle. Then Equation (2.1) becomes

$$\frac{d^2x}{dt^2} + \frac{g}{\ell}\sin(x) = 0.$$

Let $y = dx/dt$. Then

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \frac{d\mathbf{x}}{dt} = \begin{bmatrix} y \\ -\frac{g}{\ell}\sin(x) \end{bmatrix}.$$

Let $F(x,y) = y$ and $G(x,y) = (-g/\ell)\sin(x)$. Then the equilibrium points are $(n\pi, 0)$ for $n \in \mathbb{Z}$. It then follows that

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 1, \quad \frac{\partial G}{\partial x} = -\frac{g}{\ell}\cos(x), \quad \frac{\partial G}{\partial y} = 0,$$

and

$$\left.\frac{\partial g}{\partial x}\right|_{n\pi} = -\frac{g}{\ell}(-1)^n = (-1)^{n+1}\frac{g}{\ell}.$$

For n even, we have

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & 0 \end{bmatrix},$$

where $\lambda = \pm i\sqrt{g/\ell}$ is a centre. For n odd, we have

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & 0 \end{bmatrix},$$

where $\lambda = \pm\sqrt{g/\ell}$ is a saddle point. Figure 2.9 shows the phase portrait. The actual solution curves are

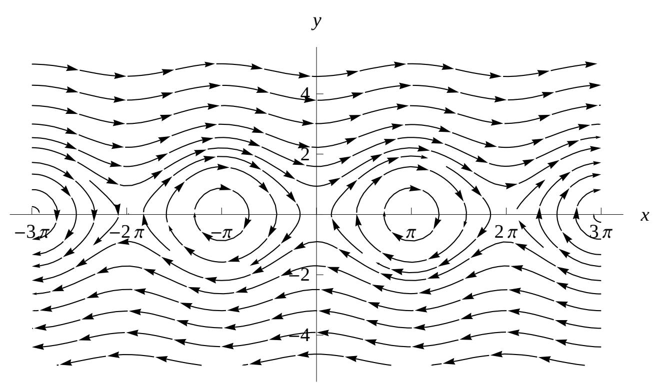


Figure 2.9: The phase portrait of a pendulum.

given by

$$x'x'' + \frac{g}{\ell}\sin(x)x' = 0.$$

Reducing its order gives us

$$\frac{(x')^2}{2} - \frac{g}{\ell}\cos(x) = C,$$

which finally gives us

$$y^2 = 2\frac{g}{\ell}\cos(x) + \tilde{C}.$$

Note that a closed loop in a phase portrait, e.g., the ones surrounding the centres in Figure 2.9, indicates a periodic solution.

4.2. The Damped Pendulum. Adding an air resistance term $-r\ell\frac{dx}{dt}$ to Equation (2.1), we have

$$m\ell\frac{d^2x}{dt^2} = -mg\sin(x) - r\ell\frac{dx}{dt},$$

$$\frac{d^2x}{dt^2} + \frac{r}{m}\frac{dx}{dt} + \frac{g}{\ell}\sin(x) = 0.$$

Letting $y = dx/dt$, we have

$$\frac{dy}{dt} = -\frac{r}{m}y - \frac{g}{\ell}\sin(x).$$

Let $F = y$ and $G = -(g/\ell)\sin(x) - (r/m)\,y$. Then the equilibrium points are $y = 0$ and $x = n\pi$, where $n \in \mathbb{Z}$. Note that

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 1, \quad \frac{\partial G}{\partial x} = -\frac{g}{\ell}\cos(x), \quad \frac{\partial G}{\partial y} = -\frac{r}{m},$$

and

$$\left.\frac{\partial G}{\partial x}\right|_{n\pi} = (-1)^{n+1}\,\frac{g}{\ell}.$$

For n even, we have

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{r}{m} \end{bmatrix},$$

which gives us

$$\lambda^2 + \frac{r}{m}\lambda + \frac{g}{\ell} = 0.$$

Solving for λ gives us

$$\lambda = \frac{-\frac{r}{m} \pm i\sqrt{\frac{4g}{\ell} - \frac{r^2}{m^2}}}{2}.$$

Assuming that $r < 2m\sqrt{g/\ell}$, this gives a stable spiral.

For n odd, we have

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & -\frac{r}{m} \end{bmatrix},$$

which gives us

$$\lambda^2 + \frac{r}{m}\lambda - \frac{g}{\ell} = 0.$$

Solving for λ gives us

$$\lambda = \frac{-\frac{r}{m} \pm \sqrt{\frac{4g}{\ell} + \frac{r^2}{m^2}}}{2}.$$

Assuming once again that $r < 2m\sqrt{g/\ell}$, this gives a saddle. Figure 2.10 shows the phase portrait of a damped pendulum.

4.3. Predator-Prey Equations.

Example 2.6 (Predator-Prey). Consider a land populated by foxes and rabbits, where the foxes prey upon the rabbits. Let $x(t)$ and $y(t)$ be the number of rabbits and foxes, respectively, at time t. In the absence of predators, at any time, the number of rabbits would grow at a rate proportional to the number of rabbits at that time. However, the presence of predators also causes the number of rabbits to decline in proportion to the number of encounters between a fox and a rabbit, which is proportional to the product

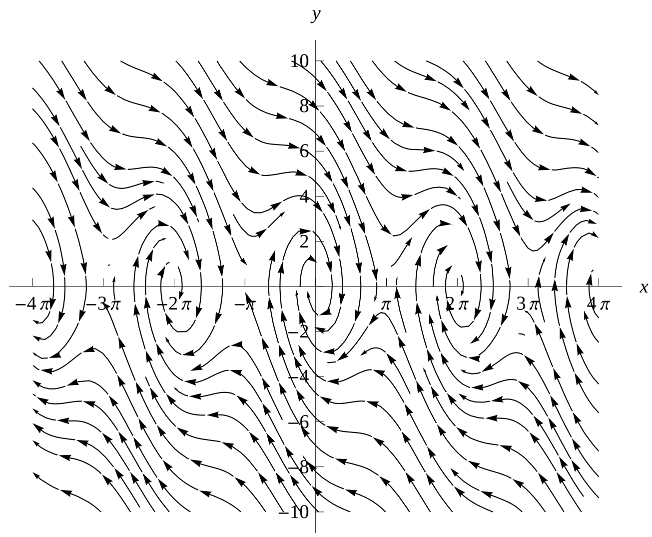


Figure 2.10: The phase portrait of a damped pendulum.

$x(t)y(t)$. Therefore, $dx/dt = Ax - Bxy$ for some positive constants a and b. For the foxes, the presence of other foxes represents competition for food, so the number declines proportionally to the number of foxes but grows proportionally to the number of encounters. Therefore $dy/dt = -Cy + Dxy$ for some positive constants c and d. The system

$$\begin{cases} \dfrac{dx}{dt} = Ax - Bxy, \\ \dfrac{dy}{dt} = -Cy + Dxy \end{cases}$$

is our mathematical model.

If we want to find the function $y(x)$, which gives the way that the number of foxes and rabbits are related, we begin by dividing to get the differential equation

$$\frac{dy}{dx} = \frac{-Cy + Dxy}{Ax - Bxy}$$

with $A, B, C, D, x(t), y(t)$ positive. In this case, we can solve explicitly as

$$\begin{aligned} \frac{dy}{dx} &= \frac{y\,(-C + Dx)}{x\,(A - By)}, \\ \frac{A - By}{y}\,dy &= \frac{-C + Dx}{x}\,dx, \\ \left(\frac{A}{y} - B\right)dy &= \left(-\frac{C}{x} + D\right)dx, \\ A\ln\big(|y|\big) - By &= -C\ln\big(|x|\big) + Dx + \tilde{C}, \\ y^A e^{-By} &= kx^{-C}e^{Dx} \end{aligned} \tag{2.2}$$

for some constant k. We can use the method of implicit differentiation* to verify that it is indeed a solution of the equation for any k.

*MATA30.

Explicitly, if $y(x)$ is the function defined implicitly by Equation (2.2), then

$$Ay^{A-1}y'e^{-By} + y^A(-B)e^{-By}y' = k(-C)x^{-C-1}e^{Dx} + kx^{-C}De^{Dx}.$$

Replacing k from Equation (2.2) gives

$$\begin{aligned} Ay^{A-1}y'e^{-By} + y^A(-B)e^{-By}y' &= -Cx^{-C-1}e^{Dx}\frac{y^Ae^{-By}}{x^{-C}e^{Dx}} + x^{-C}De^{Dx}\frac{y^Ae^{-By}}{x^{-C}e^{Dx}} \\ &= -\frac{Cy^Ae^{-By}}{x} + Dy^Ae^{-By}. \end{aligned}$$

Dividing by $y^{A-1}e^{-By}$ gives

$$Ay' + y(-B)y' = -\frac{Cy}{x} + Dy,$$

and so solving for y' gives

$$y' = \frac{-Cy + Dxy}{Ax - Bxy},$$

as desired.

The graph of a typical solution is shown in Figure 2.11.

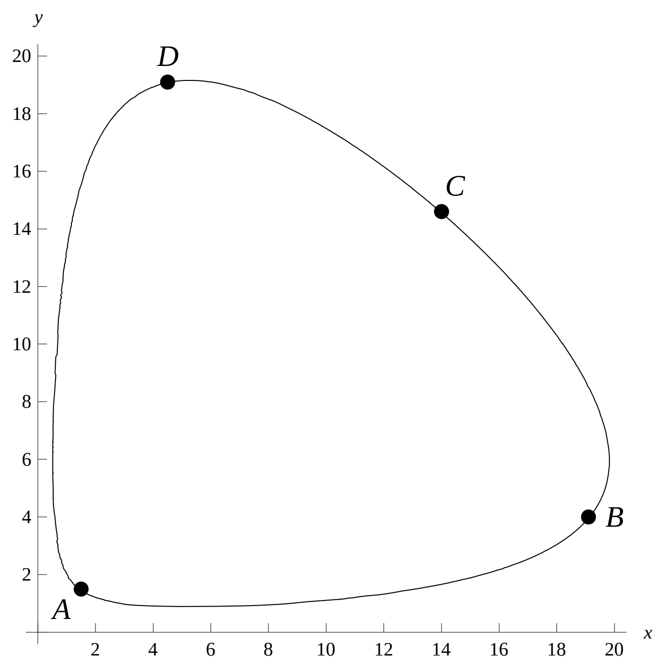


Figure 2.11: A typical solution of the Predator-Prey model with $a = 9.4$, $b = 1.58$, $c = 6.84$, $d = 1.3$, and $k = 7.54$.

Beginning at a point such as A, where there are few rabbits and few foxes, the fox population does not initially increase much due to the lack of food, but with so few predators, the number of rabbits multiplies rapidly. After a while, the point B is reached, at which time the large food supply causes the rapid increase in the number of foxes, which in turn curtails the growth of the rabbits. By the time point C is reached, the large number of predators causes the number of rabbits to decrease. Eventually, point D is reached, where the number of rabbits has declined to the point where the lack of food causes the fox population to decrease, eventually returning the situation to point A.

To find the equilibrium points, we know that we either must have $x = 0$ or $A = By$ and $y = 0$ or $C = Dx$. Therefore, the equilibrium points are

$$(0,0), \quad \left(\frac{C}{D}, \frac{A}{B}\right),$$

so we have

$$\mathbf{A} = \begin{bmatrix} A - By & -Bx \\ Dy & -C + Dx \end{bmatrix}.$$

At $(0,0)$, we have

$$\begin{bmatrix} A & 0 \\ 0 & -C \end{bmatrix},$$

giving us a saddle point. At $(C/D, A/B)$, we have

$$\begin{bmatrix} 0 & -\frac{BC}{D} \\ \frac{AD}{B} & 0 \end{bmatrix},$$

the determinant of which is AC, so $\lambda = \pm i\sqrt{AC}$, giving us a centre point. Figure 2.12 shows the phase portrait. ◇

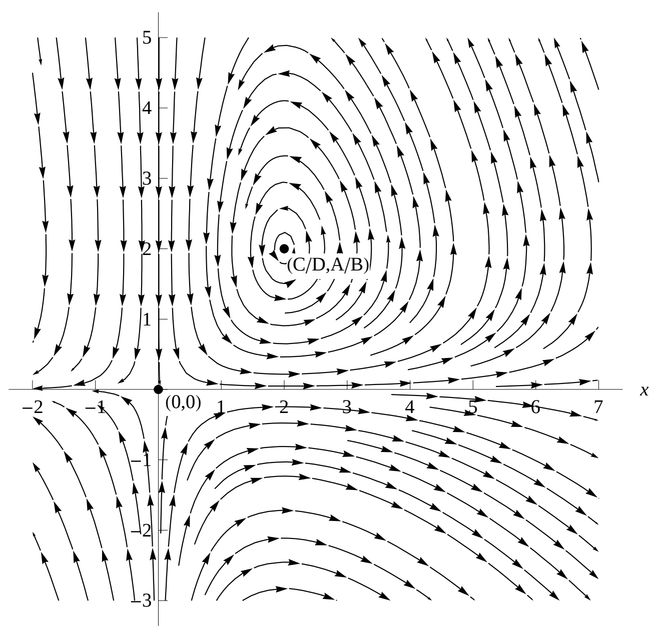


Figure 2.12: The phase portrait of Example 2.6, showing a saddle point at the origin and a centre point at $\left(\frac{C}{D}, \frac{A}{B}\right)$.

5. Liapunov's Second Method

We have been examining linearized systems about each equilibrium point to get an idea of how the original system behaves. But to what extent is it possible to conclude that the properties of linearized systems accurately reflect properties of the actual system?

Theorem 2.8a. *Consider*

$$\begin{cases} x' = F(x, y), \\ y' = G(x, y). \end{cases} \tag{2.3}$$

Let $\mathbf{V} = (F, G)$ *and let* $\mathbf{0}$ *be an equilibrium point of System* (2.3).* *Suppose there exists a function* E *with the following properties:***

(1) $E(x, y) > 0$ *for* $(x, y) \neq (0, 0)$ *and* $E(0, 0) = 0$.
(2) E *is differentiable.*
(3) For any solution $\big(x(t), y(t)\big)$ *of System* (2.3), *there exists an* $r > 0$ *such that* $\nabla E \cdot \mathbf{V} \leq 0$ *whenever* $x^2 + y^2 < r$.

Then $\mathbf{0}$ *is a stable equilibrium point of System* (2.3).

Proof. The idea of the theorem is this. Consider a contour line $E = C$, as shown in Figure 2.13. Intuitively, the hypothesis that $\nabla E \cdot \mathbf{V} \leq 0$ says that $\mathbf{V}$ points inwards, so that once a solution enters the

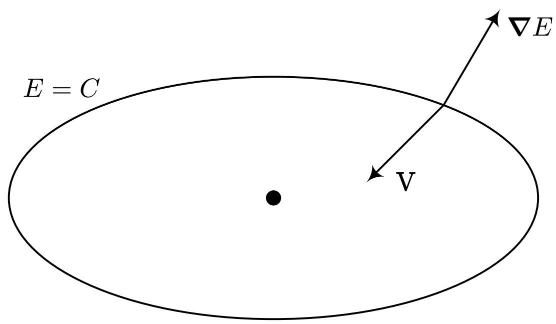


Figure 2.13: Some contour line $E = C$.

region surrounded by $E = C$ it can never leave. More precisely, if $\big(x(t), y(t)\big)$ is a solution, then

$$\begin{aligned} \frac{d}{dt} E\big(x(t), y(t)\big) &= \frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial y}\frac{dy}{dt} \\ &= \frac{\partial E}{\partial x} F + \frac{\partial E}{\partial y} G \\ &= \underbrace{\nabla E \cdot \mathbf{V}}_{\leq 0} \end{aligned}$$

So if $p_1 = \big(x(t_1), y(t_1)\big)$ and $p_2 = \big(x(t_2), y(t_2)\big)$ are points on a solution curve with $t_2 > t_1$, then

$$\begin{aligned} \int_{t_1}^{t_2} \frac{dE}{dt}\, dt &\leq 0, \\ E(x, y)\Big|_{t_1}^{t_2} &\leq 0, \\ E(p_2) - E(p_1) &\leq 0. \end{aligned}$$

Therefore, $E(p_2) \leq E(p_1)$, i.e., E decreases with t, so once it enters a region bounded by a contour line of E, it can never leave.

*We can always move our point to the origin by translation.
**Such a function is called a *Liapunov's Function* for the system.

Recall the definition of stability (p. 24) that states that given an $\epsilon > 0$, there exists a $\delta > 0$ such that any solution coming within δ of p never thereafter gets farther than ϵ of p. So given an $\epsilon > 0$, let m be the minimum value of R on $x^2+y^2=\epsilon$. Such a value exists because E is continuous and the locus of $x^2+y^2=\epsilon$ is compact. Furthermore, $m > 0$ because $E > 0$ on $x^2+y^2=\epsilon$. Then the contour line $E(x,y) = m/2$ lies entirely inside $x^2+y^2=\epsilon$, as illustrated in Figure 2.14. Since E is continuous and $E(0,0) = 0$, there

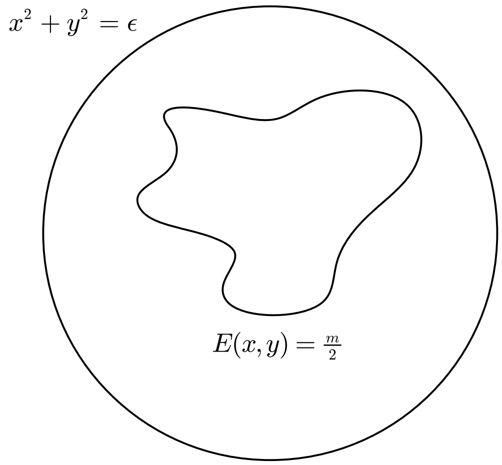


Figure 2.14: The contour line $E(x,y) = m/2$ lies entirely inside $x^2+y^2=\epsilon$. They can't touch because there is no point on $x^2+y^2=\epsilon$ where $E(x,y) = m/2$.

exists a $\delta > 0$ such that $E(x,y) < m/2$ whenever $x^2+y^2 < \delta$. Once a solution enters $x^2+y^2=\delta$, then $E(x,y) < m/2$, so it can never thereafter leave $E = m/2$ and thus can never leave $x^2+y^2=\epsilon$. $\square$

Theorem 2.8b. *Assume the hypotheses of Theorem 2.8a hold except that condition (3) is strengthened to*

(3') There exists an $r > 0$ and $\alpha > 0$ such that $\nabla E \cdot \mathbf{V} \le -\alpha E$ whenever $0 < x^2+y^2 < r^2$.

Then we get the (stronger) conclusion that $\mathbf{0}$ is asymptotically stable.

Proof. Suppose that for any solution $\big(x(t), y(t)\big)$ of System (2.3), there exists an $r > 0$ and $\alpha > 0$ such that $\nabla E \cdot \mathbf{V} \le -\alpha E$ whenever $0 < x^2+y^2 < r^2$. Then

$$\frac{dE}{dt} = \nabla E \cdot \mathbf{V} \le -\alpha E.$$

Therefore,

$$\frac{dE}{dt} + \alpha E \le 0$$

and so

$$e^{\alpha t}\frac{dE}{dt} + e^{\alpha t}E \le 0.$$

Furthermore,

$$e^{\alpha t}E \le C \Longrightarrow E \le Ce^{-\alpha t} \Longrightarrow \lim_{t\to\infty} E\big(x(t), y(t)\big) = 0,$$

that is, on each solution curve, $E \to 0$, so $(x,y) \to \mathbf{0}$. $\square$

Theorem 2.8c. *Assume the conditions of Theorem 2.8a but conditions (2) and (3) are strengthened to*

(2') E is continuously differentiable (as Boyce and Di Prima assume)

(3") There exists an $r > 0$ such that $\nabla E \cdot \mathbf{V} < 0$ whenever $0 < x^2 + y^2 < r$. Then $\mathbf{0}$ is asymptotically stable.

Proof. As in the proof of Theorem 2.8a, E is a decreasing function along each solution curve. We already showed stability. Therefore, suppose k has the property that a solution entering $x^2 + y^2 \leq k$ never leaves.

We want to show that $\lim_{t\to\infty} E\big(\mathbf{x}(t)\big) = 0$ for any solution curve $\mathbf{x}(t)$. Suppose $\mathbf{x}(t)$ is a solution curve which does not have this property. Then there exists a $c > 0$ such that $E\big(\mathbf{x}(t)\big) \geq c$ for all t. Therefore, the solution $\mathbf{x}(t)$ avoids the open set $E^{-1}\big([0, c)\big)$ and so there exists a radius R such that the solution never enters the ball $\|\mathbf{x}\| < R$. Thus, for some t_0, the solution lies in the annulus $R \leq \|\mathbf{x}\| \leq r$ for all $t \geq t_0$. Since $dE/dt = \nabla E \cdot \mathbf{V}$ is continuous (and negative), it attains a maximum $-M$ (where $M > 0$) on the compact set $R \leq \|\mathbf{x}\| \leq r$ for all $t \geq t_0$.

Therefore, for all $t > t_0$,

$$\underbrace{\int_{t_0}^{t} \frac{d}{dt} E\big(\mathbf{x}(t)\big)}_{E\left(\mathbf{x}(t)\right) - E\left(\mathbf{x}(t)\right)} \leq \int_{t_0}^{t} -M\, dt = -M\,(t - t_0)\,.$$

This implies that

$$E\big(\mathbf{x}(t)\big) \leq \underbrace{E\big(\mathbf{x}(t_0)\big) + Mt_0}_{\text{constant}} - \underbrace{Mt}_{\to -\infty}\,.$$

for all t. This is a contradiction as $E\big(\mathbf{x}(t)\big) > 0$. Therefore, $E\big(\mathbf{x}(t)\big)$ eventually gets less than any c, i.e.,

$$\lim_{t\to\infty} E\big(\mathbf{x}(t)\big) = 0 \Longrightarrow \mathbf{x}(t) \to \mathbf{0}. \qquad \square$$

Theorem 2.9. *Let $\mathbf{0}$ be an equilibrium point of System* (2.3)*. Suppose there exists a function E with the following properties:*

(1) $E(x, y) > 0$ for some (x, y) in every neighbourhood of the origin and $E(0, 0) = 0$.
(2) E is differentiable.
(3) For any solution $\big(x(t), y(t)\big)$ of System (2.3)*, there exists an r such that $\nabla E \cdot \mathbf{V} > 0$ whenever $0 < x^2 + y^2 < r$.*

Then $\mathbf{0}$ is an unstable equilibrium point of System (2.3).*

Proof. Using ideas similar to previous proofs, one can show that this is true. $\square$

Corollary 2.10. *An equilibrium point is asymptotically stable if the real parts of both eigenvalues of the corresponding linearized system are negative. It is unstable if the real part of at least one eigenvalue is positive.***

*Note that

$$\frac{d}{dt} E\big(x(t), y(t)\big) = \frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial y}\frac{dy}{dt} = \nabla E \cdot \underbrace{\left(\frac{dx}{dt}, \frac{dy}{dt}\right)}_{\mathbf{V}} = \nabla E \cdot \mathbf{V}.$$

**There is no conclusion if $\lambda_1 = 0$ while $\lambda_2 \leq 0$, e.g., if the linearized system has a centre at p, p may or may not be stable. If there exists an $E > 0$ with $E(0, 0) = 0$ such that $\nabla E \cdot \mathbf{V} > 0$, then the origin is not stable.

Proof (sketch). Suppose that the real parts of both eigenvalues are negative. Let $\mathbf{x} = [x, y]$. Write

$$\begin{cases} \dfrac{dx}{dt} = F(x,y) = Ax + By + f(x,y), \\ \dfrac{dy}{dt} = G(x,y) = Cx + Dy + g(x,y), \end{cases}$$

where f and g are continuous with $f(0,0) = 0 = g(0,0)$, and there exist constants k_1 and k_2 such that $|f(x,y)| \le k_1 \|\mathbf{x}\|$ and $|g(x,y)| \le k_2 \|\mathbf{x}\|$ whenever $\|\mathbf{x}\|$ is sufficiently small. Let

$$\mathbf{A} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} F \\ G \end{bmatrix} = \mathbf{A}\mathbf{x} + \begin{bmatrix} f \\ g \end{bmatrix}.$$

Let $p = \operatorname{tr}(\mathbf{A}) = A+D$ and $q = \det(\mathbf{A}) = AD-BC$. The characteristic equation then becomes $\lambda^2 - p\lambda + q = 0$. If the roots are real, then, by the hypothesis, they are negative, so their sum p is negative and their product q is positive. If the roots are complex, say $u \pm iv$, then, by the hypothesis, $u < 0$ and so again $p = -2u$ is negative and $q = u^2 + v^2$ is positive. Let

$$Q = (Ax+By)^2 + (Cx+Dy)^2 = \mathbf{A}\mathbf{x} \cdot \mathbf{A}\mathbf{x}.$$

and set $E = Q + q\left(x^2 + y^2\right)$. Clearly, $E > 0$ for $\mathbf{x} \ne (0,0)$ and $E(0,0) = 0$. Why is $\nabla E \cdot \mathbf{V} < 0$ for small $\|\mathbf{x}\|$?

To see this, note that

$$\nabla E = \nabla Q + q\nabla\left(x^2+y^2\right) = \nabla Q + 2q\,(x,y) = \nabla Q + 2q\mathbf{x}$$

and

$$\begin{aligned} \nabla Q &= \begin{bmatrix} 2\,(Ax+By)\,A + 2\,(Cx+Dy)\,C \\ 2\,(Ax+By)\,B + 2\,(Cx+Dy)\,D \end{bmatrix} \\ &= 2\begin{bmatrix} A & C \\ B & D \end{bmatrix}\begin{bmatrix} Ax+By \\ Cx+Dy \end{bmatrix} \\ &= 2\mathbf{A}^t\mathbf{A}\mathbf{x}. \end{aligned}$$

Therefore,

$$\begin{aligned} \nabla E \cdot \mathbf{V} &= \left(2\mathbf{A}^t\mathbf{A}\mathbf{x} + 2q\mathbf{x}\right) \cdot \left(\mathbf{A}\mathbf{x} + [f,g]\right) \\ &= 2\left(\mathbf{X}^t\mathbf{A}^t\mathbf{A}^t\mathbf{A}\mathbf{x} + q\mathbf{x}^t\mathbf{A}\mathbf{x}\right) + 2\left(\mathbf{A}^t\mathbf{A}\mathbf{x} + q\mathbf{x}\right) \cdot [f,g]\,. \end{aligned} \tag{$*$}$$

Note that by the Cayley-Hamilton Theorem, we have

$$\mathbf{A}^2 - \operatorname{tr}(\mathbf{A})\mathbf{A} + \det(\mathbf{A})\mathbf{I} = \mathbf{0}.$$

That is,

$$\mathbf{A}^2 - p\mathbf{A} + q\mathbf{I} = \mathbf{0}.$$

Taking the transpose gives

$$\begin{aligned} \left(\mathbf{A}^t\right)^2 - p\mathbf{A}^t + q\mathbf{I} &= \mathbf{0}, \\ \left(\mathbf{A}^t\right)^2 + q\mathbf{I} &= p\mathbf{A}^t. \end{aligned}$$

Therefore, Equation $(*)$ now becomes

$$\begin{aligned}\nabla E \cdot \mathbf{V} &= 2\left(\mathbf{x}^t\left(\left(\mathbf{A}^t\right)^2 + q\mathbf{I}\right)\mathbf{A}\mathbf{x}\right) + 2\left(\mathbf{A}^t\mathbf{A}\mathbf{x} + q\mathbf{x}\right)\cdot[f,g] \\ &= 2p\mathbf{x}^t\mathbf{A}^t\mathbf{A}\mathbf{x} + 2\left(\mathbf{A}^t\mathbf{A} + q\mathbf{x}\right)\cdot[f,g] \\ &= 2p\mathbf{A}\mathbf{x}\cdot\mathbf{A}\mathbf{x} + 2\left(\mathbf{A}^t\mathbf{A} + q\mathbf{x}\right)\cdot[f,g] \\ &= \underbrace{2pQ}_{<0} + 2\left(\mathbf{A}^t\mathbf{A} + q\mathbf{x}\right)\cdot[f,g]\,.\end{aligned}$$

We have $2pQ < 0$ since $p < 0$ and $Q > 0$. Using the fact that $\left\|[f,g]\right\| \leq \sqrt{k_1^2 + k_2^2}\,\|\mathbf{x}\|$, we can show that the second term is less than or equal to $|p|\,Q$ for small $\|\mathbf{x}\|$.

Therefore, it is not big enough to affect the sign of $\nabla E \cdot \mathbf{V}$, i.e., $\nabla E \cdot \mathbf{V} < 0$ for small nonzero $\|\mathbf{x}\|$. □

Example 2.11. Consider $\mathbf{V} = \left(-2xy, x^2 - y^3\right)$. Is the origin stable? ◇

Solution. First note that the only equilibrium point is $(x,y) = (0,0)$. Suppose we try $E(x,y) = ax^2 + by^2$ for suitable $a, b > 0$. Then

$$\nabla E \cdot \mathbf{V} = [2ax, 2by] \cdot \left[-2xy, x^2 - y^3\right] = -4ax^2y + 2bx^2y - 2by^4.$$

Choose $a = 1$ and $b = 2$ (so that the x^2y term will cancel). Then $\nabla E \cdot \mathbf{V} = -4y^4 \leq 0$. Therefore, by Liapunov, the origin is stable. Figure 2.15 shows the phase portrait of $\mathbf{V}$. ◇

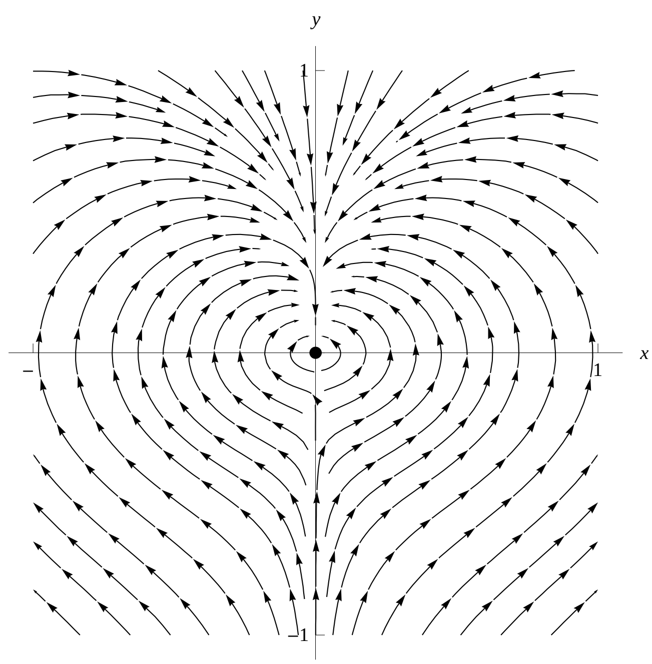


Figure 2.15: The phase portrait of $\left(-2xy, x^2 - y^3\right)$ of Example 2.11, showing that the origin is stable.

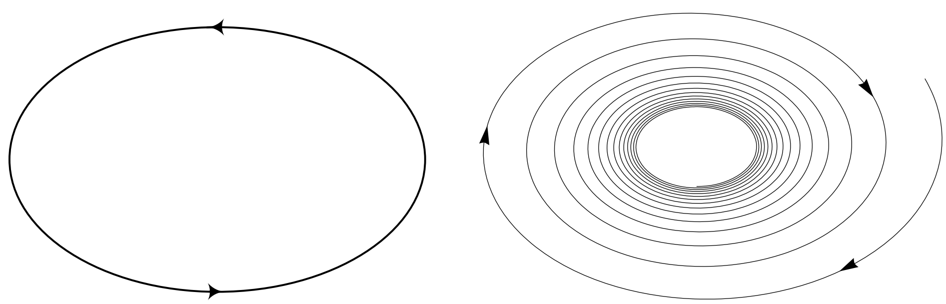
6. Periodic Solutions

THEOREM 2.12 (Poincaré-Bendixson). *Let R be a closed bounded region in $\mathbb{R}^2$. Suppose*

$$\begin{cases} \dfrac{dx}{dt} = F(x,y), \\ \dfrac{dy}{dt} = G(x,y) \end{cases} \tag{2.4}$$

has a solution $\big(x(t), y(t)\big)$ which lies in R for all $t \geq t_0$. If System (2.4) has no equilibrium points in R, then either

(1) $\big(x(t), y(t)\big)$ is a periodic solution (i.e., a closed curve or loop), as shown in Figure 2.16a or
(2) $\big(x(t), y(t)\big)$ spirals towards a periodic solution, as shown in Figure 2.16b.



(a) A periodic solution. *(b) Spirals towards a periodic solution.*

Figure 2.16:

PROOF (IDEA). Let $C = \big(x(t), y(t)\big)$ be our given solution curve. Let $p_n = \big(x(t_n + n)\,, y(t_0 + n)\big)$. Unless C is a periodic solution, the points $\{p_n\}$ are distinct, so by the Bolzano-Weierstrass Theorem, there exists an accumulation point p of $\{p_n\}$ lying in R (since R is compact).

Let C_0 be the solution curve passing through p. Note that since we assumed no equilibrium points in R, p is not an equilibrium point, so C_0 is a curve, not just the point p. Intuitively, since the solution curves cannot cross and C has points on it that approach p as a limit, C must spiral towards C_0. More precisely, we have the following.

LEMMA 2.13. *Let $C = \big(x(t), y(t)\big)$ be a solution curve to System (2.4), let $p_n = \big(x(t_n + n)\,, y(t_0 + n)\big)$, let p be an accumulation point of $\{p_n\}$ lying in R, and let C_0 be the solution curve passing through p. Then there exists a short line segment ℓ through p having the following properties:*

(1) The curves C and C_0 cross ℓ infinitely often in every neighbourhood of p.
(2) Every solution crossing ℓ does so in the same direction.

PROOF. Proof is omitted, but it uses continuity and the Jordan Curve Theorem (Theorem 2.14). $\square$

Let q be the next point at which C_0 crosses ℓ. We show that $q = p$ so that C_0 is a periodic solution. The curve C crosses ℓ near p (say, at p'), so by continuity, it must cross again near q (say, at q'). This is illustrated in Figure 2.17. But then every subsequent crossing of ℓ by C must be farther away from p than

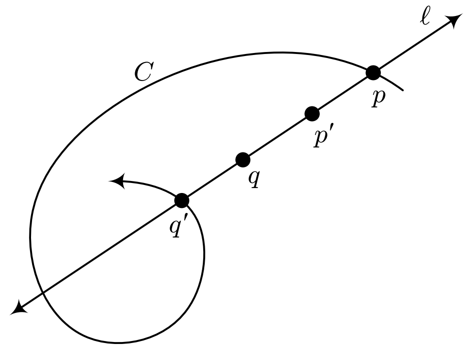


Figure 2.17: The curve C crossing ℓ.

from q', since C cannot cross itself and it cannot cross ℓ in the wrong direction. But this contradicts C crossing ℓ infinitely often in every neighbourhood of p. Therefore, $p = q$, so C is a closed curve.

The point is this. If p' is farther from p than from q', then the next crossing would be ever farther away. But if $p = q$, then q' can be closer to p than p' was, so everything is okay.

Thus, q' is closer to p than p' was and subsequent crossings are even closer. Applying this argument now to other points on C_0 and other lines, we can see that C must be approaching C_0. □

THEOREM 2.14 (Jordan Curve Theorem). *Let C be a closed curve in $\mathbb{R}^2$ which does not cross itself. Then C divides $\mathbb{R}^2$ into two disjoint non-empty connected open subsets, having C as their common boundary, namely, $\mathbb{R}^2 \setminus C = I \cup O$. One of these open sets is bounded and the other is unbounded.*[*]

EXAMPLE 2.15. Consider

$$\begin{cases} x' = x - y - x\left(x^2 + \dfrac{3}{2}y^2\right), \\ y' = x + y - y\left(x^2 + \dfrac{1}{2}y^2\right). \end{cases}$$

◇

Let

$$\begin{cases} F(x,y) = x - y - x\left(x^2 + \dfrac{3}{2}y^2\right), \\ G(x,y) = x + y - y\left(x^2 + \dfrac{1}{2}y^2\right). \end{cases}$$

Find equilibrium points. Setting $F = 0$ gives

$$x\left(1 - x^2 - \frac{3}{2}y^2\right) = y \tag{$*$}$$

and setting $G = 0$ gives

$$-y\left(1 - x^2 - \frac{1}{2}y^2\right) = x. \tag{$**$}$$

Therefore,

$$\left(1 - x^2 - \frac{3}{2}y^2\right)\left(1 - x^2 - \frac{1}{2}y^2\right) = \frac{y}{x}\left(-\frac{x}{y}\right) = -1$$

---

[*]They are called, respectively, the *inside* and *outside* of C.

unless $x = 0$ or $y = 0$. If $x = 0$, then Equation $(*)$ implies that $y = 0$. If $y = 0$, then Equation $(**)$ implies that $x = 0$. Therefore, $(0, 0)$ is one solution.

Let $a = 1 - x^2 - (1/2)\,y^2$. Then $\left(a - y^2\right) a = -1$. Therefore, one factor is positive and one is negative, which implies that $a > 0$ and $a - y^2 < 0$.

$$\begin{aligned} a^2 - ay^2 &= -1, \\ a^2 - ay^2 + 1 &= 0. \end{aligned}$$

To have real solutions for a, we need

$$y^4 - 4 \geq 0 \Longrightarrow y^2 \geq 2.$$

But then

$$a > 0 \Longrightarrow x^2 + \frac{1}{2}y^2 < 1 \Longrightarrow y^2 < 2,$$

which is a contradiction. Therefore, no solution exists other than $(0, 0)$.

Let $\mathbf{V} = (F, G)$. Consider the behaviour of $\mathbf{V}$ on circles $x^2 + y^2 = c^2$, as shown in Figure 2.18.

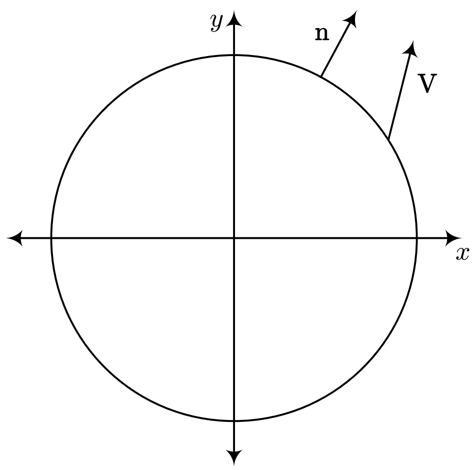


Figure 2.18: The vector $\mathbf{V}$ on a circle $x^2 + y^2 = c^2$.

To determine if $\mathbf{V}$ points into the circle or out of the circle, we look at $\mathbf{V} \cdot \mathbf{n}$. Then

- $\mathbf{V} \cdot \mathbf{n} > 0$ implies that $\mathbf{V}$ is pointing out.
- $\mathbf{V} \cdot \mathbf{n} = 0$ implies that $\mathbf{V}$ is tangent to the circle.
- $\mathbf{V} \cdot \mathbf{n} < 0$ implies that $\mathbf{V}$ is pointing in.

To find out which condition it satisfies, we compute

$$\begin{aligned} \mathbf{V} \cdot \mathbf{n} = (F, G) \cdot (x, y) &= Fx + Gy \\ &= x^2 - \cancel{xy} - x^2\left(x^2 + \frac{3}{2}y^2\right) + \cancel{xy} + y^2 - y^2\left(x^2 + \frac{1}{2}y^2\right) \\ &= x^2 - x^4 - \frac{3}{2}x^2y^2 + y^2 - x^2y^2 - \frac{1}{2}y^4 \\ &= x^2 + y^2 - x^4 - \frac{1}{2}y^4 - \frac{5}{2}x^2y^2 \\ &= r^2 - x^4 - 2x^2y^2 - y^4 + \frac{1}{2}y^4 - \frac{1}{2}x^2y^2 \end{aligned}$$

$$
\begin{aligned}
&= r^2 - \left(x^2+y^2\right)^2 + \frac{1}{2}y^2\left(y^2-x^2\right) \\
&= r^2 - r^4 - \frac{1}{2}y^2\left(x^2-y^2\right) \\
&= r^2 - r^4 - \frac{1}{2}r^2\sin^2(\theta)\left(r^2\cos^2(\theta) - r^2\sin^2(\theta)\right) \\
&= r^2 - r^4 - \frac{1}{2}r^4\sin^2(\theta)\cos(2\theta) \\
&= r^2 - r^4\left(1+\frac{1}{2}\sin^2(\theta)\cos(2\theta)\right) \\
&= r^2\left(1 - r^2\left(1+\frac{1}{2}\sin^2(\theta)\cos(2\theta)\right)\right).
\end{aligned}
$$

Note that

$$-\frac{1}{2} \le \frac{1}{2}\sin^2(\theta)\cos(2\theta) \le \frac{1}{2} \implies \frac{1}{2} \le 1+\frac{1}{2}\sin^2(\theta)\cos(2\theta) \le \frac{3}{2}.$$

If $r=2$, then $r^2=4$, so

$$
\begin{aligned}
r^2\left(1+\frac{1}{2}\sin^2(\theta)\cos(2\theta)\right) \ge 2 &\implies 1 - r^2\left(1+\frac{1}{2}\sin^2(\theta)\cos(2\theta)\right) < 0 \\
&\implies \mathbf{V}\cdot\mathbf{n} < 0.
\end{aligned}
$$

If $r=1/2$, then $r^2=1/4$, so

$$
\begin{aligned}
r^2\left(1+\frac{1}{2}\sin^2(\theta)\cos(2\theta)\right) \le \frac{3}{8} &\implies 1 - r^2\left(1+\frac{1}{2}\sin^2(\theta)\cos(2\theta)\right) > 0 \\
&\implies \mathbf{V}\cdot\mathbf{n} > 0.
\end{aligned}
$$

This situation is illustrated in Figure 2.19. So once a solution comes within $r=2$, it stays within $r=2$, but a solution outside $r=1/2$ stays outside $r=1/2$. Therefore, let R be the region between $r=1/2$ and $r=2$. This region contains no equilibrium points, but any solution which enters it stays within it.

So applying the Poincaré-Bendixson Theorem (Theorem 2.12, p. 38) shows that any solution within R spirals towards a periodic solution within R, as shown in Figure 2.19. Figure 2.20 shows the phase portrait

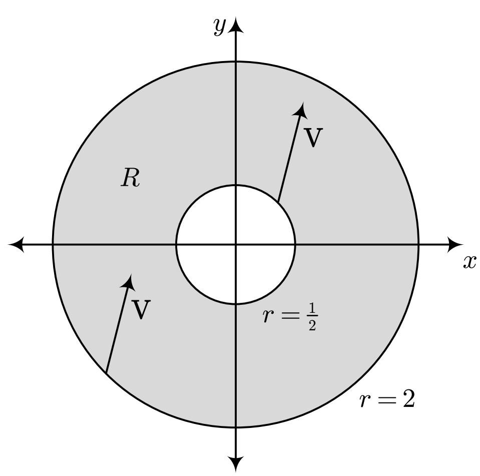


Figure 2.19: If $r=2$, then $\mathbf{V}$ points out. If $r=1/2$, then it points in.

of the system.

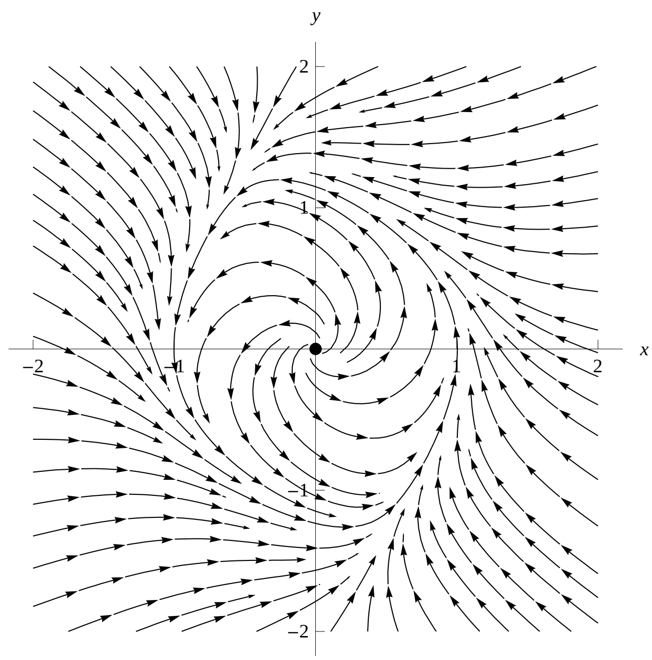


Figure 2.20: The phase portrait of Example 2.15, showing that the origin is the only equilibrium point.

EXAMPLE 2.16. Consider

$$\begin{cases} x' = -y + \dfrac{x}{\sqrt{x^2+y^2}}\Big(1-\big(x^2+y^2\big)\Big), \\ y' = x + \dfrac{y}{\sqrt{x^2+y^2}}\Big(1-\big(x^2+y^2\big)\Big). \end{cases}$$

$\diamond$

Immediately, note that $(x, y) \neq (0, 0)$ as we must enforce $\sqrt{x^2+y^2} \neq 0$. To find equilibrium points, solve

$$-y + \frac{x}{\sqrt{x^2+y^2}}\Big(1-\big(x^2+y^2\big)\Big) = 0, \tag{$*$}$$

$$x + \frac{y}{\sqrt{x^2+y^2}}\Big(1-\big(x^2+y^2\big)\Big) = 0. \tag{$**$}$$

It follows that

$$y^2 \underbrace{=}_{\text{Eq. }(*)} \frac{xy}{\sqrt{x^2+y^2}}\Big(1-\big(x^2+y^2\big)\Big) \underbrace{=}_{\text{Eq. }(**)} -x^2.$$

Therefore, there is no solution in the domain of $\mathbf{V}$, which is $\mathbb{R}^2 \setminus \{\mathbf{0}\}$.

Consider $\mathbf{V} \cdot \mathbf{n}$ on the circle $x^2 + y^2 = c^2$. Then

$$\begin{aligned} \mathbf{V} \cdot \mathbf{n} &= Fx + Gy \\ &= -\cancel{xy} + \frac{x^2}{\sqrt{x^2+y^2}}\Big(1-\big(x^2+y^2\big)\Big) \end{aligned}$$

$$
\begin{aligned}
&+\cancel{xy}+\frac{y^2}{\sqrt{x^2+y^2}}\Big(1-\left(x^2+y^2\right)\Big)\\
&=\sqrt{x^2+y^2}\Big(1-\left(x^2+y^2\right)\Big)\\
&=r\left(1-r^2\right).
\end{aligned}
$$

Therefore, if $r>1$, then $\mathbf{V}\cdot\mathbf{n}<0$, while if $r<1$, then $\mathbf{V}\cdot\mathbf{n}>0$. So the solutions entering the annulus $1/2\le r\le 3/2$ stay there. Since there are no equilibrium points in this annulus, by the Poincaré-Bendixson Theorem (Theorem 2.12, p. 38), it has a periodic solution.

In fact, let $r^2=x^2+y^2$. Then

$$
\begin{aligned}
2rr' &= 2xx'+2yy',\\
rr' &= xx'+yy'\\
&= xF+yG\\
&= r\left(1-r^2\right),\\
r' &= 1-r^2,
\end{aligned}
$$

where $r\neq 0$. Now, we have

$$
\begin{aligned}
\frac{dr}{dt} &= 1-r^2,\\
\frac{dr}{1-r^2} &= dt,\\
\int\left(\frac{1/2}{1-r}+\frac{1/2}{1+r}\right)dr &= \int dt,\\
\int\left(\frac{1}{1-r}+\frac{1}{1+r}\right)dr &= 2\int dt,\\
\ln\left(\left|\frac{1+r}{1-r}\right|\right) &= 2t+C,\\
\frac{1+r}{1-r} &= ke^{2t},\\
r &= \frac{ke^{2t}-1}{ke^{2t}+1}.
\end{aligned}
$$

Therefore, $\lim_{t\to\infty} r=k/k=1$, i.e., all solutions spiral towards $r=1$ as $t\to\infty$, as Figure 2.21 shows.

Example 2.17 (van der Pol Equation). Consider

$$
x''+\mu\left(x^2-1\right)x'+x=0,\qquad \mu>0. \tag{2.5}
$$

◇

Let

$$
\begin{cases}
x'=y=F,\\
y'=x''=-\mu\left(x^2-1\right)y-x=G.
\end{cases}
$$

To find equilibrium points, we let

$$
y=0,
$$

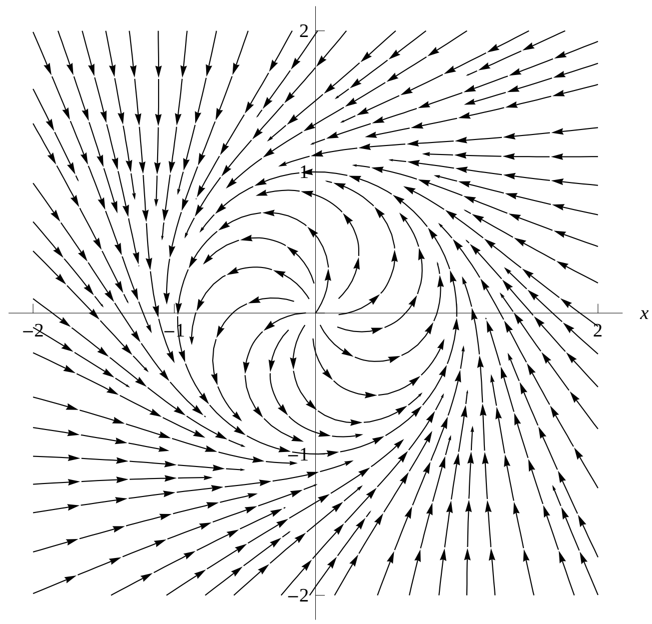


Figure 2.21: The phase portrait of Example 2.16, showing that all solutions spiral towards $r = 1$ as $t \to \infty$.

$$-\mu\left(x^2 - 1\right)y - x = 0.$$

Since $y = 0$, immediately $x = 0$. Therefore, $(0, 0)$ is the only equilibrium point.

Consider the linearized system

$$\begin{cases} x' = y, \\ y' = -x + \mu y. \end{cases}$$

Then the matrix of the system is

$$\begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix}.$$

The characteristic equation is $\lambda^2 - \mu\lambda + 1 = 0$. The solutions of this are

$$\lambda = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}.$$

Note that

- $\mu > 2$ gives us an unstable node (distinct real roots).
- $\mu = 2$ gives us an unstable node (repeated real positive roots).
- $\mu < 2$ gives us an unstable spiral (complex roots).

Does it have any periodic solutions? We can attempt to find out with

$$\begin{aligned} \mathbf{V} \cdot \mathbf{n} &= Fx + Gx \\ &= xy - \mu\left(x^2 - 1\right)y^2 - xy \\ &= -\mu\left(x^2 - 1\right)y^2. \end{aligned}$$

But this indicates nothing. Thus, we need to try a different-looking region R (not an annulus). To carry on with this solution, we need a new tool: Liénard's Theorem.

THEOREM 2.18 (Liénard's Theorem). *Let $f, h : \mathbb{R} \to \mathbb{R}$ and let $g(x) = \int_0^x f(t)\,dt$. Suppose that*

(1) f is continuous.
(2) f is even.
(3) there exists an $a > 0$ such that
- *$g(a) < 0$ for $0 < x < a$.*
- *$g(x) > 0$ for $x > a$.*
- *$f(x) > 0$ for $x > a$.*

(4) $\lim_{x\to\infty} g(x) = \infty$.
(5) h is odd and $h(x) > 0$ for $x > 0$.

Then $x'' + f(x)x' + h(x) = 0$ has a unique periodic solution and every other solution spirals towards it.

PROOF. We convert $x'' + f(x)x' + h(x) = 0$ to a system. Let $y = x' + g(x)$. Therefore,

$$y' = x'' + \frac{dg}{dx}x' = -f(x)x' - h(x) + f(x)x' = -h(x).$$

So the system is

$$\begin{cases} x' = y - g(x), \\ y' = -h(x). \end{cases}$$

Note that f is even implies that g is odd. Therefore, replacing x by $-x$ and y by $-y$ leaves the equations unchanged, so the solutions are symmetric about the origin. Hence, if we know the solutions for $x \geq 0$, we can get those with $x \leq 0$ by reflection about the origin.

So assume that $x \geq 0$. Let γ be the graph of $y = g(x)$ with $x \geq 0$. Let (x_0, y_0) lie on γ and let C_{x_0} be the solution which passes through (x_0, y_0) at $t = 0$.

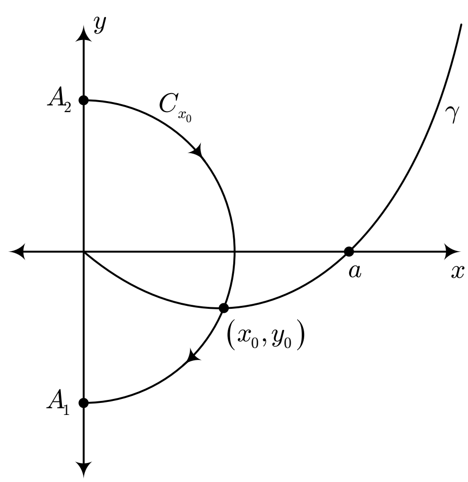


Figure 2.22: The solution C_{x_0} reaching the y-axis at both ends.

LEMMA 2.19. *As t increases from 0, x decreases and y decreases until eventually the y-axis is reached. As t decreases from 0, x decreases and y increases until eventually the y-axis is reached.*

PROOF. Since $y' = -h(x) < 0$, y always decreases as t increases. On γ, $\mathbf{V} = (0, -h(x))$, so C_{x_0} leaves γ heading straight down, and after passing (x_0, y_0) can never get above γ.

So $x' = y - g(x) < 0$ when $t \geq 0$. Therefore, x decreases as t increases from 0. Similarly, C_{x_0} enters γ heading straight down, so it was never below γ for $t \leq 0$.

So $x' = y - g(x) > 0$ when $t \leq 0$. Therefore, x decreases as t decreases from 0. It remains to be shown that C_{x_0} actually reaches the y-axis on both ends. From what we have shown so far, it might look like the situation illustrated in Figure 2.23.

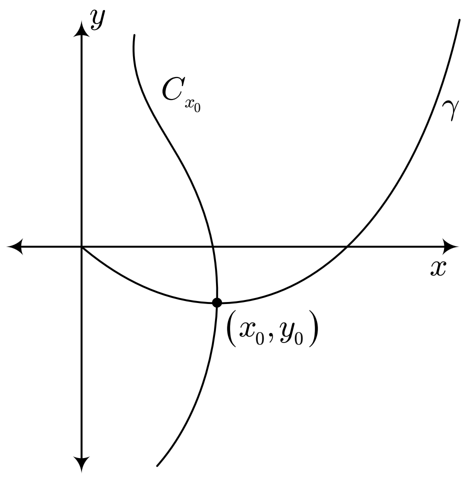


Figure 2.23:

Let

$$k(x) = 2\int_0^x h(x)\,dx$$

so that $k(x) \geq 0$ for $x \geq 0$. Given a constant b, let

$$r_b^2 = k(x) + (y-b)^2 .$$

Then differentiating with respect to t gives

$$\begin{aligned} 2r_b r_b' &= \frac{dk}{dx}x' + 2\,(y-b)\,y' \\ &= 2h(x)\big(y - g(x)\big) + 2\,(y-b)\,\big(-h(x)\big) \\ &= 2h(x)\big(b - g(x)\big). \end{aligned}$$

Therefore, $r_b r_b' = h(x)\big(b - g(x)\big)$.

Since g is continuous and $[0, x_0]$ is compact, g has both a minimum m and a maximum M on $[0, x_0]$. Choosing $b = m$ gives $g(x) - b \geq 0$ for all $x \in [0, x_0]$, so $r_m r_m' < 0$. Therefore, $r_m < 0$, so r_m decreases with t. But $r_m \geq (y-m)^2$, so $(y-m)^2$ does not go to ∞ as in the diagram we wish to rule out.

Similarly, choosing $b = M$ gives $g(x) - b \leq 0$, so r_m increases with increasing t. Equivalently, r_m decreases with decreasing t, so the distance from C_{x_0} to $(0, M)$ decreases as $t \to -\infty$, and in particular does not go to ∞. Thus, the y-axis is reached on this side also. ■

Given x_0, let $y_1(x_0)$ and $y_2(x_0)$ be the values of y where C_{x_0} crosses the y-axis. Reflection about the origin gives another section of the solution curve C_{x_0} as shown in Figure 2.24. We wish to show that there is a value of x_0 for which $y_2(x_0) = -y_1(x_0)$ so that the two halves piece together to give a periodic solution.

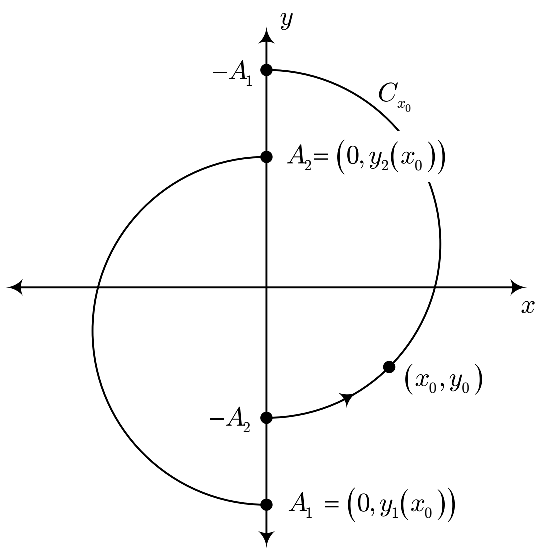


Figure 2.24: The solution curve C_{x_0} crosses y at $y_1(x_0)$ and $y_2(x_0)$.

Lemma 2.20. *There exists a unique x^* such that*

$$x_0 < x^* \Longrightarrow y_2(x_0) < -y_1(x_0),$$
$$x_0 = x^* \Longrightarrow y_2(x_0) = -y_1(x_0),$$
$$x_0 > x^* \Longrightarrow y_2(x_0) > -y_1(x_0).$$

Proof. Recall that

$$k(x) := 2\int_0^x h(x)\,dx$$

and that given a constant b,

$$r_b^2 := k(x) + (y-b)^2 .$$

We choose $b = 0$ to obtain $r^2 = k(x) + y^2$. Therefore,

$$rr' = -h(x)g(x) = g(x)\frac{dy}{dt}$$

holds on any solution curve. Let $\omega = g(x)\,dy$, a first order differential form. Let

$$I(x_0) := \int_{C_{x_0}} \omega$$

with C_{x_0} directed "backwards" from A_1 to A_2. Let t_1 and t_2 be the values of t at A_1 and A_2, respectively. Then

$$\begin{aligned} I(x_0) &= \int_{C_{x_0}} \omega = \int_{C_{x_0}} g(x)\,dy = \int_{t_1}^{t_2} g(x)\frac{dy}{dt}\,dt \\ &= \int_{t_1}^{t_2} r\frac{dr}{dt}\,dt = \frac{r^2}{2}\bigg|_{t=t_1}^{t=t_2} = \frac{1}{2}\left(r(t_2)^2 - r(t_1)^2\right) \\ &= \frac{y_2^2 + k(0) - y_1^2 - k(0)}{2} = \frac{y_2^2 - y_1^2}{2}. \end{aligned}$$

We now need to show the following:

(1) $I(x_0) < 0$ if $x < a$.

(2) $I(x_0)$ strictly increases with increasing x_0 for $x_0 > a$.

(3) $\lim_{x_0 \to \infty} I(x_0) = \infty$.

To show (1), if $x_0 < a$, then $g(x) < 0$ for all x on C_{x_0}, and dy/dt is increasing in the direction of C_{x_0} we are following. So

$$I(x_0) = \int_{C_{x_0}} g(x)\, dy < 0$$

for $x_0 < a$.

To show (2), consider $a < x_0 < \tilde{x}_0$. Let

$$C_{x_0} = C_1 \cup C_2 \cup C_3,$$
$$C_{\tilde{x}_0} = \tilde{C}_1 \cup \tilde{C}_2 \cup \tilde{C}_3$$

as shown in Figure 2.25. Then

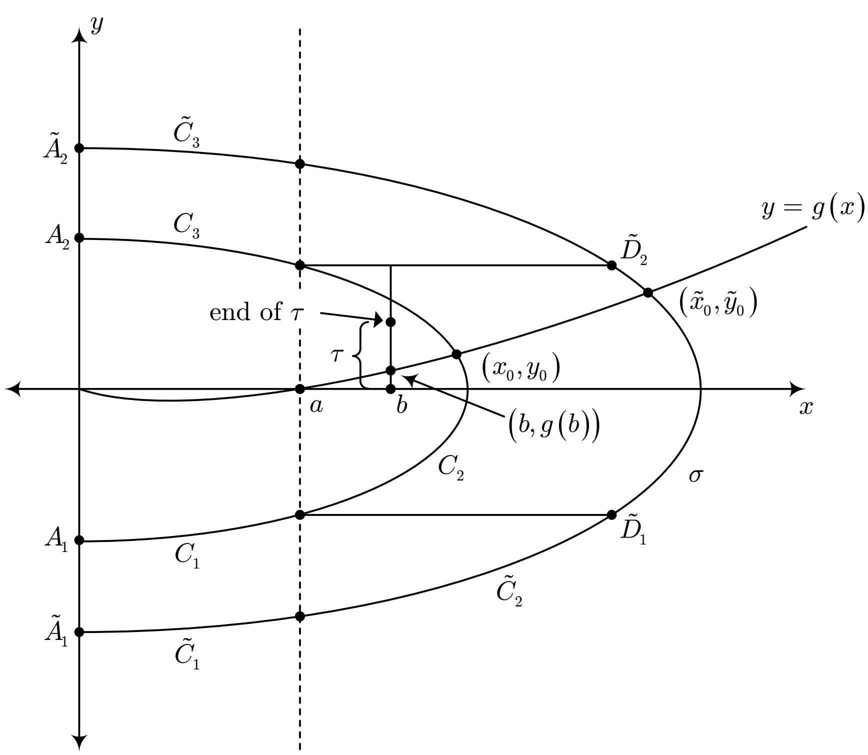


Figure 2.25:

$$\int_{C_1} \omega = \int_{C_1} g(x)\, dy = \int_0^a g(x) \frac{dy}{dx}\, dx = \int_0^a g(x) \frac{dy/dt}{dx/dt}\, dx$$
$$= \int_0^a g(x) \frac{-h(x)}{y - g(x)}\, dx = \int_0^a g(x) \frac{h(x)}{g(x) - y}\, dx.$$

On $\tilde{C}_1$, $g(x) - y$ is larger than it is on C_1, so $\big(g(x) - y\big)^{-1}$ is smaller. But $g(x) \le 0$ when $x \in [0, a]$, so $g(x)h(x)\big(g(x) - y\big)^{-1}$ is larger (less negative) on $\tilde{C}_1$ than it is on C_1. Therefore

$$\int_{C_1} \omega \le \int_{\tilde{C}_1} \omega.$$

Similarly,

$$\int_{C_3} \omega = \int_a^0 g(x)\frac{dy}{dx}\,dx = -\int_0^a g(x)\frac{-h(x)}{y - g(x)}\,dx = \int_0^a g(x)\frac{h(x)}{y - g(x)}\,dx.$$

On $\tilde{C}_3$, $y - g(x)$ is larger than it is on C_3, so $\big(y - g(x)\big)^{-1}$ is smaller. But $g(x) \le 0$, so $g(x)h(x)\big(y - g(x)\big)^{-1}$ is larger (less negative) on $\tilde{C}_3$ than on C_3. Therefore,

$$\int_{C_3} \omega < \int_{\tilde{C}_3} \omega.$$

Finally, let σ be the portion of $\tilde{C}_2$ between $\tilde{D}_1$ and $\tilde{D}_2$. Since $f(x) = dg/dx > 0$ when $x > a$, each point on σ has a larger value of $g(x)$ than the corresponding point (the one with the same y-coordinate) on C_2. Therefore,

$$\int_{C_2} \omega < \int_{\sigma} \omega < \int_{\tilde{C}_2} \omega,$$

where the second inequality comes from the fact that since $g(x) > 0$ on $x > a$, the integral over $\tilde{C}_2 - \sigma$ is positive. Therefore

$$\underbrace{\int_{C_1} \omega + \int_{C_2} \omega + \int_{C_3} \omega}_{I(x_0)} < \underbrace{\int_{\tilde{C}_1} \omega + \int_{\tilde{C}_2} \omega + \int_{\tilde{C}_3} \omega}_{I(\tilde{x}_0)},$$

i.e., $I(x_0)$ strictly increases with increasing x_0 when $x_0 > a$.

To show (3), select b so that $a < b$ and b is less than the x-coordinate of the point where C_{x_0} crosses the x-axis. Let τ be the vertical line segment through b as shown in Figure 2.25. Then

$$\int_{\tau} \omega < \int_{C_2} \omega$$

by the argument we used to show $\int_{C_2} \omega < \int_{\tilde{C}_2} \omega$. Note that

$$\begin{aligned}\int_{\tau} \omega &= \int_{\tau} g(x)\,dy = g(b)\int_{\tau} dy \\ &= g(b)\,(\text{length of } \tau) \\ &= g(b)y_0 = g(b)g(x_0).\end{aligned}$$

Since $\lim_{x\to\infty} g(x) = \infty$, we have $\int_{C_2} \omega \to \infty$ as $x_0 \to \infty$. Therefore,

$$\lim_{x_0\to\infty} I(x_0) = \infty.$$

It follows from (1), (2), (3), and from continuity that there exists a unique x^* such that

$$\begin{aligned}&I(x^*) = 0, \\ &I(x_0) < 0, \quad x_0 < x^*, \\ &I(x_0^*) > 0, \quad x_0 > x^*.\end{aligned}$$

So C_{x^*} pieces together with its reflection about the origin to form a periodic solution, as shown in Figure 2.26. Also, $x_0 < x^* \Rightarrow -y_1 > y_2$. Therefore, the solutions inside C^* spiral out to C^*. Similarly, solutions outside

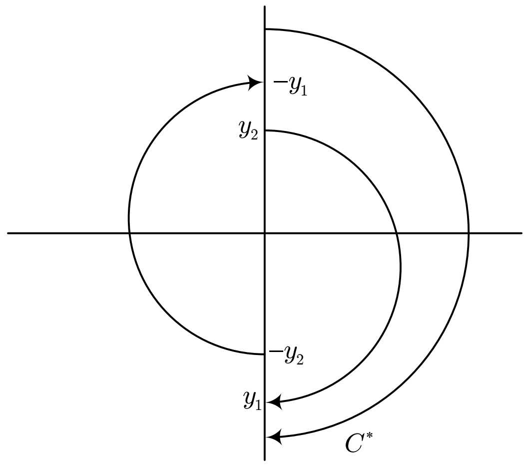


Figure 2.26:

C^* spiral in towards C^*. Therefore, C^* is the unique periodic solution. □

In van der Pol's Equation, i.e., Equation (2.5), we have

$$\begin{aligned} f(x) &= \mu\left(x^2-1\right), \\ g(x) &= \mu\left(\frac{x^3}{3}-x\right), \\ h(x) &= x. \end{aligned}$$

Let $a=\sqrt{3}$. Therefore, by Liénard's Theorem (Theorem 2.18, p. 44), we know that Equation (2.5) has a unique solution and that every other solution spiral towards it. □

7. Index Theory

Let $\mathbf{V}:\mathbb{R}^2\to\mathbb{R}^2$ be a vector field $\mathbf{V}=(F,G)$, where $(x,y)\mapsto(u,v)$ with $u=F(x,y)$ and $v=G(x,y)$. Let $\gamma\subset\mathbb{R}^2$ be a simple closed curve oriented counterclockwise with no critical points of $\mathbf{V}$ on γ, i.e., $\mathbf{V}(X)\neq\mathbf{0}$ for $X\in\gamma$. This is shown in Figure 2.27a.

Definition 2.21 (Index). Define $I_{\mathbf{V}}(\gamma)$ to be the *winding number* of $\mathbf{V}(\gamma)$ about $\mathbf{0}$. We call $I_{\mathbf{V}}(\gamma)$ the *index of* γ *for* $\mathbf{V}$. Unless considering more than one $\mathbf{V}$, we usually write $I(\gamma)$, where $\mathbf{V}$ is understood implicitly. ◊

The geometric interpretation of $I_{\mathbf{V}}(\gamma)$ is a follows.

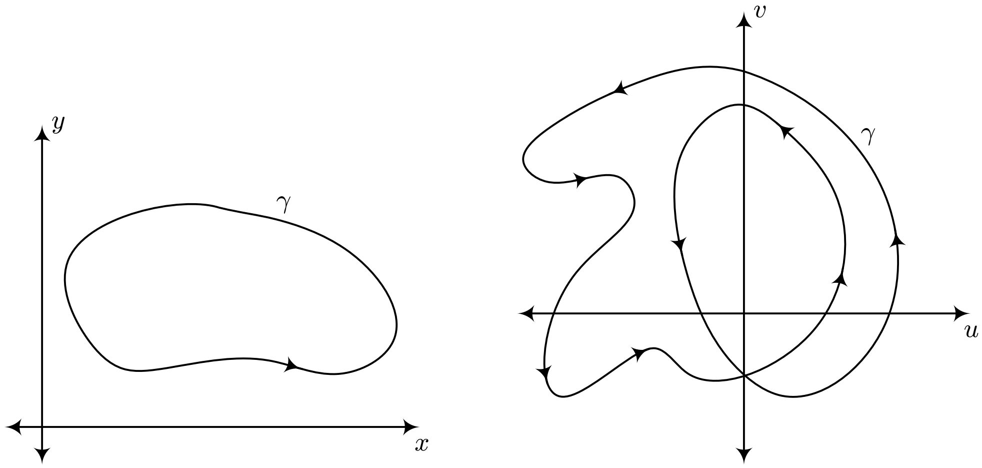
Proposition 2.22. *At each point $X\in\gamma$, there is an associated vector $\mathbf{V}(X)$ which makes an angle ϕ with the horizontal, as shown in Figure 2.28. Start with ϕ_0 at X_0. As X moves around the curve, ϕ gradually changes, returning to $\phi_0+2\pi n$ when we get back to X_0. Then $I(\gamma)=n$.*

Proof. We have

$$n=\frac{1}{2\pi}\left(\phi_{\text{end of }\mathbf{V}(\gamma)}-\phi_{\text{start of }\mathbf{V}(\gamma)}\right)=\frac{1}{2\pi}\int_{\mathbf{V}(\gamma)}d\phi.$$

Note that

$$\phi=\tan^{-1}\left(\frac{G}{F}\right)=\tan^{-1}\left(\frac{v}{u}\right).$$



(a) The curve γ is a simple closed curve oriented counterclockwise.

(b) The winding number of γ is 2 because it encircles the origin twice.

Figure 2.27:

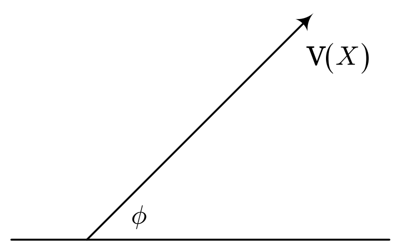


Figure 2.28: The vector $\mathbf{V}(X)$ *makes an angle* ϕ *with the horizontal.*

Strictly speaking, this holds only when $F \neq 0$. Thus

$$d\phi = \frac{1}{1+\left(\frac{v}{u}\right)^2}\frac{u\,dv - v\,du}{u^2} = \frac{-v\,du + u\,dv}{u^2+v^2}.$$

But by continuity, the following conclusion holds even for $u = 0$ (provided that $v \neq 0$ also). Therefore, we have

$$n = \frac{1}{2\pi}\int_{\mathbf{V}(\gamma)} \frac{-v\,du + u\,dv}{u^2+v^2},$$

which is the winding number of $\mathbf{V}(\gamma)$ about $\mathbf{0}$. □

PROPOSITION 2.23. *If* $\mathbf{V}$ *is never zero on the annular region between the curves* γ_1 *and* γ_2*, then* $I_{\mathbf{V}}(\gamma_1) = I_{\mathbf{V}}(\gamma_2)$.

PROOF. By applying a homotopy argument, $I_{\mathbf{V}}(\gamma)$ changes continuously, so it cannot jump from one integer to another. □

Let P be a critical point of $\mathbf{V}$. Define $I_{\mathbf{V}}(P) = I_{\mathbf{V}}(\gamma)$, where γ is any simple closed curve circling P once counterclockwise (i.e., having the winding number 1 about P) but not containing any other critical points of $\mathbf{V}$, e.g., γ could be a small circle about P. Proposition 2.23 implies that the answer is independent of the choice of γ.

PROPOSITION 2.24. *If* $\mathbf{V}$ *and* $\mathbf{W}$ *never have opposite directions on* γ*, then* $I_{\mathbf{V}}(\gamma) = I_{\mathbf{W}}(\gamma)$.

PROOF. Suppose that $\mathbf{V}$ and $\mathbf{W}$ never have opposite directions on γ. Consider $\mathbf{H}_s = s\mathbf{V} + (1-s)\,\mathbf{W}$ so that $\mathbf{H}_0 = \mathbf{V}$ and $\mathbf{H}_1 = \mathbf{W}$. Since $\mathbf{V}$ and $\mathbf{W}$ never have opposite directions on γ, it follows that $\mathbf{H}_s \neq \mathbf{0}$ on γ. However, $I_{\mathbf{H}_s}(\gamma)$ changes continuously with s, so being an integer, it must be constant. □

COROLLARY 2.25. *Suppose* P *is a critical point of* $\mathbf{V}$*. Let* $\mathbf{W}$ *be the linear approximation of* $\mathbf{V}$ *at* P*, that is,* $\mathbf{W} = \mathbf{A}\,(\mathbf{X} - P)$*, where*

$$\mathbf{A} = \begin{bmatrix} F_x(P) & F_y(P) \\ G_x(P) & G_y(P) \end{bmatrix}.$$

Assume that $\det(\mathbf{A}) \neq 0$*. Then* $I_{\mathbf{V}}(P) = I_{\mathbf{W}}(P)$.

PROOF. By translation, we may assume that $P = \mathbf{0}$. Let $\mathbf{V} = \mathbf{A}\mathbf{x} + \mathbf{h}$, where

$$\lim_{\|\mathbf{x}\|\to 0} \frac{\mathbf{h}(\mathbf{x})}{\|\mathbf{x}\|} = 0.$$

Pick γ to be a circle $x^2 + y^2 = r^2$ small enough to contain no other critical points of $\mathbf{V}$ ($\mathbf{W}$ doesn't have any other critical points). Suppose that $s\mathbf{W} + \mathbf{V} = \mathbf{0}$ at someplace on γ, i.e., $\mathbf{W}$ and $\mathbf{V}$ are in opposite directions. Note that

$$\begin{aligned} s\mathbf{W} + \mathbf{V} = s\mathbf{W} + \mathbf{W} + \mathbf{h} &\Longrightarrow (1+s)\,\mathbf{W} = -\mathbf{h} \\ &\Longrightarrow (1+s)^2\,\|\mathbf{W}\|^2 = \|\mathbf{h}\|^2\,. \end{aligned}$$

But

$$(1+s)^2\, m^2 r^2 \leq (1+s)^2\,\|\mathbf{W}\|^2\,,$$

where

$$m := \min_{\|\mathbf{x}\|=1} \Big(\big\|\mathbf{A}(\mathbf{x})\big\|\Big) > 0$$

since $\det(\mathbf{A}) \neq 0$. This implies that

$$\frac{\|\mathbf{h}\|^2}{r^2} \geq (1+s)^2\, m^2,$$

contradicting $\lim_{r\to 0}\big(\|\mathbf{h}\|/r\big) = 0$. Therefore, $\mathbf{W}$ and $\mathbf{V}$ never have opposite signs on γ once r is sufficiently small. Thus, $I_{\mathbf{V}}(\mathbf{0}) = I_{\mathbf{W}}(\mathbf{0})$. □

THEOREM 2.26. *We have*

$$I_{\mathbf{V}}(\gamma) = \sum_{P\in S} I_{\mathbf{V}}(P)\,,$$

where S *is the set of the critical points of* $\mathbf{V}$ *lying inside* γ.

PROOF. Subdivide the interior of γ into regions each containing only one critical point, as shown in Figure 2.29. The extra curves added cancel out when doing the sum of integrals to get the winding numbers. □

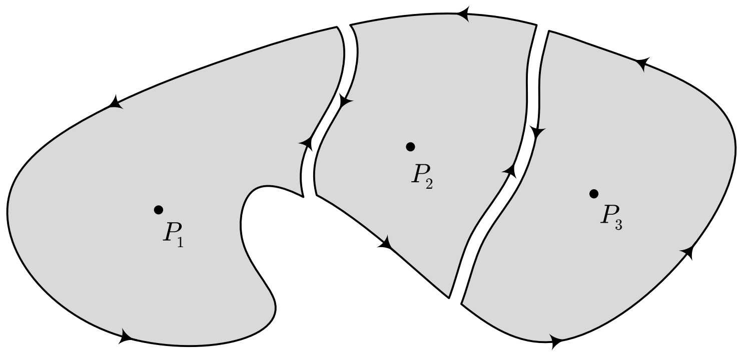


Figure 2.29: The interior of γ is subdivided into regions, each containing only one critical point.

THEOREM 2.27. *Suppose that γ is a counterclockwise-oriented periodic solution* to the system*

$$\begin{cases} x' = F(x, y), \\ y' = G(x, y). \end{cases} \tag{2.6}$$

Set $\mathbf{V} = (F, G)$. Then $I_{\mathbf{V}}(\gamma) = 1$.

PROOF. Suppose that γ is a counterclockwise-oriented periodic solution to System (2.6). Since γ is a solution curve, we have $\mathbf{V} = (x', y')$. Then

$$I_{\mathbf{V}}(\gamma) = \frac{1}{2\pi} \int_{\mathbf{V}(\gamma)} d \tan^{-1}\left(\frac{v}{u}\right),$$

where $u = F(x, y)$ and $v = G(x, y)$. Then

$$\frac{1}{2\pi} \int_{\mathbf{V}(\gamma)} d \tan^{-1}\left(\frac{v}{u}\right) = \frac{1}{2\pi} \int_a^b \frac{d}{dt} \tan^{-1}\left(\frac{G\big(x(t), y(t)\big)}{F\big(x(t), y(t)\big)}\right) dt$$

and it follows that

$$\begin{aligned} I_{\mathbf{V}}(\gamma) &= \frac{1}{2\pi} \int_\gamma d \tan^{-1}\left(\frac{G}{F}\right) = \frac{1}{2\pi} \int_\gamma d \tan^{-1}\left(\frac{y'}{x'}\right) \\ &= \frac{1}{2\pi} \int_\gamma d\theta = \frac{1}{2\pi}\left(\theta_{\gamma_{\text{end}}} - \theta_{\gamma_{\text{start}}}\right) \\ &= \frac{1}{\cancel{2\pi}}(\cancel{2\pi}) = 1, \end{aligned}$$

where θ is the angle between the tangent to γ and the x-axis, as shown in Figure 2.30. ☐

COROLLARY 2.28. *If γ (counterclockwise) is a periodic solution to $\mathbf{X}' = \mathbf{V}$, then*

$$\sum_{P \in S} I_{\mathbf{V}}(P) = 1,$$

where S is the set of the critical points of $\mathbf{V}$ lying inside γ.

COROLLARY 2.29. *Any periodic solution encloses at least one critical point.*

*Solutions to differential equations cannot intersect themselves, i.e., γ is a simple closed curve.

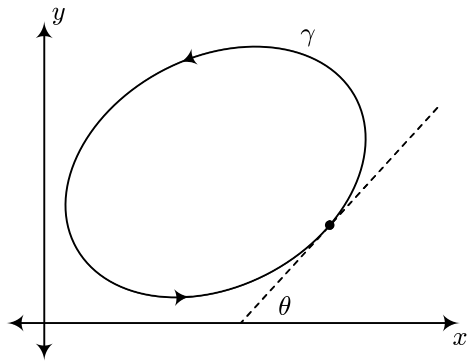


Figure 2.30: The angle θ is the angle between the tangent to γ and the x-axis.

COROLLARY 2.30. *Let $\mathbf{X}' = \mathbf{V}(X)$. Suppose R is a closed bounded simply connected region without critical points. Then any solution entering R must leave again. (It can later come back, but then must leave again.)*

PROOF. Let $\mathbf{X}' = \mathbf{V}(X)$. Suppose R is a closed bounded simply connected region without critical points. If a solution $\mathbf{X}(t)$ stayed in R, then by the Poincaré-Bendixson Theorem (Theorem 2.12, p. 38), it would either be periodic or spiral towards a periodic solution. In either case, R would contain a periodic solution, which must therefore surround a critical point. This is a contradiction to our hypothesis. □

COROLLARY 2.31. *Let $\mathbf{X}(t)$ be a solution of $\mathbf{X}' = \mathbf{V}(\mathbf{X})$. If $\lim_{t\to\infty} \mathbf{X}(t)$ exists, then it is a critical point.*[*]

PROOF. Let $P = \lim_{t\to\infty} \mathbf{X}(t)$. If P is not a critical point, then there exists a small closed disk D around P with no critical points. To say that $\lim_{t\to\infty} \mathbf{X}(t) = P$ is to say that $\mathbf{X}(t)$ eventually enters D and never leaves. This is a contradiction. Therefore, P is a critical point. □

Figure 2.31 shows some possible types of solution curves.

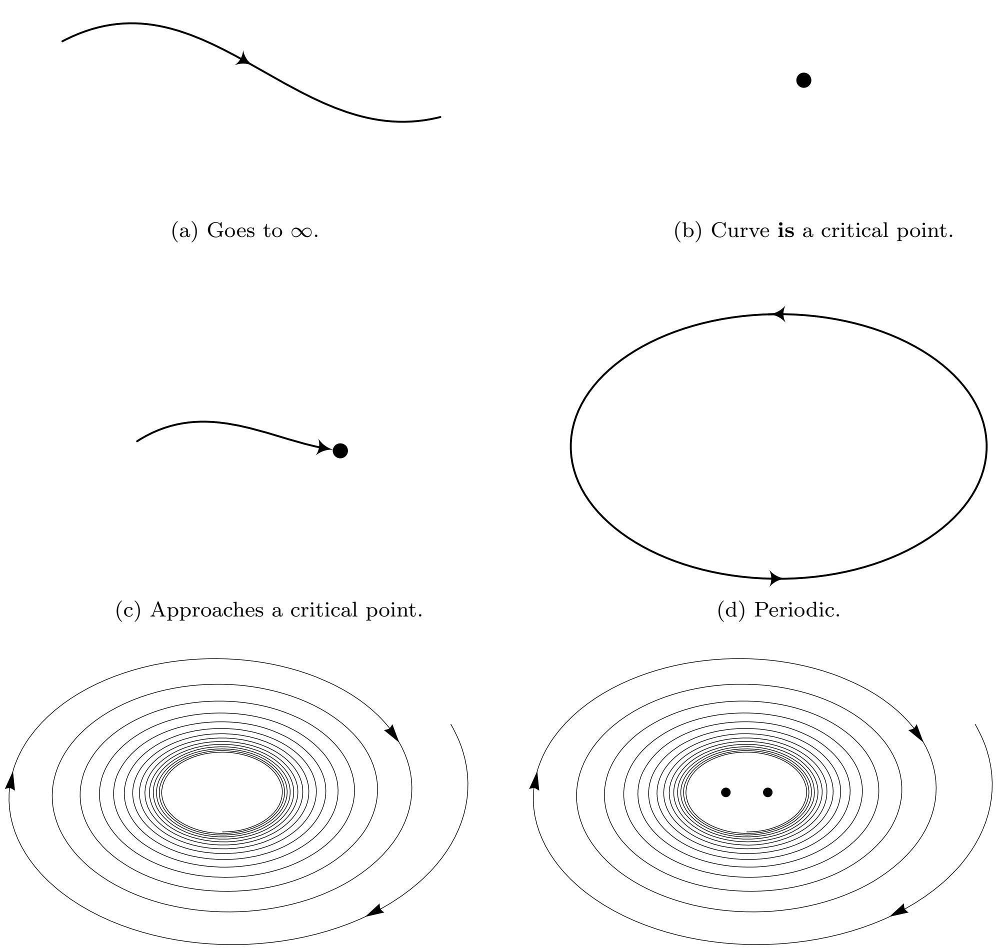
EXAMPLE 2.32.

(1) *Unstable node.* By changing variables (which rotates and stretches curves, but doesn't change their index), we have
$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$
Pick γ to be a counterclockwise circle about $\mathbf{0}$. Therefore, $\mathbf{V}(\gamma) = \gamma$ and $I_{\mathbf{V}}(\gamma) = 1$, i.e., $I_{\mathbf{V}}(\mathbf{0}) = 1$.

(2) *Stable node.* We may assume that
$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$
This rotates γ by 180°. Therefore, γ is still oriented counterclockwise, so $I_{\mathbf{V}}(0) = 1$.

(3) *Saddle.* We may assume that
$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

[*]By a symmetrical argument, by replacing t by $-t$, the same holds for $\lim_{t\to-\infty} \mathbf{X}(t)$.



(a) Goes to ∞.

(b) Curve **is** a critical point.

(c) Approaches a critical point.

(d) Periodic.

(e) Spirals towards a periodic solution.

(f) Spirals around some critical points.

Figure 2.31: Some possible types of solution curves.

which is a reflection about the x-axis. Therefore, $\mathbf{A}(\gamma)$ is oriented clockwise, so $I_{\mathbf{V}}(\mathbf{0}) = -1$.

(4) *Spiral*. We may assume that

$$\mathbf{A} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix},$$

which is a rotation by θ, so $I_{\mathbf{V}}(\mathbf{0}) = 1$. Note that $\theta < 0$ indicates a clockwise spiral of the solutions to the differential equation, but $\mathbf{V}$ itself rotates by θ, which preserves orientation.

Note that in all cases, $I_{\mathbf{V}}(\mathbf{0}) = \operatorname{sgn}\big(\det(\mathbf{A})\big)$. ◇

Example 2.33 (Example 2.6, p. 29). Consider the Predator-Prey equations

$$\begin{cases} x' = ax - bxy, \\ y' = -cy + dxy \end{cases}$$

◇

The critical points are $(0,0)$ and $(c/d, a/b)$, which are a saddle ($I_{\mathbf{V}}(\mathbf{0}) = -1$) and centre ($I_{\mathbf{V}}(P) = 1$), respectively, as shown in Figure 2.32. There cannot be any periodic solutions containing the origin since we

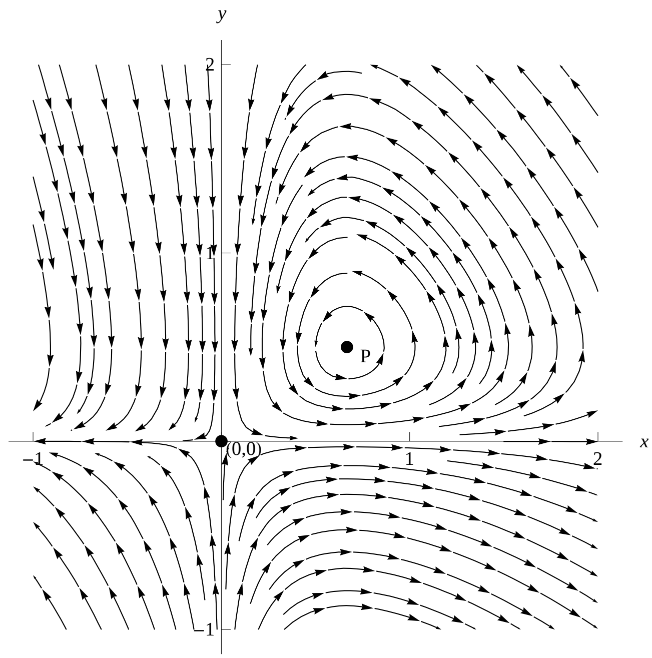
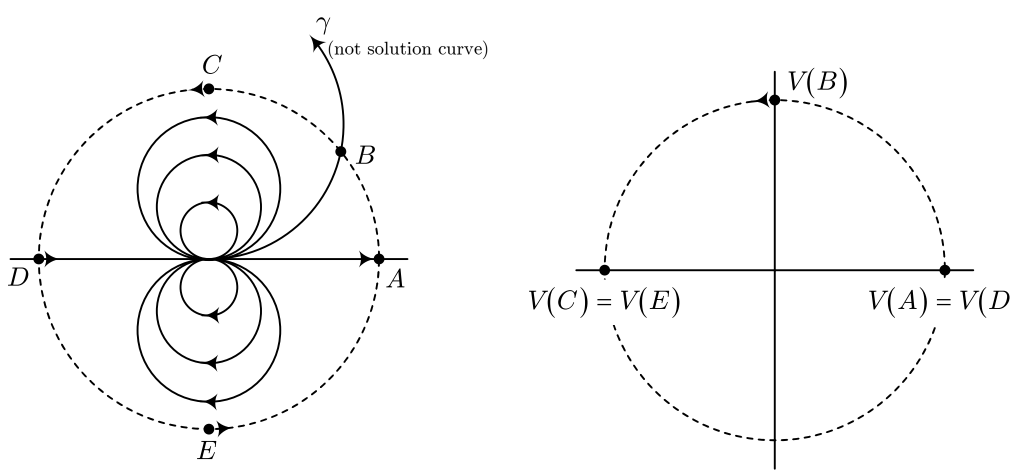


Figure 2.32: The phase portrait of Example 2.33, where $P = (c/d, a/b)$.

can't make a total index sum of $+1$ if we include the origin. Hence, by Theorem 2.27, there cannot be a periodic solution.

Figure 2.33 shows a critical point with index 2.



(a) A critical point with index 2.

(b) Image under V.

Figure 2.33:

CHAPTER 3

Boundary Value Problems

1. Boundary Value Problems

Consider

$$\begin{cases} y'' + P(x)y' + Q(x)y = R(x), \\ \alpha y(a) + \beta y'(a) = r_1, \\ \gamma y(b) + \delta y'(b) = r_2. \end{cases}$$

In `MATB44`, we considered the special case where $a = b$ and $\beta = 0 = \gamma$, which gives an intermediate value problem (IVP) as shown in Figure 3.1. Unlike IVPs, there is no existence and uniqueness theorem in the general case.

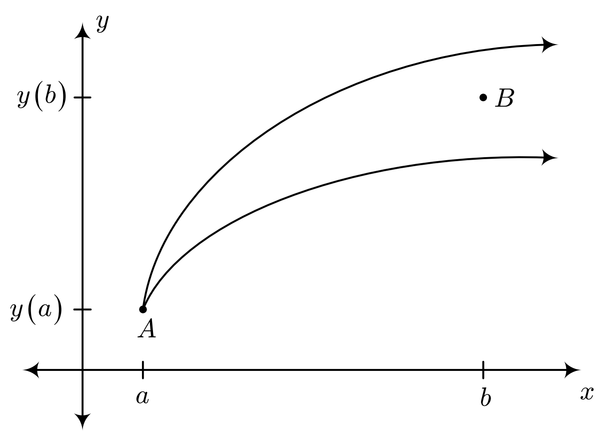


Figure 3.1: The IVP case when $\beta = \delta$.

If we look at the solutions throughout A (with various slopes), are there any which pass though B? In other words, can we hit the target?

1.1. Sample Application Leading to BVPs.

Consider an insulated wire of length L as shown in Figure 3.2. Let $u(x,t)$ be the temperature of the

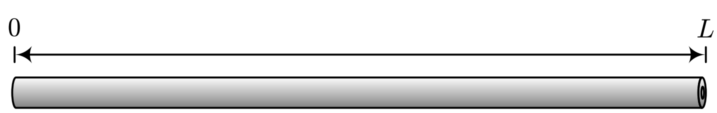


Figure 3.2: An insulated wire of length L.

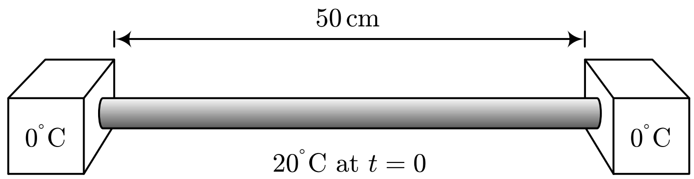


Figure 3.3: An insulated wire with both ends touching material mantained at 0° C.

wire at x at time t. Then we naturally have $0 \leq x \leq L$ and $0 \leq t < \infty$. Suppose heat may flow within the wire but may not enter or leave anywhere except at the ends. If we know

- $u(x,0)$, the starting temperature at all points,
- $u(0,t)$, the temperature at the left end at all times (which also determines the heat gain/loss at this end at all times), and
- $u(L,t)$, the temperature at the right end at all times (which also determines the heat gain/loss at this end at all times),

then, intuitively, this should determine $u(x,t)$.

Newton's Law of Cooling states that

> Given an object A at temperature T_1 and a neighbourhood B at a distance d away at a higher temperature T_2, A gains heat from B at a rate proportional to $(T_2 - T_1)/d$.

So at any time t, the rate $H(x)$ of heat transfer from the left end to the right end at x is proportional to $-u_x(x,t)$, e.g., if u is decreasing at x (with u_x being negative), then it is hotter to the left of x than it is to the right of x, so the heat flow from the left to the right is positive. Therefore

$$H(x) = -ku_x(x,t).$$

For a segment $[x, x+\Delta x]$, the heat entering in time Δt is

$$\big(H(x,t) - H(x+\Delta x,t)\big)\Delta t = k\Delta t\big(u_x(x+\Delta t,t) - u_x(x,t)\big).$$

The difference is absorbed, producing a change in the temperature near x, i.e.,

$$k\Delta t\big(u_x(x+\Delta t,t)\big) - u_x(x,t) \underbrace{=}_{\text{change in heat content causes a change in temperature}} \overbrace{c}^{\text{specific heat of metal}} m\Delta u = c\rho\Delta x\Delta u.$$

where ρ is the density (mass per unit length). Therefore,

$$\frac{\Delta u}{\Delta t} = K\frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x},$$

where $K = k/(c\rho)$, and so we have the *Heat Equation*

$$u_t = Ku_{xx}, \tag{3.1}$$

where K depends only on the properties of the wire and the units used.

Example 3.1 (Insulated wire). An insulated wire of length 50 cm is placed with its ends touching a material maintained at 0° C. This is shown in Figure 3.3. Suppose the wire is made of material for which $K = 1$. Initially, the wire had uniform temperature 20° C. Find $u(x,t)$. ◇

SOLUTION. Expressing this problem as a boundary value problem (BVP) gives

$$\begin{cases} u_t = u_{xx}, \\ u(0,t) = 0, \quad \forall t, \\ u(50,t) = 0, \quad \forall t, \\ u(x,0) = 20 \quad \text{for } 0 < x < 50. \end{cases}$$

Trial and error suggests looking for solutions of the form

$$u(x,t) = X(x)T(t).$$

The partial differential equation (see §4) itself has many other solutions (it is not possible to write down the general solution), e.g., $u(x,t) = x^2 + 2t$ is a solution of $u_{xx} = u_t$ not having the form $X(x)T(t)$ (but it does not satisfy our conditions). Substituting into $u_{xx} = u_t$ gives

$$X''(x)T(t) = X(x)T'(t),$$
$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)}.$$

Now the left side of the equation depends only on x and the right depends only on t. For equality to hold, they must be constant. It is convenient to call this constant $-\sigma$ (it turns out to be negative). Therefore

$$\frac{X''}{X} = -\sigma = \frac{T'}{T},$$

and from this we obtain

$$X'' + \sigma X = 0, \tag{3.2a}$$
$$T' + \sigma T = 0. \tag{3.2b}$$

Also,

$$u(0,t) = 0 \Longrightarrow X(0)T(t) = 0, \quad \text{for all } t, \tag{B1}$$
$$u(50,t) = 0 \Longrightarrow X(50)T(t) = 0, \quad \text{for all } t, \tag{B2}$$
$$u(x,0) = 20 \Longrightarrow X(x)T(0) = 20, \quad \text{for } 0 < x < 50. \tag{B3}$$

Equation (B3) implies that $T(0) \neq 0$, so setting $t = 0$ in Equation (B1) and (B2) gives $X(0) = 0$ and $X(50) = 0$, respectively.

Considering Equation (3.2a), we have

$$\begin{cases} X'' + \sigma X = 0, \\ X(0) = 0, \\ X(50) = 0. \end{cases} \tag{$*$}$$

One solution of the BVP $(*)$ is $X \equiv 0$, but this will not satisfy Equation (B3). We need a nonzero solution. The general solution of Equation (3.2a) has the form

$$X = c_1 X_1 + c_2 X_2.$$

Can we choose c_1 and c_2 to obtain $X(0) = 0$ and $X(50) = 0$? We must consider the three cases, namely,

(1) $\sigma < 0$.
(2) $\sigma = 0$.
(3) $\sigma > 0$.

Case 1: $\sigma < 0$

Let $\sigma = -a^2$. Then the solution of $X'' - a^2X = 0$ is

$$X = c_1 e^{ax} + c_2 e^{-ax}.$$

For $X(0) = 0$, we have

$$c_1 e^0 + c_2 e^0 = 0 \Longrightarrow c_1 + c_2 = 0.$$

For $X(50) = 0$, we have

$$c_1 e^{50a} + c_2 e^{-50a} = 0.$$

One solution is $c_1 = 0 = c_2$, but this implies that $X \equiv 0$, which is a contradiction to Equation (B3) as previously noted. Therefore, this solution is no good. For a nonzero solution, we require

$$\begin{vmatrix} 1 & 1 \\ e^{50a} & e^{-50a} \end{vmatrix} = 0 \Longrightarrow \begin{cases} e^{-50a} - e^{50a} = 0, \\ e^{-50a} = e^{50a}. \\ e^{100a} = 1. \end{cases}$$

But this means that $a = 0$, and so $\sigma = 0$, which contradicts our original supposition that $\sigma < 0$. Therefore $a = 0$ is also no good, which means that $\sigma \not< 0$.

Case 2: $\sigma = 0$

Then $X'' = 0$ and the solution is $X = c_1 + c_2 x$. Using the fact that $X(0) = 0 = X(50)$, we have

$$c_1 + 0 = 0,$$
$$c_1 + 50c_2 = 0,$$

and so $c_1 = c_2 = 0$, meaning that $X \equiv 0$, which is a contradiction as before.

Case 3: $\sigma > 0$

Then $X'' + a^2X = 0$ and the general solution is $x = c_1 \cos(ax) + c_2 \sin(bx)$. Using the fact that $X(0) = 0 = X(50)$, we have,

$$c_1 + 0 = 0,$$
$$c_1 \cos(50a) + c_2 \sin(50a) = 0.$$

For a nonzero solution,

$$\begin{vmatrix} 1 & 0 \\ \cos(50a) & \sin(50a) \end{vmatrix} = 0,$$

giving $\sin(50a) = 0$. Therefore $50a = n\pi$ and we have $a = n\pi/50$ for some n. Correspondingly,

$$\sigma = a^2 = \frac{n^2\pi^2}{50^2}$$

for some n, i.e., the nonzero solutions of System $(*)$ have the form

$$X(x) = c \sin\left(\frac{n\pi}{50}x\right)$$

for some n and some c.

Considering $u(x,t) = X(x)T(t)$, now that we have the solution $X(x)$, what is the corresponding $T(t)$? Equation (3.2b) becomes

$$\frac{dT}{dt} + \frac{n^2\pi^2}{50^2}T = 0 \Longrightarrow T = Ae^{-\frac{n^2\pi^2}{50^2}t}.$$

Therefore, for any n,

$$u(x,t) = \underbrace{B}_{Ac} e^{-\frac{n^2\pi^2}{50^2}t}\sin\left(\frac{n\pi}{50}x\right)$$

satisfies $u_t = u_{xx}$ with $u(0,t) = 0$ and $u(50,t) = 0$.

What about the condition $u(x,0) = 20$? Write

$$u_n(x,t) = e^{-\frac{n^2\pi^2}{50^2}t}\sin\left(\frac{n\pi}{50}x\right).$$

Notice that for any constants b_n, $u(x,t) = \sum_{n=1}^{\infty} b_n u_n$ also satisfies $u_t = u_{xx}$ with $u(0,t) = 0 = u(50,t)$. Can we choose $\{b_n\}$ so that $u(x,20) = 20$? Note that

$$u_n(x,0) = \sin\left(\frac{n\pi}{50}x\right).$$

Therefore, we need

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{50}x\right) \equiv 20,$$

i.e., the Fourier series expansion of the function $f(x) \equiv 20$ on $(0,50)$. Since we have only sine terms, extend $f(x)$ to an odd function by

$$f(x) \equiv \begin{cases} 20, & 0 < x < 50, \\ -20, & -50 < x < 0, \end{cases}$$

where the period is $P = 100$, one period of which is shown in Figure 3.4. Therefore,

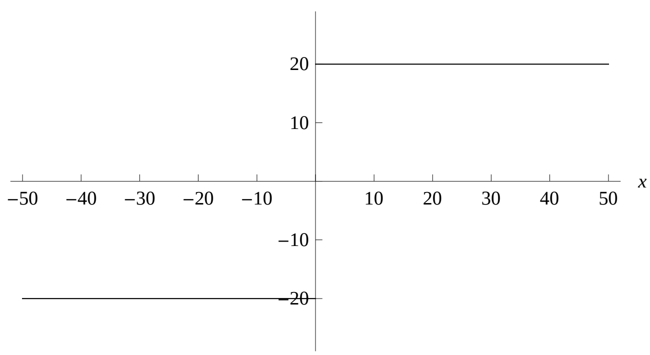


Figure 3.4: An odd function with $P = 100$.

$$\begin{aligned} b_n &= \frac{2}{P}\int_{-50}^{50} f(x)\sin\left(\frac{2\pi n}{P}x\right)dx \\ &= \underbrace{\frac{1}{50}\int_{-50}^{50} f(x)\sin\left(\frac{\pi n}{50}x\right)dx}_{\text{use symmetry}} \\ &= \frac{2}{50}\int_{0}^{50} 20\left(\sin\left(\frac{\pi n}{50}x\right)\right)dx \end{aligned}$$

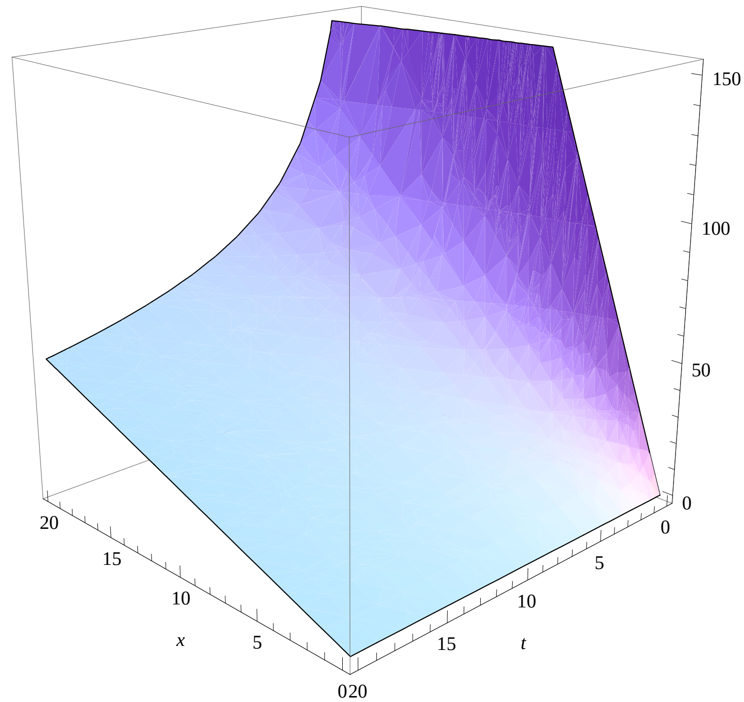


Figure 3.5: The plot of $u(x,t)$ of Example 3.1.

$$
\begin{aligned}
&= \frac{4}{5}\left(-\frac{50}{\pi n}\cos\Big(\frac{\pi n}{50}x\Big)\right)\bigg|_0^{50} \\
&= \frac{40}{\pi n}\big(-\cos(n\pi)+1\big) \\
&= \begin{cases} \frac{80}{\pi n}, & n \text{ is odd,} \\ 0, & n \text{ is even.} \end{cases}
\end{aligned}
$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} b_n u_n = \sum_{n \text{ odd}} \frac{80}{\pi n} e^{-\frac{n^2\pi^2}{50^2}t} \sin\Big(\frac{n\pi}{50}x\Big),$$

i.e., n takes on odd values, or put in a more self-contained way, we have

$$u(x,t) = \sum_{n=0}^{\infty} \frac{80}{\pi\,(2n+1)} e^{-\frac{(2n+1)^2\pi^2}{50^2}t} \sin\left(\frac{(2n+1)\,\pi}{50}x\right).$$

Figure 3.5 shows the plot of $u(x,t)$. ◇

The preceding method in Example 3.1 depended upon the fact that $u(0,t)=0$ and $u(50,t)=0$, namely, they equal zero. This is because for solutions f and g of

$$\begin{cases} X'' + \sigma X = 0, \\ X(0) = 0, \\ X(50) = 0, \end{cases}$$

it implies that $c_1 f_1 + c_2 f_2$ are solutions. What if $u(0,t) \neq 0$ and $u(50,t) \neq 0$?

EXAMPLE 3.2 (Insulated wire, nonzero ends). Solve Example 3.1 with $u(0,t) = 5$ and $u(50,t) = 15$. $\diamond$

SOLUTION. Let

$$v(x,t) = 5 + x\frac{15-5}{50} = 5 + \frac{x}{5}.$$

Then we consider $v_{xx} = 0 = v_t$ with $v(0,t) = 5$ and $v(50,t) = 15$. Let $w = u - v$. Thus, we have transformed our problem into

$$\begin{cases} w_{xx} = w_t, \\ w(0,t) = 0, \\ w(50,t) = 0, \\ w(x,0) = 15 - \dfrac{x}{5}, \end{cases}$$

which is the same type of problem as in Example 3.1, and so it is solvable the same way. Solving it, we obtain

$$w(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2}{50^2}t} \sin\left(\frac{n\pi}{50}x\right),$$

where $\{b_n\}$ are the Fourier coefficients in the series for $15 - x/5$. Finding b_n, we have

$$\begin{aligned} b_n &= \frac{2}{100}\int_{-50}^{50} f(x)\sin\left(\frac{2\pi n}{100}x\right)dx \\ &= \frac{2}{50}\int_0^{50}\left(15 - \frac{x}{5}\right)\sin\left(\frac{\pi n}{50}x\right)dx \\ &= \frac{1}{25}\begin{cases} \frac{15\cdot 50^2 - 10}{n\pi}, & n \text{ is odd,} \\ -\frac{50^2}{n\pi}, & n \text{ is even.} \end{cases} \end{aligned}$$

Then $u = w + v$, with w and v as above, is the solution. $\diamond$

2. Homogeneous Boundary Value Problems

2.1. Introduction.

DEFINITION 3.3 (Homogeneous BVP). A BVP of the form

$$y'' + P(x)y' + Q(x)y = 0, \tag{3.3a}$$

$$\alpha y(a) + \beta y'(a) = 0, \tag{3.3b}$$

$$\gamma y(b) + \delta y'(b) = 0 \tag{3.3c}$$

is called *homogeneous*. $\diamondsuit$

If $u(x)$ and $v(x)$ satisfy the above system, then so do $c_1u(x) + c_2v(x)$, i.e., solutions to the system form a vector space. Let y_1 and y_2 be linearly independent solutions of Equation (3.3a). Then the general solution is $y = c_1y_1 + c_2y_2$. We wish to choose c_1 and c_2 (if possible) so that y satisfies the boundary conditions given in Equations (3.3b) and (3.3c). One solution is when $c_1 = 0$ and $c_2 = 0$, in which case $y = 0$. Are there any other solutions? By Equation (3.3b), we have

$$\alpha c_1 y_1(a) + \alpha c_2 y_2(a) + \beta c_1 y_1'(a) + \beta c_2 y_2'(a) = 0,$$

and by Equation (3.3c), we have

$$\gamma c_1 y_1(b) + \gamma c_2 y_2(b) + \delta c_1 y_1'(b) + \delta c_2 y_2'(b) = 0.$$

Therefore,

$$\begin{aligned} c_1 \overbrace{\big(\alpha y_1(a) + \beta y_1'(a)\big)}^{B_a(y_1)} + c_2 \overbrace{\big(\alpha c_2 y_2(a) + \beta y_2'(a)\big)}^{B_a(y_2)} &= 0, \\ c_1\big(\gamma y_1(b) + \delta y_1'(b)\big) + c_2\big(\gamma c_2 y_2(b) + \delta y_2'(b)\big) &= 0, \end{aligned}$$

where for notational simplicity we define

$$B_a(u) := \alpha u(a) + \beta u'(a), \tag{3.4a}$$

$$B_b(u) := \gamma u(b) + \delta u'(b). \tag{3.4b}$$

Then

$$\begin{bmatrix} B_a(y_1) & B_a(y_2) \\ B_b(y_1) & B_b(y_2) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

There exist nonzero solutions for c_1 and c_2 if and only if

$$\underbrace{\begin{vmatrix} B_a(y_1) & B_a(y_2) \\ B_b(y_1) & B_b(y_2) \end{vmatrix}}_{\triangle} = 0.$$

If the differential equation involves a parameter λ, then $\triangle = 0$ is an equation that can be solved to give the λ's for which there are nonzero solutions (this is reminiscent of finding eigenvalues).

THEOREM 3.4. *If $u(x)$ is a nonzero solution of*

$$\begin{cases} y'' + P(x)y' + Q(x)y = 0, \\ \alpha y(a) + \beta y'(a) = 0, \\ \gamma y(b) + \delta y'(b) = 0, \end{cases} \tag{H}$$

then all solutions to the system are given by $y = cu(x)$ for some constant c.

PROOF. Let $v(x)$ be a solution of System (H). Then

$$\begin{bmatrix} u(a) & u'(a) \\ v(a) & v'(a) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So

$$\underbrace{\begin{vmatrix} u(a) & u'(a) \\ v(a) & v'(a) \end{vmatrix}}_{W(u,v)(a)} = 0.$$

But if a Wronskian is 0 at one point, it is zero everywhere. Therefore, u and v are linearly dependent, i.e., $v(x) = cu(x)$ for some constant c. $\square$

2.2. Eigenvalue Problems (Sturm-Liouville).

The heat equation leads to BVP

$$\begin{cases} y'' = \lambda y, \\ y(0) = 0, \\ y(1) = 1 \end{cases}$$

for some unknown constant λ. This is an example of what we call an *eigenvalue problem.*

DEFINITION 3.5 (Differential operator). An operator formed as a combination of differentiation and multiplication operators is called a *differential operator.* ♢

For example, a second order differential operator L has the the form

$$L(y) = -\big(y'' + P(x)y' + Q(x)y\big).$$

The explanation for the convention of including the minus sign is given below.

A second order differential operator can be regarded as a linear transformation on the vector space of twice differentiable functions.

If the boundary conditions are homogeneous, namely,

$$\begin{cases} \alpha y(a) + \beta y'(a) = 0, \\ \gamma y(b) + \delta y'(b) = 0, \end{cases} \tag{B}$$

then the set of twice differentiable functions satisfying them forms a vector space V_B. Given such a set of boundary conditions and a differential operator L, a value of λ for which there exists a nonzero f satisfying $Lf = \lambda f$ and conditions (B) (i.e., a nonzero solution to the BVP) is called an *eigenvalue* for $Lf = \lambda f$ (relative to the conditions (B)), and f is called an *eigenvector* or *eigenfunction.* The negative sign convention was introduced because with it the eigenvalues usually come out to be positive.

EXAMPLE 3.6. We have shown that if $Ly = -y''$, then, relative to the conditions $y(0) = y(50) = 0$, the eigenvalues of $Ly = \lambda y$ are

$$\lambda_n = \frac{n^2\pi^2}{50^2}$$

for $n = 1, 2, 3, \ldots$, and an eigenfunction for the eigenvalue $\lambda = n^2\pi^2/50^2$ is

$$y = \sin\left(\frac{n\pi x}{50}\right).$$

◇

EXAMPLE 3.7. Consider

$$\begin{cases} L(y) = -\left(y'' + 2y'\right), \\ y'(0) = 0, \\ y(1) = 0. \end{cases}$$

Find real eigenvalues of $Ly = \lambda y$ and the eigenfunctions. ◇

SOLUTION. Since $y'' + 2y' + \lambda y = 0$, we have

$$\begin{aligned} m^2 + 2m + \lambda &= 0, \\ m^2 + 2m + 1 - 1 + \lambda &= 0, \\ (m+1)^2 - 1 + \lambda &= 0. \end{aligned}$$

Therefore, $m = \pm\sqrt{1-\lambda} - 1$. We divide into three cases.

Case 1: $\lambda < 1$.

Let $-(1-\lambda) = k^2$ for $k > 0$ so that $m = k-1$ or $m = -k-1$.

$$\begin{aligned} y_1 &= e^{(k-1)x}, & y_2 &= e^{(-k-1)x}, \\ y_1' &= (k-1)\,e^{(k-1)x}, & y_2' &= (-k-1)\,e^{(-k-1)x}, \\ y_1'(0) &= k-1, & y_2'(0) &= -k-1, \\ y_1(1) &= e^{k-1}, & y_2(1) &= e^{-k-1}. \end{aligned}$$

Thus,

$$\triangle = \begin{vmatrix} k-1 & -k-1 \\ e^{k-1} & e^{-k-1} \end{vmatrix} = (k-1)\,e^{-k-1} + (k+1)\,e^{k-1}.$$

Note that $\triangle = 0$ implies that

$$\begin{aligned} (1-k)\,e^{-k-1} &= (k+1)\,e^{k-1}, \\ \underbrace{\left(\frac{1-k}{1+k}\right)}_{<1} &= e^{k-1+k+1} = \underbrace{e^{2k}}_{>1 \text{ (since } k>0)}. \end{aligned}$$

which is a contradiction. Therefore, there are no solutions for $\lambda < 1$.

Case 2: $\lambda = 1$.

Then $m = -1$ is a double root and we have

$$\begin{aligned} y_1 &= e^{-x}, & y_2 &= xe^{-x}, \\ y_1' &= -e^{-x}, & y_2' &= e^{-x} - xe^{-x}, \\ y_1'(0) &= -1, & y_2'(0) &= 1, \\ y_1(1) &= e^{-1}, & y_2(1) &= e^{-1}. \end{aligned}$$

Thus,

$$\triangle = \begin{vmatrix} -1 & 1 \\ e^{-1} & e^{-1} \end{vmatrix} = -\frac{2}{e} \neq 0.$$

Therefore, $\lambda = 1$ is not an eigenvalue.

Case 3: $\lambda > 1$.

Set $\lambda - 1 = k^2$ for $k > 0$. Then $m = -1 \pm ki$ and we have

$$\begin{aligned} y_1 &= e^{-x}\cos(kx), & y_2 &= e^{-x}\sin(kx), \\ y_1' &= -e^{-x}\cos(kx) - ke^{-x}\sin(kx), & y_2' &= -e^{-x}\sin(kx) + ke^{-x}\cos(kx), \\ y_1'(0) &= -1, & y_2'(0) &= k, \\ y_1(1) &= \frac{1}{e}\cos(k), & y_2(1) &= \frac{1}{e}\sin(k). \end{aligned}$$

Thus,

$$\triangle = \begin{vmatrix} -1 & k \\ \frac{1}{e}\cos(k) & \frac{1}{e}\sin(k) \end{vmatrix} = -\frac{1}{e}\big(\sin(k) + k\cos(k)\big).$$

Note that $\triangle = 0 \Rightarrow k = -\tan(k)$. Figure 3.6 shows where $y = -\tan(k)$ and $y = k$ intersect. Therefore, the eigenvalues are the λ's satisfying $\lambda = k^2+1$, where $k > 0$ is a solution to $k = -\tan(k)$.

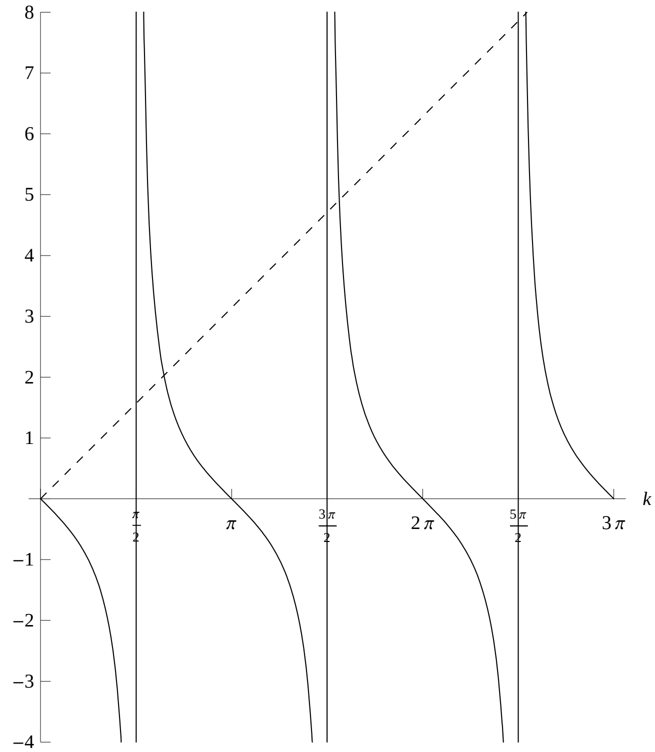


Figure 3.6: The plot of $y = -\tan(k)$ and $y = k$, showing that they intersect at $k^2 + 1$.

We have

$$\begin{bmatrix} -1 & k \\ \frac{1}{e}\cos(k) & \frac{1}{e}\sin(k) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow -c_1 + kc_2 = 0 \Longrightarrow -c_1 = kc_2.$$

Therefore, the eigenfunction for $\lambda = k^2 + 1$ is $ke^{-x}\cos(kx) + e^{-x}\sin(kx)$. ◇

DEFINITION 3.8 (Inner product). An *inner product* on a real vector space V is a function $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$ satisfying

(1) $\langle v, w\rangle = \langle w, v\rangle$ for all $v, w \in V$.
(2) $\langle \alpha v, w\rangle = \alpha\langle v, w\rangle = \langle v, \alpha w\rangle$ for all $\alpha \in \mathbb{R}$ and $v, w \in V$.
(3) $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ and $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$ for all $u, v, w \in V$.
(4) $\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0 \Leftrightarrow v = 0$ for all $v \in V$.

◊

EXAMPLE 3.9. Given $w : [a, b] \to \mathbb{R}$, set

$$\langle f, g\rangle_w := \int_a^b w(x)f(x)g(x)\,dx.$$

This defines an inner product if $w(x) > 0$ for all x and, in fact, we can allow $w = 0$ at finitely many points. ◇

DEFINITION 3.10 (Orthogonality). Elements $v, w \in V$ are called *orthogonal* with respect to $\langle\ ,\ \rangle$ if $\langle v, w\rangle = 0$. $\lozenge$

EXAMPLE 3.11. Let $[a,b] = [0,1]$ and $w(x) = 1$. Then the solutions of

$$\begin{cases} -y'' = \lambda y, \\ y(a) = 0, \\ y(b) = 0 \end{cases}$$

are $\sin(n\pi x)$, which are orthogonal with respect to $\langle\ ,\ \rangle_w$ on $[a,b]$. $\diamond$

PROPOSITION 3.12. *If $V \subset \mathcal{C}(X)$, w is continuous, and $\langle f, f\rangle_w = 0$, then $f \equiv 0$.*

PROOF. If $0 = \langle f, f\rangle_w = \int_a^b f^2(x)w(x)\,dx$, then, since f is continuous and $w > 0$, we have $f \equiv 0$. $\square$

Given

$$\begin{cases} Ly = \lambda y, \\ \alpha y(a) + \beta y'(a) = 0, \\ \gamma y(b) + \delta y'(b) = 0, \end{cases}$$

we want to find w such that the solutions to the BVP are orthogonal with respect to $\langle\ ,\ \rangle_w$ on $[a,b]$.

DEFINITION 3.13 (Self-adjoint). Given an inner product $\langle\ ,\ \rangle$, an operator T is called *self-adjoint* (with respect to the inner product) if $\langle Tv, w\rangle = \langle v, Tw\rangle$ for all $v, w \in V$. $\lozenge$

THEOREM 3.14. *If T is self-adjoint with respect to $\langle\ ,\ \rangle$, then the eigenvectors corresponding to distinct eigenvalues of T are orthogonal with respect to $\langle\ ,\ \rangle$.*

PROOF. Suppose $Tv = \lambda_1 v$ and $Tw = \lambda_2 w$, where $\lambda_1 \neq \lambda_2$. Then

$$\lambda_1 \langle v, w\rangle = \langle \lambda_1 v, w\rangle = \langle Tv, w\rangle = \langle v, Tw\rangle = \langle v, \lambda_2 w\rangle = \lambda_2 \langle v, w\rangle \tag{3.5}$$

Therefore, $\lambda_1 \neq \lambda_2 \Rightarrow \langle v, w\rangle = 0$. $\square$

COROLLARY 3.15. *If L is self-adjoint with respect to $\langle\ ,\ \rangle_w$, then the solutions to the eigenvalue problem $Ly = \lambda y$ with boundary conditions*

$$\begin{cases} \alpha y(a) + \beta y'(a) = 0, \\ \gamma y(b) + \delta y'(b) = 0 \end{cases}$$

are mutually orthogonal with respect to $\langle\ ,\ \rangle_w$.

Note that by Theorem 3.4, eigenspaces are one-dimensional.

By convention, to say that L is self-adjoint implicitly means that L is self-adjoint with respect to $\langle\ ,\ \rangle_1$.

Given $w : [a,b] \to (0,\infty)$ and a differential operator L, define a new operator L_w by $L_w := w(x)Ly$. If y is a solution to $Ly = \lambda y$, then

$$L_w y = w(x)Ly = \lambda w(x)y,$$

so the equation $L_w y = \lambda w(x) y$ is equivalent to the eigenvalue problem $Ly = \lambda y$.

DEFINITION 3.16 (Sturm-Liouville problem). A problem of the form $Ly = \lambda w(x)y$, where L is self-adjoint with respect to $\langle\ ,\ \rangle_1$, is called a *Sturm-Liouville problem.* $\lozenge$

PROPOSITION 3.17. *The differential operator L is self-adjoint with respect to $\langle\ ,\ \rangle_w$ if and only if L_w is self-adjoint with respect to $\langle\ ,\ \rangle_1$.*

PROOF. Observe that

$$\langle f, g\rangle_w := \int_a^b w(x)f(x)g(x)\,dx = \langle wf, g\rangle\,,$$

(where we write $\langle\ ,\ \rangle$ for $\langle\ ,\ \rangle_1$). Therefore

$$\langle L_w f, g\rangle = \langle wLf, g\rangle = \langle Lf, g\rangle_w\,,$$
$$\langle f, L_w g\rangle = \langle f, Lg\rangle_w\,,$$

so $\langle L_w f, g\rangle = \langle f, L_w g\rangle$, or equivalently, $\langle Lf, g\rangle_w = \langle f, Lg\rangle_w$. □

COROLLARY 3.18. *If L is self-adjoint (with respect to $\langle\ ,\ \rangle_1$), then the solutions to $Ly = \lambda wy$ are orthogonal with respect to $\langle\ ,\ \rangle_w$.**

PROOF. Ignoring a finite set of points on which w is zero, consider $L_{1/w}$ and let $L = \left(L_{1/w}\right)_w$. Then solutions to $Ly = \lambda wy$ are solutions to $L_{1/w}y = \lambda y$. It also follows that if L is self-adjoint, then $L_{1/w}$ is self-adjoint with respect to $\langle\ ,\ \rangle_w$. The solutions to $Ly = \lambda wy$ are the solutions to $L_{1/w}y = \lambda y$, and so are mutually orthogonal with respect to $\langle\ ,\ \rangle_w$.

Now suppose that $w(x) = 0$ for some values of x. Subdivide $[a, b]$ into subintervals on which $w(x) \neq 0$. Then the proof follows, as in the above, for each of these subintervals. □

Given homogeneous boundary conditions

$$\begin{cases} \alpha y(a) + \beta y'(a) = 0, \\ \gamma y(b) + \delta y'(b) = 0 \end{cases} \tag{B}$$

and a continuous function $w : [a, b] \to (0, \infty)$, we wish to find conditions on L such that L is self-adjoint with respect to $\langle\ ,\ \rangle_w$ on V_B, where V_B is the set of all f satisfying conditions (B).

LEMMA 3.19. *(Lagrange formula) For every $f, g \in V_B$, we have*

$$\langle f'', g\rangle_w - \langle f, g''\rangle_w = -\langle f', g\rangle_{w'} + \langle f, g'\rangle_{w'}\,. \tag{3.6}$$

PROOF. First note that $\langle f'', g\rangle_w = \int_a^b wf''g\,dx$. Integrating by parts, let

$$u = wg, \qquad dv = f''\,dx,$$
$$du = (w'g + wg')\,dx, \qquad v = f'.$$

Then

$$\langle f'', g\rangle_w = wgf'\Big|_a^b - \int_a^b w'f'g\,dx - \int_a^b wf'g'\,dx.$$

Therefore,

$$\langle f'', g\rangle_w - \langle f, g''\rangle_w = wgf'\Big|_a^b - \int_a^b w'f'g\,dx - \cancel{\int_a^b wf'g'\,dx}$$
$$- wfg'\Big|_a^b + \int_a^b w'fg'\,dx + \cancel{\int_a^b wf'g'\,dx} \quad = w\left(f'g - fg'\right)\Big|_a^b - \langle f', g\rangle_{w'} + \langle f, g'\rangle_{w'}\,.$$

*Phrased differently, solutions to Sturm-Liouville problems are mutually orthogonal with respect to $\langle\ ,\ \rangle_w$.

But

$$f'(b)g(b) - f(b)g'(b) = -\frac{\gamma}{\delta}f(b)g(b) + \frac{\gamma}{\delta}f(b)g(b) = 0, \quad \delta \neq 0,$$

while if $\delta = 0$, then $f(b) = g(b) = 0$, so $f'(b)g(b) - f(b)g'(b) - f(b)g'(b) = 0$. Similarly, $f'(a)g(a) - f(a)g'(a) = 0$. □

LEMMA 3.20. *If $p : [a, b] \to \mathbb{R}$ is continuous and $\int_a^b p(x)\,(f'g - g'f)\,dx = 0$ for all $f, g \in V_B$, then $p \equiv 0$.*

PROOF. By changing variables, we may assume that $a = 0$ and $b = 1$, i.e., $x = a + (b-a)\,\bar{x}$, where $\bar{x} = (x-a)\,/\,(b-a)$. We will also assume that $\gamma, \delta \neq 0$. The argument needs slight modification otherwise. Now f and g must satisfy

$$\begin{aligned} \frac{f'(0)}{f(0)} &= \frac{\beta}{\alpha} = \frac{g'(0)}{g(0)}, \\ \frac{f'(1)}{f(1)} &= \frac{\delta}{\gamma} = \frac{g'(1)}{g(1)}. \end{aligned}$$

Suppose we try $f := e^{mx^2+nx}$. Then

$$\frac{f'}{f} = \frac{(2mx+b)\,e^{mx^2+nx}}{e^{mx^2+bx}} = 2mx + n.$$

Therefore, choose $n = \beta/\alpha$ and $2m + n = \frac{\delta}{\gamma}$ so that $m = \frac{\sigma/\gamma - \beta/\gamma}{2}$ to make $f \in V_B$.

Choose g so that $f'g - fg' = p\sin(x)$ as follows:

$$f'g - fg' = p\sin(\pi x) \Longleftrightarrow \underbrace{\frac{f'g - fg'}{f^2}}_{d\left(\frac{g}{f}\right)/dx} = \frac{p\sin(\pi x)}{f^2}.$$

So we want

$$\frac{g}{f} = \int_0^x \frac{p\sin(\pi t)}{f^2}\,dt \Longrightarrow g = f\int_0^x \frac{p\sin(\pi t)}{f^2}\,dt.$$

Define g by

$$g(x) := f\int_0^x \frac{p\sin(\pi t)}{f^2}\,dt,$$

so that it will satisfy $f'g - fg' = p\sin(\pi x)$. But then

$$\sin(0) = 0 \Longrightarrow f'(0)g(0) - f(0)g'(0) = 0,$$

so

$$\frac{g(0)}{g'(0)} = \frac{f(0)}{f'(0)} = \frac{\beta}{\alpha}.$$

Similarly, $\sin(\pi) = 0$, so

$$\frac{g(1)}{g'(1)} = \frac{f(1)}{f'(1)} = \frac{\delta}{\gamma}.$$

Therefore, $g \in V_B$ also. Note that

$$\int_0^1 p(x)\,(f'g - g'f)\,dx = \int_0^1 p(x)^2\sin(\pi x)\,dx$$

by construction, and so the hypothesis that $\int_a^b p(x)\,(f'g - g'f)\,dx = 0$ gives

$$\int_0^1 p(x)^2\sin(\pi x)\,dx = 0.$$

But $p(x)^2 \sin(\pi x) \geq 0$ for all $x \in [0,1]$, so

$$\int_0^1 p(x)^2 \sin(\pi x)\, dx = 0 \underbrace{\Longrightarrow}_{\text{continuity of integrand}} p(x)^2 \sin(\pi x) = 0$$

for all $x \in [0,1]$. But $\sin(\pi x) \neq 0$ for $x \in (0,1)$. Therefore, $p(x)^2 = 0$ for all $x \in (0,1) \Rightarrow p(x) = 0$ for all $x \in (0,1)$. Therefore by continuity, $p(x) = 0$ for all $x \in [0,1]$. $\square$

THEOREM 3.21. *Let $Ly := -\left(a_0(x)y'' + a_1(x)y' + a_2 y\right)$. Then L is self-adjoint with respect to $\langle\ ,\ \rangle_1$ on V_B if and only if $a_1(x) = a_0'(x)$.*

PROOF. Let $Ly := -\left(a_0(x)y'' + a_1(x)y' + a_2 y\right)$ and suppose that L is self-adjoint with respect to $\langle\ ,\ \rangle_1$ on V_B. Then for all $f, g \in V_B$, we have

$$\begin{aligned}
0 &= -\langle Lf, g\rangle + \langle g, Lf\rangle \\
&= \langle a_0 f'', g\rangle + \langle a_1 f', g\rangle + \langle a_2 f, g\rangle - \langle f, a_0 g''\rangle - \langle f, a_1 g'\rangle - \langle f, a_2 g\rangle \\
&= \langle f'', g\rangle_{a_0} - \langle f, g''\rangle_{a_0} + \langle a_1 f', g\rangle - \langle f, a_1 g'\rangle + \cancel{\langle f, g\rangle_{a_2}} - \cancel{\langle f, g\rangle_{a_2}} \\
&= -\langle f', g\rangle_{a_0'} + \langle f, g'\rangle_{a_0'} + \langle a_1 f', g\rangle - \langle f, a_1 g'\rangle \\
&= \int_a^b \left(-a_0' f' g + a_0' f g' + a_1 f' g - a_1 f g'\right) dx \\
&= \int_a^b \left(a_1 - a_0'\right)\left(f' g - f g'\right) dx.
\end{aligned}$$

So by Lemma 3.20 applied to $p = a_1 - a_0'$, we have $a_1 = a_0'$.

Conversely, if $a_1 = a_0'$, then the same calculation shows that $\langle Lf, g\rangle = \langle g, Lf\rangle$ for all $f, g \in V_B$. $\square$

COROLLARY 3.22. *The differential operator L is self-adjoint with respect to $\langle\ ,\ \rangle_w$ if and only if L_w is self-adjoint with respect to $\langle\ ,\ \rangle_1$ if and only if $a_1 w = (a_0 w)'$.*

So given L and conditions (B) (p. 69), we can now choose w such that L_w is self-adjoint on V_B as follows. We need $a_1 w = (a_0 w)'$, so it follows that

$$\begin{aligned}
a_1 w &= \overbrace{a_0' w + a_0 w'}^{(a_0 w)'}, \\
a_0 w' &= \left(a_1 - a_0'\right) w, \\
\frac{w'}{w} &= \frac{a_1 - a_0'}{a_0},
\end{aligned}$$

resulting in

$$w = A e^{\int \frac{a_1 - a_0'}{a_0}\, dx}. \tag{3.7}$$

This is summarized in the following theorem.

THEOREM 3.23. *If $Ly = -\left(a_0 y'' + a_1 y' + a_2 y\right)$, then the solution to the eigenvalue problem $Ly = \lambda y$ satisfying the boundary conditions* (B) *are orthogonal with respect to $\langle\ ,\ \rangle_w$, where*

$$w = e^{\int \frac{a_1 - a_0'}{a_0}\, dx}.$$

EXAMPLE 3.24. Suppose $a_0 = 1$ and $a_1 = 0$. Then $w = e^{0\, dx} = e^C$. Pick $C = 1$. $\diamond$

Given the eigenvalue problem $Ly = \lambda y$ with conditions (B), we can rephrase it as $L_w y = \lambda w y$ for any w we choose. By choosing w as in Equation (3.7), it becomes $L_w y = \lambda w y$ with L_w self-adjoint, and thus the solutions are orthogonal with respect to w.

EXAMPLE 3.25. Consider

$$\left(1-x^2\right)y'' - xy' + (\lambda+1)\,y = 0,$$

where $|x| < 1$. ◇

Write the equation as $Ly = \lambda y$, where

$$Ly = -\Big(\left(1-x^2\right)y'' - xy' + y\Big).$$

Let $a_0 := 1 - x^2$, so $a_0' = -2x$. But $a_1 := -x \neq a_0'$. Therefore, L is not self-adjoint with respect to $\langle\ ,\ \rangle_1$. By Equation (3.7), we have

$$\begin{aligned} w &= e^{\int \frac{a_1 - a_0'}{a_0}\,dx} = e^{\int \frac{-x+2x}{1-x^2}\,dx} = e^{\int \frac{x}{1-x^2}\,dx} \\ &= e^{-\ln\left(1-x^2\right)/2} = \frac{1}{\sqrt{1-x^2}}. \end{aligned}$$

Now L is self-adjoint with respect to $\langle\ ,\ \rangle_w$. Multiplying by w gives

$$\underbrace{\sqrt{1-x^2}\,y'' - \frac{x}{\sqrt{1-x^2}}y' + \frac{1}{\sqrt{1-x^2}}y}_{\text{self-adjoint with respect to } \langle\ ,\ \rangle_1} = \frac{1}{\sqrt{1-x^2}}\lambda y.$$

For any boundary conditions* $B_a(y)$ and $B_b(y)$ with $a, b \in (-1, 1)$, the eigenfunctions of

$$\begin{cases} \left(1-x^2\right)y'' - xy' + (\lambda+1)\,y = 0, \\ \hfill B_a(y) = 0, \\ \hfill B_b(y) = 0 \end{cases}$$

are orthogonal with respect to

$$\langle f, g\rangle := \int_a^b \frac{1}{\sqrt{1-x^2}} f g\,dx.$$

Note that changing B_a and B_b changes the eigenvalues and the eigenfunctions, but it doesn't change the inner product with respect to which they are orthogonal, i.e., w is determined entirely by

$$\left(1-x^2\right)y'' - xy' + (\lambda+1)\,y$$

and the inner product is determined by w, a, and b.

THEOREM 3.26 (Properties of Sturm-Liouville problems). *Let $L = -\left(a_0 y'' + a_1 y' + a_2 y\right)$. Consider the Sturm-Liouville problem $Ly = \lambda w(x) y$, along with boundary conditions $B_a(y) = 0$ and $B_b(y) = 0$, where L is self-adjoint with respect to $\langle\ ,\ \rangle_1$ and $a_0(x) > 0$ on $[a, b]$. Assume $w_0 \geq 0$ on $[a, b]$. Then the following hold:*

1. *The eigenvalues $\{\lambda_n\}$ of $Ly = \lambda w(x) y$ are real and form a countably infinite sequence $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ with $\lim_{n\to\infty} \lambda_n = \infty$.*
2. *For each eigenvalue λ_n, there is (up to scalar multiples), a unique eigenfunction ϕ_n. The eigenfunctions ϕ_n has exactly $n-1$ zeros on (a, b).*
3. *The eigenfunctions form an orthogonal basis with respect to the inner product $\langle\ ,\ \rangle_w$.*

*Recall the definition of this notation in Equations (3.4a) and (3.4b) on page 64.

(4) *If g is twice differentiable on$[a, b]$, then*

$$g(x) = \sum_{n=1}^{\infty} c_n \phi_n,$$

where

$$c_n := \frac{\langle g, \phi_n \rangle_w}{\langle \phi_n, \phi_n \rangle_w}$$

is called the nth generalized Fourier coefficient of g.

EXAMPLE 3.27. In our earlier Example 3.6 (p. 65), where $w(x) = 1$ with conditions $y(0) = 1$ and $y(0) = 50$, we had $\lambda_n = \pi^2 n^2/50^2$ and $\phi_n = \sin\left(\frac{n\pi x}{50}\right)$. Note that $\phi_n = 0$ for

$$x \in \left\{0, \frac{50}{n}, \frac{2 \cdot 50}{n}, \ldots, \frac{n \cdot 50}{n} = 50\right\},$$

so it has $n + 1$ zeros. ◇

PROOF OF THEOREM 3.26 (IDEA).

(1) Showing that the eigenvalues are real is essentially the same as the proof that the eigenvalues of a symmetric matrix are real. It is much harder to show that there exists a smallest eigenvalue λ_1 and that they form a sequence going to ∞.
(2) We already proved this. See Theorem 3.4.
(3) We already showed that $\langle \phi_i, \phi_j \rangle_w = 0$, but this says more. It claims that $\{\phi_n\}$ is a maximal orthogonal set, i.e., there does not exist a nonzero g such that $\langle g, \phi_n \rangle = 0$ for all n. This is harder to prove.
(4) This is analogous to Dirichlet's Theorem for (standard) Fourier series and the proof is comparable (we will not prove this).

We postpone the formal proof until §6. □

EXAMPLE 3.28. Solve

$$\begin{cases} u_{xx} = x u_t, \\ u(0, t) = 0, \\ u(1, t) = 0, \\ u(x, 0) = x \text{ for } x \in (0, 1). \end{cases}$$

◇

SOLUTION. We try finding solutions of the form $u(x, t) = X(x)T(t)$. Then

$$\left.\begin{aligned} u_{xx} &= X''T, \\ u_t &= xXT' \end{aligned}\right\} \implies \frac{X''}{xX} = -\lambda = \frac{T'}{T}$$

for some constant λ. Therefore,

$$X'' = -\lambda x X, \tag{$*$}$$

$$T' = -\lambda T, \tag{$**$}$$

$$X(0)T(t) = 0 \quad \forall t, \tag{B1}$$

$$X(1)T(t) = 0 \quad \forall t, \tag{B2}$$

$$X(x)T(0) = x \text{ for } x \in (0,1). \tag{B3}$$

To have $T(0) = 0$ would contradict Equation (B3), so $T(0) \neq 0$ and therefore Equation (B1) implies that $X(0) = 0$, and Equation (B2) implies that $X(1) = 0$. Therefore, consider

$$\begin{cases} y'' = -\lambda xy, \\ y(0) = 0, \\ y(1) = 0. \end{cases}$$

Letting

$$y = \sum_{n=0}^{\infty} a_n x^n \Longrightarrow y'' = \sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2} x^n$$
$$\Longrightarrow \underbrace{a_{n+2} = -\lambda \frac{a_{n-1}}{(n+2)(n+1)}}_{\text{recurrence relation}}.$$

Suppose $y = c_1 y_1 + c_2 y_2$. Then y_1 has $a_0 = 1$ and $a_1 = 0$ while y_2 has $a_0 = 0$ and $a_1 = 1$. Thus, $y(0) = 0 \Rightarrow c_1 = 0$ and so

$$y = c y_2.$$

In y_2, we have

$$a_1 = 1, \quad a_4 = -\lambda \frac{1}{3 \cdot 4}, \quad a_7 = \lambda^2 \frac{1}{3 \cdot 4 \cdot 6 \cdot 7}, \quad a_{10} = -\lambda^3 \frac{1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10}, \ldots .$$

In general, we have

$$a_{3n+1} = (-1)^n \lambda^n \frac{1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3n)(3n+1)}$$

with all other terms equal to zero. Therefore,

$$y_2 = x - \lambda \frac{x^4}{3 \cdot 4} + \lambda^2 \frac{x^7}{3 \cdot 4 \cdot 6 \cdot 7} - \lambda^3 \frac{x^{10}}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} + \cdots .$$

Note that

$$0 = \triangle = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1(1) & y_2(1) \end{vmatrix} = \begin{vmatrix} 1 & y_2(0) \\ 0 & y_2(1) \end{vmatrix} = y_2(1)$$
$$= 1 - \frac{\lambda}{3 \cdot 4} + \frac{\lambda^2}{3 \cdot 4 \cdot 6 \cdot 7} - \frac{\lambda^3}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} + \cdots .$$

Clearly $\lambda \leq 0 \Rightarrow \triangle > 0$, so all eigenvalues are positive. With the aid of a computer program, we find that the first few eigenvalue approximations are

$$\lambda_1 \approx 18.956, \quad \lambda_2 \approx 81.887, \quad \lambda_3 \approx 189.217, \quad \lambda_4 \approx 340.967, \ldots .$$

The eigenfunctions are y_2 with these values of λ. The eigenfunctions should be orthogonal with respect to

$$\langle f, g \rangle := \int_0^1 x f(x) g(x)\, dx.$$

Thus,

$$\langle \phi_k, \phi_n \rangle_x = \int_0^1 x \phi_k(x) \phi_n(x)\, dx,$$

$$= \int_0^1 \left(x - \lambda_k \frac{x^4}{3\cdot 4} + \lambda_k^2 \frac{x^7}{3\cdot 4\cdot 6\cdot 7}\right)\left(x - \lambda_n \frac{x^4}{3\cdot 4} + \lambda_n^2 \frac{x^7}{3\cdot 4\cdot 6\cdot 7} + \cdots\right) dx$$
$$= \int_0^1 x\left(x^2 - (\lambda_k + \lambda_n)\frac{x^5}{3\cdot 4} + \left(\lambda_k^2 + \lambda_n^2\right)\frac{x^8}{3\cdot 4\cdot 6\cdot 7} + \lambda_k\lambda_n \frac{x^8}{(3\cdot 4)^2} - \cdots\right) dx$$
$$= \frac{1}{4} - \frac{\lambda_k}{84} - \frac{\lambda_n}{84} + \frac{\lambda_k\lambda_n}{1440} + \frac{\lambda_k^2 + \lambda_n^2}{5040} + \cdots .$$

Expand $f(x) = x$ in eigenfunctions $\phi_1, \phi_2, \ldots$. Then

$$x = \sum_{n=1}^{\infty} b_n \phi_n,$$

where

$$b_n = \frac{\langle x, \phi_n\rangle_x}{\langle \phi_n, \phi_n\rangle_x}.$$

The first few approximations are

$$b_1 \approx 1.859, \quad b_2 \approx -0.0156, \quad b_3 \approx -0.0031.$$

Note that $T' = -\lambda T$, and so $T = Ae^{-\lambda t}$. Therefore, the solution is given by

$$u(x,t) = \sum_{n=0}^{\infty} b_n e^{\lambda_n t}\left(x - \lambda_n \frac{x^4}{3\cdot 4} + \lambda_n^2 \frac{x^7}{3\cdot 4\cdot 6\cdot 7} - \lambda_n^3 \frac{x^{10}}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10} + \cdots\right),$$

where

$$b_1 \approx 1.859, \quad b_2 \approx -0.0156, \quad b_3 \approx -0.0031, \ldots$$

and

$$\lambda_1 \approx 18.956, \quad \lambda_2 \approx 81.817, \quad \lambda_3 \approx 189.217, \ldots .$$

◇

Consider

$$-\left(a_0 y'' + a_0 y' + a_2 y\right) = \lambda w y,$$

where $a_1 = a_0'$. Until now, we required that $a_0(x) > 0$ on $[a,b]$. Suppose that $a_0(b) = 0$. Consider

$$\begin{cases} a_0 y'' + a_1 y' + a_2 y + \lambda w y = 0, \\ B_a(y) = 0, \\ \left|B_b(y)\right| < \infty. \end{cases}$$

This is not our standard form for boundary conditions, but $V_B := \left\{f : B_a(f) = 0 \text{ and } \left|B_b(f)\right| < \infty\right\}$ is nevertheless a vector space. Our theory applies equally well to this situation. It also applies to the conditions

$$\begin{cases} \left|B_a(y)\right| < \infty, \\ \left|B_b(y)\right| < \infty. \end{cases}$$

Example 3.29. Solve Legendre's equation given by

$$\begin{cases} \left(1 - x^2\right) y'' - 2xy' + \lambda y = 0, \\ \left|y(-1)\right| < \infty, \\ \left|y(1)\right| < \infty. \end{cases}$$

$\diamond$

SOLUTION. Since $\left(1-x^2\right)' = -2x$, the Legendre operator is self-adjoint with respect to $\langle\ ,\ \rangle_1$. By inspection, we see that

- $\lambda = 0$ is an eigenvalue with eigenfunction $y = 1$, so let $\phi_0(x) \equiv 1$.
- $\lambda = 2$ is an eigenvalue with eigenfunction $y = x$, so let $\phi_1(x) := x$.

Observe that

$$\int_{-1}^{1} \phi_0(x)\phi_1(x)\,dx = \int_{-1}^{1} x\,dx = \left.\frac{x^2}{2}\right|_{-1}^{1} = 0,$$

which is consistent with our theorems that the eigenfunctions should be orthogonal with respect to $\langle\ ,\ \rangle_w$.

Let's try to guess another eigenfunction. Since ϕ_0 and ϕ_1 were polynomials of degree 0 and 1, respectively, let's look for a quadratic eigenfunction ϕ_2. Thus, we now wish to find a quadratic ϕ_2 such that

$$\int_{-1}^{1} \phi_0(x)\phi_2(x)\,dx = 0, \qquad \int_{-1}^{1} \phi_1(x)\phi_2(x)\,dx = 0.$$

Note that being a polynomial automatically makes $\phi_2(-1)$ and $\phi_2(1)$ bounded. Let $\phi_2(x) := x^2 + Cx + D$. Then

$$\int_{-1}^{1} \phi_0(x)\phi_1(x)\,dx = \left.\left(\frac{x^3}{3} + C\frac{x^2}{2} + Dx\right)\right|_{-1}^{1} = \frac{2}{3} + 2D \Longrightarrow D = -\frac{1}{3}$$

and

$$\begin{aligned}\int_{-1}^{1} \phi_1(x)\phi_2(x)\,dx &= \int_{-1}^{1} \left(x^3 + Cx^2 + Dx\right)dx \\ &= \left.\left(\frac{x^4}{4} + C\frac{x^3}{3} + D\frac{x^2}{2}\right)\right|_{-1}^{1} \\ &= \frac{2C}{3} \Longrightarrow C = 0.\end{aligned}$$

Therefore,

$$\phi_2(x) = x^2 - \frac{1}{3}.$$

A more systematic way of finding ϕ_2 is through the *Gram-Schmidt* process. Given a vector x, the component of x perpendicular to u is found by subtracting the projection of x onto u. The projection of x onto u is

$$\left(x \cdot \frac{u}{\|u\|^2}\right) u = \frac{\langle x, u\rangle}{\langle u, u\rangle} u.$$

To check for orthogonality, set

$$y = x - \frac{\langle x, u\rangle}{\langle u, u\rangle} u.$$

Then

$$\langle y, u\rangle = \langle x, u\rangle - \frac{\langle x, u\rangle}{\langle u, u\rangle} \langle u, u\rangle = 0.$$

To get ϕ_2, we start with x^2 and remove projections onto ϕ_0 and ϕ_1, i.e., the Gram-Schmidt process. Therefore

$$\phi_2(x) = x^2 - \frac{\langle x^2, \phi_0\rangle}{\langle \phi_0, \phi_0\rangle}\phi_0 - \frac{\langle x^2, \phi_1\rangle}{\langle \phi_1, \phi_1\rangle}\phi_1$$

$$
\begin{aligned}
&= x^2 - \left(\frac{\displaystyle\int_{-1}^{1} x^2 \, dx}{\displaystyle\int_{-1}^{1} dx} \right) 1 - \left(\frac{\displaystyle\int_{-1}^{1} x^2 x \, dx}{\displaystyle\int_{-1}^{1} x^2 \, dx} \right) x \\
&= x^2 - \frac{2 \displaystyle\int_{0}^{1} x^2 \, dx}{2} 1 - 0x \\
&= x^2 - \frac{1}{3},
\end{aligned}
$$

as before. Because of its orthogonality with the eigenfunctions ϕ_1 and ϕ_2, we might conjecture that ϕ_2 is also an eigenfunction. To determine whether ϕ_2 is an eigenfunction, we need to see if $\phi_2(x)$ solves the differential equation for some λ. To find out, we have

$$
\begin{aligned}
\left(1 - x^2\right) \phi_2''(x) - 2x\phi_2'(x) + \lambda\phi_2(x) &= \left(1 - x^2\right) 2 - 2x \cdot 2x + \lambda \left(x^2 - \frac{1}{3} \right) \\
&= 2 - 2x^2 - 4x^2 + \lambda x^2 - \frac{\lambda}{3} \\
&= (\lambda - 6)\, x^2 + \left(2 - \frac{\lambda}{3} \right).
\end{aligned}
$$

Since

$$
(\lambda - 6)\, x^2 + \left(2 - \frac{\lambda}{3} \right) = 0
$$

when $\lambda = 6$, we conclude that $\lambda = 6$ is an eigenvalue with ϕ_2 as an eigenfunction. We know that $\lambda = 6$ appears no earlier than third on the list of eigenvalues because we already found two eigenvalues less than it. But have we missed some other eigenvalue less than 6? To decide, we apply Theorem 3.26 (p. 72), which tells us that the nth eigenfunction has $n-1$ zeros on $(-1,1)$. Since $\phi_2(x) = x^2 - 1/3$ has 2 zeros on $(-1,1)$, we conclude that $\lambda = 6$ is the third eigenvalue.

It turns out that the eigenvalues are $\lambda = n\,(n+1)$ for $n = 0, 1, 2, 3, \ldots$. Since ϕ_n turns out to be a degree n polynomial, it has at most n zeros on $(-1,1)$, so the eigenvalue $n\,(n+1)$ is no later than $(n+1)$st on the list of eigenvalues. So it is exactly number $n+1$ since we know of n preceding it. Therefore, $\left\{n\,(n+1)\right\}$ is the complete list of eigenvalues, and all zeros of ϕ_n occur on $(-1,1)$.

The polynomials $\{\phi_n\}$ are called the *Legendre polynomials* and can be found inductively by Gram-Schmidt as above, or by the power series methods of `MATB44`. ◇

Example 3.30. Give the first three terms in the expansion of $f(x) := e^x$ in Legendre polynomials. ◇

Solution. Note that $f(-1)$ and $f(1)$ are bounded since $f(x)$ is continuous on $[-1,1]$. Therefore, such an expansion is possible. Let

$$
e^x = \sum_{n=0}^{\infty} c_n \phi_n,
$$

where $c_n = \langle f, \phi_n \rangle / \langle \phi_n, \phi_n \rangle$. To find c_0, we have

$$
c_0 = \frac{\displaystyle\int_{-1}^{1} e^x \, dx}{\displaystyle\int_{-1}^{1} dx} = \frac{e - e^{-1}}{2}.
$$

To find c_1, we have

$$c_1 = \frac{\displaystyle\int_{-1}^{1} xe^x\,dx}{\displaystyle\int_{-1}^{1} x^2\,dx} = \frac{(xe^x - e^x)\big|_{-1}^{1}}{x^3/3\big|_{-1}^{1}} = \frac{2e^{-1}}{2/3} = \frac{3}{e}.$$

To find c_2, we have

$$\begin{aligned}
c_2 &= \frac{\displaystyle\int_{-1}^{1}\left(x^2 - \frac{1}{3}\right)e^x\,dx}{\displaystyle\int_{-1}^{1}\left(x^2 - \frac{1}{3}\right)^2 dx} \\
&= \frac{\left(x^2e^x - 2xe^x + \frac{5}{3}e^x\right)\big|_{-1}^{1}}{\left(\frac{x^5}{5} - \frac{2x^3}{9} + \frac{x}{9}\right)\big|_{-1}^{1}} \\
&= \frac{\frac{2}{3}e - \frac{14}{3}e^{-1}}{8/45} \\
&= \frac{15}{4}e - \frac{105}{4}e^{-1}.
\end{aligned}$$

Therefore,

$$e^x \approx \frac{e - e^{-1}}{2} + 3e^{-1}x + \frac{15e - 105e^{-1}}{4}\left(x^2 - \frac{1}{3}\right) + \cdots.$$

◇

3. Nonhomogeneous Boundary Value Problems

Consider

$$\begin{cases} Ly = f(x), \\ B_a y = 0, \\ B_b y = 0, \end{cases}$$

where $Ly = py'' + qy' + ry$. Until now, we have only considered $f(x) \equiv 0$. The solution of $Ly = f(x)$ is

$$y = c_1y_1 + c_2y_2 + y_p,$$

where y_1 and y_2 are linearly independent solutions of $Ly = 0$ and y_p is a solution of $Ly = f(x)$. We have

$$y_p = v_1y_1 + v_2y_2, \tag{3.11}$$

where

$$\begin{aligned}
v_1' &= -\frac{fy_2}{pW}, & v_2' &= \frac{fy_1}{pW}, \\
v_1 &= \int_a^x -\frac{f(t)y_2(t)}{p(t)W\big(y_1(t), y_2(t)\big)}\,dt, & v_2 &= \int_a^x \frac{f(t)y_1(t)}{p(t)W\big(y_1(t), y_2(t)\big)}\,dt.
\end{aligned}$$

Note that using a different lower limit of integration changes the solution only by the addition of $k_1y_1 + k_2y_2$, which can be incorporated into the homogeneous term of the solution. It is convenient to choose a as the lower limit. Therefore,

$$y_p(x) = y_1(x)\int_a^x -\frac{f(t)y_2(t)}{p(t)W\big(y_1(t), y_2(t)\big)}\,dt + y_2(x)\int_a^x \frac{f(t)y_1(t)}{p(t)W\big(y_1(t), y_2(t)\big)}\,dt$$

$$= \int_a^x \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{p(t)W\big(y_1(t), y_2(t)\big)} f(t)\,dt.$$

Then $y_p(a) = 0$. We have

$$\begin{aligned} y_p'(x) &= \cancel{\frac{y_1(x)y_2(x) - y_1(x)y_2(x)}{p(x)W\big(y_1(x), y_2(x)\big)} f(x)} + \int_a^x \frac{y_1(t)y_2'(x) - y_1'(x)y_2(t)}{p(t)W\big(y_1(t), y_2(t)\big)}\,dt \\ &= \int_a^x \frac{y_1(t)y_2'(x) - y_1'(x)y_2(t)}{p(t)W\big(y_1(t), y_2(t)\big)}\,dt. \end{aligned}$$

Therefore, $y_p'(a) = 0 + 0 = 0$.

Hence,

$$B_a(y_p) = \alpha y_p(a) + \beta y_p'(a) = 0 + 0 = 0$$

and

$$\begin{aligned} B_b(y_p) &= \gamma y_p(b) + \delta y_p'(b) \\ &= \gamma \int_a^b \frac{y_1(t)y_2(b) - y_1(b)y_2(t)}{p(t)W\big(y_1(t), y_2(t)\big)} f(t)\,dt + \delta \int_a^b \frac{y_1(t)y_2'(b) - y_1'(b)y_2(t)}{p(t)W\big(y_1(t), y_2(t)\big)} f(t)\,dt \\ &= \int_a^b \frac{y_1(t)B_b(y_2) - B_b(y_1)y_2(t)}{p(t)W\big(y_1(t), y_2(t)\big)} f(t)\,dt. \end{aligned} \tag{3.12}$$

Also, $W = e^{-\int \frac{q}{p}\,dx}$. If L is self-adjoint, then $q = p'$, so

$$W = e^{-\int \frac{p}{q}\,dx} = e^{-\int \frac{p}{p'}\,dx} = \frac{A}{p(x)}.$$

Therefore, pW is a constant in this case.

We need

$$\begin{cases} B_a(y) = c_1 B_a(y_1) + c_2 B_a(y_2) + B_a(y_p) = 0, \\ B_b(y) = c_1 B_b(y_1) + c_2 B_b(y_2) + B_b(y_p) = 0, \end{cases}$$
$$\underbrace{\begin{vmatrix} B_a(y_1) & B_a(y_2) \\ B_b(y_1) & B_b(y_2) \end{vmatrix}}_{\triangle} \neq 0, \tag{3.13}$$

Then System (3.13) can be solved (uniquely) for c_1 and c_2. But if $\triangle = 0$, then System (3.13) has either no solution (equations inconsistent) or many solutions (the second equation is redundant). Now, $\triangle = 0$ if and only if the corresponding homogeneous problem

$$\begin{cases} Ly = f(x), \\ B_a(y) = 0, \\ B_b(y) = 0 \end{cases}$$

has nontrivial solutions.

THEOREM 3.31. *Consider*

$$\begin{cases} Ly = f(x), \\ B_a(y) = 0, \\ B_b(y) = 0 \end{cases} \tag{3.14}$$

and let

$$\triangle = \begin{vmatrix} B_a(y_1) & B_a(y_2) \\ B_b(y_1) & B_b(y_2) \end{vmatrix}.$$

If $\triangle \neq 0$*, then System* (3.14) *has a unique solution. If* $\triangle = 0$*, then System* (3.14) *either has no solution or many solutions.*

We divide into the two possible cases: $\triangle \neq 0$ and $\triangle = 0$.

3.1. Determinant Cases.

3.1.1. *The Case When* $\triangle \neq 0$. Suppose that $\triangle \neq 0$. Then by Theorem 3.31, we know that a unique solution exists. To find it, we proceed as follows. Since $\triangle \neq 0$, we cannot find a $y \not\equiv 0$ satisfying System (3.14) with $f(x) \equiv 0$. But we can find a

- y_1 such that $Ly_1 = 0$ and $B_a(y_1) = 0$ with $B_b(y_1) \neq 0$.
- y_2 such that $Ly_1 = 0$ and $B_b(y_2) = 0$ with $B_a(y_2) \neq 0$.

Therefore, System (3.13) becomes

$$\begin{aligned} c_2 B_a(y_2) &= 0, \\ c_1 B_b(y_1) + B_b(y_p) &= 0. \end{aligned}$$

So choose $c_2 = 0$ and

$$\begin{aligned} c_1 &= -\frac{B_b(y_p)}{B_b(y_1)} = -\frac{1}{B_b(y_1)} \int_a^b \frac{\cancelto{0}{B_b(y_2)} y_1(t) - B_b(y_1) y_2(t)}{p(t) W\big(y_1(t), y_2(t)\big)} f(t)\, dt \\ &= \int_a^b \frac{y_2(t) f(t)}{p(t) W\big(y_1(t), y_2(t)\big)}\, dt. \end{aligned}$$

Therefore,

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 + y_p \\ &= \int_a^b \frac{y_2(t) y_1(x)}{p(t) W\big(y_1(t), y_2(t)\big)} f(t)\, dt + \int_a^x \frac{y_1(t) y_2(x) - y_1(x) y_2(t)}{p(t) W\big(y_1(t), y_2(t)\big)} f(t)\, dt \\ &= \int_a^x \frac{y_2(t) y_1(x)}{p(t) W\big(y_1(t), y_2(t)\big)} f(t)\, dt + \int_x^b \frac{y_2(t) y_1(x)}{p(t) W\big(y_1(t), y_2(t)\big)} f(t)\, dt + \int_a^x \frac{y_1(t) y_2(x) - y_1(x) y_2(t)}{p(t) W\big(y_1(t), y_2(t)\big)} f(t)\, dt \\ &= \int_x^b \frac{y_2(t) y_1(x)}{p(t) W\big(y_1(t), y_2(t)\big)} f(t)\, dt + \int_a^x \frac{y_1(t) y_2(x)}{p(t) W\big(y_1(t), y_2(t)\big)} f(t)\, dt. \end{aligned}$$

Let

$$g(x,t) := \begin{cases} \dfrac{y_2(t) y_1(x)}{p(t) W\big(y_1(t), y_2(t)\big)}, & x \leq t, \\[2ex] \dfrac{y_1(t) y_2(x)}{p(t) W\big(y_1(t), y_2(t)\big)}, & t \leq x, \end{cases} \tag{3.15}$$

This is called the *Green's Function* for L (see §3.2 below). Note that it depends only on L and B and not on f.

THEOREM 3.32. *The solution of* (**??**) *in the case* $\Delta \neq 0$ *is*

$$y = \int_a^b g(x,t) f(t)\, dt, \tag{3.16}$$

where g is as above with y_1 and y_2 being independent solutions of $Ly = 0$ such that y_1 satisfies $B_a(y_1) = 0$ and y_2 satisfies $B_b(y_2) = 0$.

EXAMPLE 3.33. Solve

$$\begin{cases} y'' = f(x), \\ y(0) = 0, \\ y(1) = 0. \end{cases}$$

◇

SOLUTION. We have $a = 0$, $b = 1$, and

$$\begin{cases} Ly = y'', \\ B_a(y) = y(0), \\ B_b(y) = y(1). \end{cases}$$

The solutions of $Ly = 0$ are in the form $y = mx + b$. Using $y_1 = 1$ and $y_2 = x$, we get

$$\triangle = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix},$$

so we are indeed in the case $\triangle \neq 0$. However, we want

- y_1 such that $Ly_1 = 0$ with $B_0(y_1) = 0$,
- y_2 such that $Ly_2 = 0$ with $B_1(y_1) = 0$,

so although any y_1 and y_2 is acceptable for computing $\triangle$ (which is independent of the choice), $y_1 = 1$ and $y_2 = x$ is not the choice required to proceed with the method. Instead, pick

$$y_1 = x, \qquad y_2 = x - 1.$$

Note that $p(t) \equiv 1$ and

$$W(y_1, y_2) = \begin{vmatrix} x & x-1 \\ 1 & 1 \end{vmatrix} = x - (x-1) = x - x + 1 = 1.$$

Therefore,

$$g(x,t) = \begin{cases} (t-1)\,x, & x \leq t, \\ t\,(x-1)\,, & t \leq x \end{cases}$$

and so

$$\begin{aligned} y(x) &= \int_0^1 g(x,t) f(t)\,dt \\ &= \int_0^x g(x,t) f(t)\,dt + \int_x^1 g(x,t) f(t)\,dt \\ &= \int_0^x t\,(x-1)\,f(t)\,dt + \int_x^1 x\,(t-1)\,f(t)\,dt. \end{aligned}$$

For example, if $f(x) := x^2$, we get

$$y(x) = \int_0^x (x-1)\,t^3\,dt + \int_x^1 x\left(t^3 - t^2\right)\,dt$$

$$
\begin{aligned}
&= (x-1)\left.\frac{t^4}{4}\right|_0^x + x\left.\left(\frac{t^4}{4}-\frac{t^3}{3}\right)\right|_x^1 \\
&= (x-1)\frac{x^4}{4} + x\left(\frac{1}{4}-\frac{1}{3}\right) - x\left(\frac{x^4}{4}-\frac{x^3}{3}\right) \\
&= \cancel{\frac{x^5}{4}} - \frac{x^4}{4} - \frac{x}{12} - \cancel{\frac{x^5}{4}} + \frac{x^4}{3} \\
&= \frac{x^4}{12} - \frac{x}{12},
\end{aligned}
$$

which, in this simple case, could have easily been found by other methods. To check, note that $y'' = x^2$ and $y(0) = 0 - 0 = 0$ and $y(1) = 1/12 - 1/12 = 0$. ◇

3.1.2. *The Case When* $\triangle = 0$.

In this case, there exists a $y_1(x)$ satisfying $Ly = 0$ and both boundary conditions. Having chosen $y_p(x)$ as in Equation (3.11), we have $y_p = 0$, so System (3.13) becomes

$$c_2 B_a(y_2) = 0, \tag{3.17a}$$

$$c_2 B_b(y_2) + B_b(y_p) = 0, \tag{3.17b}$$

and we have

$$
\underbrace{\begin{bmatrix} y_1(a) & y_1'(a) \\ y_2(a) & y_2'(a) \end{bmatrix}}_{\mathbf{M}}
\begin{bmatrix} \alpha \\ \beta \end{bmatrix}
=
\begin{bmatrix} \alpha y_1(a) + \beta y_1'(a) \\ \alpha y_2(a) + \beta y_2'(a) \end{bmatrix}
\begin{bmatrix} B_a(y_1) \\ B_a(y_2) \end{bmatrix}
=
\begin{bmatrix} 0 \\ B_a(y_2) \end{bmatrix}.
$$

Since $\det(\mathbf{M}) = W(y_1, y_2) \neq 0$ (since y_1 and y_2 are linearly independent), we have

$$M \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore, $B_a(y_2) \neq 0$, so Equation (3.17a) implies that $c_2 = 0$. Similarly, $B_b(y_2) \neq 0$. Equation (3.17b) becomes $B_b(y_p) = 0$, i.e., a solution exists if and only if $B_p(y_p) = 0$.

If $B_b(y_p) = 0$, then a solution is found as follows. We need

$$
\begin{aligned}
0 &= B_b(y_p) \\
&= \underbrace{\int_a^b \frac{y_1(t)B_b(y_2) - \cancel{B_b(y_1)y_2(t)}}{p(t)W\big(y_1(t), y_2(t)\big)} f(t)\,dt}_{\text{Eq. (3.12)}}
\end{aligned}
$$

by (3.12)

$$= B_b(y_2)\int_a^b \frac{y_1(t)f(t)}{p(t)W\big(y_1(t), y_2(t)\big)}\,dt.$$

Since $B_b(y_2) \neq 0$, we must have

$$\int_a^b \frac{y_1(t)f(t)}{p(t)W\big(y_1(t), y_2(t)\big)}\,dt = 0.$$

The general solution is

$$y = c_1 y_1 + \cancel{c_2 y_2} + y_p$$

$$
\begin{aligned}
&= c_1 y_1(x) + \int_a^x \frac{y_1'(t)y_2(x) - y_1(x)y_2(t)}{p(t)W\big(y_1(t), y_2(t)\big)} f(t)\,dt \\
&= cy_1(x) + \underbrace{\left(\int_a^b \frac{y_2(t)f(t)}{p(t)W\big(y_1(t), y_2(t)\big)}\,dt\right)}_{\tilde{c}} y_1(x) + \int_a^x \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{p(t)W\big(y_1(t), y_2(t)\big)} f(t)\,dt, \\
&= cy_1(x) + \int_x^b \frac{y_1(x)y_2(t)}{p(t)W\big(y_1(t), y_2(t)\big)}\,dt + \int_a^x \frac{y_1(t)y_1(x)}{p(t)W\big(y_1(t), y_2(t)\big)} f(t)\,dt \\
&= cy_1(x) + \int_a^b g(x,t)f(t)\,dt,
\end{aligned}
$$

where $c = c_1 - \tilde{c}$, and

$$
g(x,t) = \begin{cases} \dfrac{y_2(t)y_1(x)}{p(t)W\big(y_1(t), y_2(t)\big)}, & x \le t, \\[2ex] \dfrac{y_1(t)y_2(x)}{p(t)W\big(y_1(t), y_2(t)\big)}, & t \le x, \end{cases}
$$

as is defined in Equation (3.15).

THEOREM 3.34. *The solution of (**??**) in the case* $\Delta = 0$ *is*

$$
y = cy_1(x) + \int_a^b g(x,t)f(t)\,dt, \tag{3.18}
$$

where g *is as defined in Equation* (3.15), y_1 *and* y_2 *are linearly independent solutions of* $Ly = 0$, *and* y_1 *satisfies both* $B_a(y_1) = 0$ *and* $B_b(y_1) = 0$.

However, to actually be a solution, the condition

$$
\int_a^b \frac{y_1(t)f(t)}{p(t)W\big(y_1(t), y_2(t)\big)}\,dt = 0
$$

must be satisfied. Otherwise, there is no solution. Note that in the case that L *is self-adjoint,* pW *is a constant, so the condition simplifies to*

$$
\int_a^b y_1(t)f(t)\,dt = 0
$$

in this case.

EXAMPLE 3.35. Solve

$$
\begin{cases} y'' = f(x), \\ y(0) = 0, \\ y(1) - y'(1) = 0. \end{cases}
$$

◇

SOLUTION. The solutions of $y'' = 0$ are $y = mx + b$. Using $y_1 = 1$ and $y_2 = x$ gives

$$
\triangle = \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0,
$$

so we see that we are in the case $\triangle = 0$. Now choose a new y_1 satisfying both $B_a(y_1) = 0$ and $B_b(y_1) = 0$, namely, $y_1 = x$. Choose any linearly independent solution for y_2, so let $y_2 = 1$. Then $p \equiv 1$ and

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

We need

$$0 = \int_a^b \frac{y_1(t)f(t)}{p(t)W(t)}\,dt = -\int_0^1 y_1(t)f(t)\,dt = -\int_0^1 tf(t)\,dt.$$

If this is satisfied, then

$$g(x,t) = \begin{cases} -1x, & x \le t, \\ -1t, & x \ge t, \end{cases} = \begin{cases} -x, & x \le t, \\ -t, & x \ge t. \end{cases}$$

Therefore,

$$y = cx + \int_0^1 g(x,t)f(t)\,dt = cx - \int_0^x tf(t)\,dt - \int_x^1 xf(t)\,dt.$$

For example, if $f(x) := x^2 - 1/2$, we see that

$$\int_0^1 y_1(t)f(t)\,dt = \int_0^1 t\left(t^2 - \frac{1}{2}\right)dt = \left.\left(\frac{t^4}{4} - \frac{t^2}{4}\right)\right|_0^1 = 0,$$

so the condition is satisfied. Therefore,

$$\begin{aligned} y &= cx - \int_0^x t\left(t^2 - \frac{1}{2}\right)dt - \int_x^1 x\left(t^2 - \frac{1}{2}\right)dt \\ &= cx - \left.\left(\frac{t^4}{4} - \frac{t^4}{4}\right)\right|_0^x - x\left.\left(\frac{t^3}{3} - \frac{t}{2}\right)\right|_x^1 \\ &= cx - \frac{x^4}{4} + \frac{x^2}{4} - x\left(\frac{1}{3} - \frac{1}{2}\right) + x\left(\frac{x^3}{3} - \frac{x}{2}\right) \\ &= cx - \frac{x^4}{4} + \frac{x^2}{4} + \frac{x}{6} + \frac{x^4}{3} - \frac{x^2}{2} \\ &= \left(c + \frac{1}{6}\right)x - \frac{x^2}{4} + \frac{x^4}{12} \\ &= \tilde{c}x - \frac{x^2}{4} + \frac{x^4}{12}. \end{aligned}$$

To check, note that

$$y' = \tilde{c} - \frac{x}{2} + \frac{x^3}{3}, \qquad y'' = -\frac{1}{2} + x^2.$$

Then $y(0) = 0$ and

$$y(1) - y'(1) = \left(\tilde{c} - \frac{1}{4} + \frac{1}{12}\right) - \left(\tilde{c} - \frac{1}{2} + \frac{1}{3}\right) = \tilde{c} - \frac{1}{6} - \tilde{c} + \frac{1}{6} = 0,$$

showing that we indeed have the correct y. ◇

3.2. Green's Functions. Given a second order differential operator L, we can select two linearly independent solutions $y_1(x)$ and $y_2(x)$ of $Ly = 0$ and use them to define a Green's function

$$g(x,t) := \begin{cases} \dfrac{y_2(t)y_1(x)}{p(t)W\big(y_1(t),y_2(t)\big)}, & x \le t, \\[2ex] \dfrac{y_1(t)y_2(x)}{p(t)W\big(y_1(t),y_2(t)\big)}, & t \le x, \end{cases}$$

as previously defined in Equation (3.15) on page 80. The function $g(x,t)$ depends on the choice of y_1 and y_2, not just on L. Note that the formulas agree when $x = t$, and so $g(x,t)$ is a well-defined continuous function.

Having chosen g (by choosing y_1 and y_2), for $f : [a,b] \to \mathbb{R}$, define

$$\begin{aligned} Gf(x) &:= \int_a^b g(x,t)f(t)\,dt \\ &= \int_a^x \frac{y_1(t)y_2(x)}{p(t)W\big(y_1(t),y_2(t)\big)} f(t)\,dt + \int_x^b \frac{y_1(x)y_2(t)}{p(t)W\big(y_1(t),y_2(t)\big)} f(t)\,dt. \end{aligned}$$

Then it follows that

$$\begin{aligned} (Gf)'(x) = {} & \cancel{\frac{y_1(x)y_2(x)}{p(x)W\big(y_1(x),y_2(x)\big)}} f(x) - \cancel{\frac{y_1(x)y_2(x)}{p(x)W\big(y_1(x),y_2(x)\big)}} f(x) \\ & + \int_a^x \frac{y_1(t)y_2'(x)}{p(t)W\big(y_1(t),y_2(t)\big)} f(t)\,dt + \int_x^b \frac{y_1'(x)y_2(t)}{p(t)W\big(y_1(t),y_2(t)\big)} f(t)\,dt, \end{aligned}$$

and differentiating a second time gives

$$\begin{aligned} (Gf)''(x) &= \frac{y_1(x)y_2'(x)}{p(x)W\big(y_1(x),y_2(x)\big)} f(x) - \frac{y_1'(x)y_2(x)}{p(x)W\big(y_1(x),y_2(x)\big)} f(x) \\ &\quad + \int_a^x \frac{y_1(t)y_2''(x)}{p(t)W\big(y_1(t),y_2(t)\big)} f(t)\,dt + \int_x^b \frac{y_1''(x)y_2(t)}{p(t)W\big(y_1(t),y_2(t)\big)} f(t)\,dt \\ &= \frac{f(x)}{p(x)} + \int_a^x \frac{y_1(t)y_2''(x)}{p(t)W\big(y_1(t),y_2(t)\big)} f(t)\,dt + \int_x^b \frac{y_1''(t)y_2(t)}{p(t)W\big(y_1(t),y_2(t)\big)} f(t)\,dt. \end{aligned}$$

Therefore,

$$\begin{aligned} L(Gf) &= p(x)\frac{f(x)}{p(x)} + \int_a^x \frac{y_1(t)L(y_2)}{p(t)W\big(y_1(t),y_2(t)\big)}\,dt + \int_x^b \frac{L(y_1)y_2(t)}{p(t)W\big(y_1(t),y_2(t)\big)} f(t)\,dt \\ &= f(x) + 0 + 0 = f(x) \end{aligned}$$

regardless of the choice of y_1 and y_2. What about the boundary conditions?

We have

$$Gf(a) = 0 + \int_a^b \frac{y_1(a)y_2(t)}{p(t)W\big(y_1(t),y_2(t)\big)} f(t)\,dt = y_1(a)\int_a^b \frac{y_2(t)f(t)}{p(t)W\big(y_1(t),y_2(t)\big)}\,dt$$

and

$$Gf(b) = y_2(b)\int_a^b \frac{y_1(t)f(t)}{p(t)W\big(y_1(t),y_2(t)\big)}\,dt,$$

so it follows that

$$(Gf)'(a) = y_1'(a)\int_a^b \frac{y_2(t)f(t)}{p(t)W\big(y_1(t),y_2(t)\big)}\,dt,$$

$$(Gf)'(b) = y_2'(b) \int_a^b \frac{y_1(t)f(t)}{p(t)W\big(y_1(t), y_2(t)\big)}\, dt.$$

So given boundary conditions B_a and B_b, we have

$$B_a(Gf) = B_a(y_1) \int_a^b \frac{y_2(t)f(t)}{p(t)W\big(y_1(t), y_2(t)\big)}\, dt,$$
$$B_b(Gf) = B_b(y_2) \int_a^b \frac{y_1(t)f(t)}{p(t)W\big(y_1(t), y_2(t)\big)}\, dt.$$

Therefore, $B_a(Gf)$ and $B_b(Gf)$ **do** depend on the choice of y_1 and y_2.

If $\triangle \neq 0$ (thereby giving us a unique solution to the system 3.31 by Theorem 3.31, p. 79), by choosing a y_1 such that $B_a(y_1) = 0$ and a y_2 such that $B_b(y_2) = 0$, we obtain Gf satisfying both boundary conditions. If $\triangle = 0$, then we can choose y_1 such that $B_a(y_1) = 0$ and then we can solve the system 3.31 only if

$$\int_a^b \frac{y_1(t)f(t)}{p(t)W\big(y_1(t), y_2(t)\big)}\, dt = 0.$$

4. Partial Differential Equations

Most partial differential equations (PDEs) have so many solutions that computing the general solutions is impossible. Usually, we are satisfied to find a family of solutions, perhaps with a view to find one one satisfying particular conditions.

Consider the wave equation $y_{xx} = ky_{tt}$. Earlier, in Example 3.1 (p. 58), we considered how to find solutions of the form $y = u(x)v(t)$. We found a solution of this form satisfying our conditions. But the equation also has many other solutions, e.g., $y = x + t$ is a solution which does not have the form $u(x)v(t)$. Finding solutions of the form $u(x)v(t)$ is relatively tractable. We have

$$y = u(x)v(t) \Longrightarrow \begin{cases} y_{xx} = u_{xx}v, \\ y_{tt} = uv_{tt}. \end{cases}$$

Therefore, $u_{xx}v = kuv_{tt}$, so

$$\frac{u_{xx}}{u} = k\frac{v_{tt}}{v}.$$

Since u is a function only of x, and v is a function only of t, for equality to hold, both sides must be constant. Therefore,

$$u_{xx} = Au, \tag{$*$}$$
$$v_{tt} = \frac{A}{k}v. \tag{$**$}$$

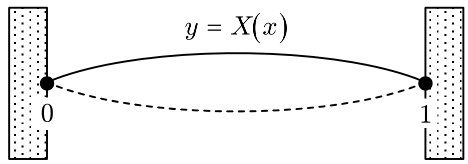
For given boundary conditions, there will be only certain values of A for which Equation ($*$) can be solved for u: those where A is an eigenvalue. With A known, one can then determine k so that A/k is an eigenvalue of Equation ($**$). For example,

$$u(0) = 0 = u(L) \Longrightarrow u = c_n \sin\left(\frac{n\pi}{L}x\right)$$

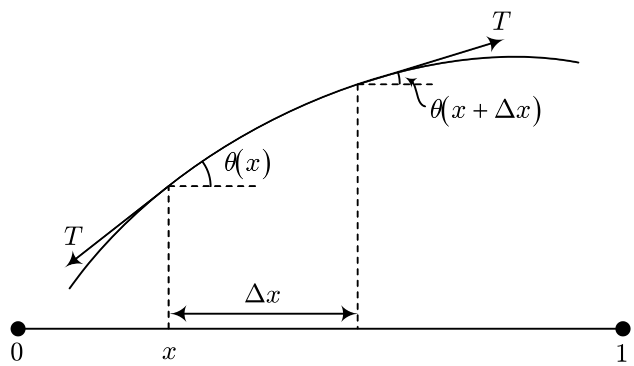
for $n = 1, 2, \ldots$, i.e., $A = n^2\pi^2/L^2$. Therefore,

$$v = b_1 \cos\left(\frac{n\pi}{L\sqrt{k}}x\right) + b_2 \sin\left(\frac{n\pi}{L\sqrt{k}}x\right).$$

4.1. Vibrating String. A string lies on the x-axis fastened at 0 and 1. It is displaced slightly in the y-direction so that, at time $t = 0$, it has the equation $y = X(x)$ and it is then released. This is shown in Figure 3.7a. We wish to find the equation of motion.



(a) A vibrating string with equation $y = X(x)$.



(b) A vibrating string with tension T.

Figure 3.7:

Let $y(x, t)$ be the y-coordinate at time t of the point whose x-coordinate is x. See Figure 3.7b. The main force acting is the tension T in the string. For simplicity, we will ignore the other forces. At time t, $\theta(x)$ will be the angle between the tangent to the string at x and the horizontal. The vertical component of T on the segment between x and $x + \Delta x$ is

$$-T\sin\big(\theta(x)\big) + T\sin\big(\theta(x + \Delta x)\big)$$

on the segment between x and $x + \Delta x$. The mass of the segment between x and $x + \Delta x$ is $\rho\Delta x$, where ρ is the density of the string, i.e., mass divided by length. Therefore,

$$T\Big(\sin\big(\theta(x + \Delta x)\big) - \sin\big(\theta(x)\big)\Big) \approx \rho\Delta x\frac{\partial^2 y}{\partial t^2},$$
$$\frac{\partial^2 y}{\partial t^2} \approx \frac{T}{\rho}\frac{\sin\big(\theta(x + \Delta x)\big) - \sin\big(\theta(x)\big)}{\Delta x}.$$

We will assume that the displacement is small so that dy/dx is small. Then

$$\sin(\theta) = \frac{dy}{ds} = \frac{dy}{dx}\frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \approx \frac{dy}{dx},$$

and it follows that

$$\frac{\partial^2 y}{\partial t^2} \approx \frac{T}{\rho}\frac{\frac{\partial y}{\partial x}(x + \Delta x) - \frac{\partial y}{\partial x}(x)}{\Delta x}.$$

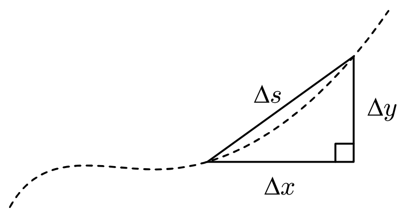


Figure 3.8: A small segment of the string.

So taking the limit gives

$$\frac{\partial^2 y}{\partial t^2} = \lim_{\Delta x \to 0} \frac{T}{\rho} \frac{\frac{\partial y}{\partial x}(x+\Delta x) - \frac{\partial y}{\partial x}(x)}{\Delta x} = a\frac{\partial^2 y}{\partial x^2},$$

where $a = T/P$.

Since the endpoints are fixed, we have $y(0,t) = 0 = y(1,t)$ for all t. Also since the string is released from rest, we have

$$y(x,0) = X(x)$$

and

$$\frac{\partial}{\partial y} y(x,0) = 0,$$

We look for the solutions of the form $y = X(x)T(t)$, i.e., the general shape of the initial displacement is maintained, but the amplitude varies with time. Then

$$y_{tt} = a y_{xx},$$

$$\underbrace{X(x)\frac{d^2T}{dt^2}}_{=y_{tt}} = \underbrace{aT(t)\frac{d^2X}{dx^2}}_{=ay_{xx}},$$

i.e.,

$$\frac{X''}{X} = \frac{1}{a} = \frac{T''}{T}.$$

Since the left function depends only on x and the right function only on t, to be equal, they must both be identically equal to a constant. Therefore,

$$\frac{X''}{X} = k, \qquad \frac{1}{a}\frac{T''}{T} = k.$$

This gives us two boundary value problems

$$\begin{cases} X'' - kX = 0, \\ X(0) = 0, \\ X(1) = 0, \end{cases} \qquad \begin{cases} T'' - akT = 0, \\ T'(0) = 0. \end{cases}$$

To proceed, we need to know the solution to the following BVP,

EXAMPLE 3.36. Solve

$$\begin{cases} y'' - ky = 0, \\ y(0) = 0, \\ y(1) = 0, \end{cases}$$

where k is a constant. ◇

SOLUTION. Consider its three cases.

Case 1: $k > 0$

Then

$$y_1 = e^{\sqrt{k}x}, \qquad y_2 = e^{-\sqrt{k}x},$$

and we have

$$\begin{aligned} B_0(y_1) &= y_1(0) = 1, \\ B_0(y_2) &= y_2(0) = 1, \\ B_1(y_1) &= y_1(1) = e^{\sqrt{k}}, \\ B_1(y_2) &= y_2(1) = e^{-\sqrt{k}}. \end{aligned}$$

Therefore,

$$\triangle = \begin{vmatrix} 1 & 1 \\ e^{\sqrt{k}} & e^{-\sqrt{k}} \end{vmatrix} = e^{-\sqrt{k}} - e^{\sqrt{k}}.$$

If $e^{-\sqrt{k}} = e^{\sqrt{k}}$, then $e^{2\sqrt{k}} = 1 \Rightarrow k = 0$, contradicting our assumption that $k > 0$. Therefore, $\triangle \neq 0$ and there is no solution for $k > 0$.

Case 2: $k = 0$

Then

$$y_1 \equiv 1, \qquad y_2 = x,$$

and we have

$$\begin{aligned} B_0(y_1) &= y_1(0) = 1, \\ B_0(y_2) &= y_2(0) = 0, \\ B_1(y_1) &= y_1(1) = 1, \\ B_1(y_2) &= y_2(1) = 1. \end{aligned}$$

Hence,

$$\triangle = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \neq 0.$$

Therefore, there is no solution, and therefore $k \neq 0$.

Case 3: $k < 0$

Let $\lambda = -k$. Then

$$y_1 = \cos\left(\sqrt{\lambda}x\right), \qquad y_2 = \sin\left(\sqrt{\lambda}x\right),$$

and we have

$$\begin{aligned} B_0(y_1) &= y_1(0) = 1, \\ B_0(y_2) &= y_2(0) = 0, \\ B_1(y_1) &= y_1(1) = \cos\left(\sqrt{\lambda}\right), \\ B_1(y_2) &= y_2(1) = \sin\left(\sqrt{\lambda}\right) \end{aligned}$$

and

$$\triangle = \begin{vmatrix} 1 & 0 \\ \cos\left(\sqrt{\lambda}\right) & \sin\left(\sqrt{\lambda}\right) \end{vmatrix} = \sin\left(\sqrt{\lambda}\right).$$

Note that

$$\sin\left(\sqrt{\lambda}\right) = 0 \Longleftrightarrow \sqrt{\lambda} = n\pi \Longleftrightarrow \lambda = n^2\pi^2.$$

Therefore, there is no solution unless $\lambda = n^2\pi^2$ for some integer $n > 0$. If $\lambda = n^2\pi^2$, then

$$\begin{bmatrix} 1 & 0 \\ (-1)^n & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow c_1 = 0.$$

Therefore

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

To summarize: there is no nonzero solution unless $k = -n^2\pi^2$ for some integer $n > 0$. In this case, the solution is

$$y = c\sin(n\pi x).$$

◇

Returning to the string, in order to be able to solve for X, we must have $k = -n^2\pi^2$ for some $n = 1, 2, 3, \ldots$. Therefore,

$$T'' + an^2\pi^2 T = 0, \qquad T'(0)$$

gives us

$$T = c_n \cos\left(\sqrt{a}n\pi t\right).$$

So we have the solutions $y_1, y_2, \ldots,$ where

$$y_n(x,t) = \sin(n\pi x)\cos\left(\sqrt{a}n\pi t\right)$$

satisfying all conditions except $y(x,0) = f(x)$. Observe that the sum of solutions is also a solution, so we want to choose the right c_n's so that

$$y(x,t) = c_1 y_1 + c_2 y_2 + \cdots$$

satisfies $y(x,0) = f(x)$ as well. Since $y_n(x,0) = \sin(n\pi x)$, we want

$$f(x) = c_1 \sin(\pi x) + c_2 \sin(2\pi x) + \cdots.$$

To find c_n, extend f to an odd function (Figure 3.9) and take the Fourier series of that function.

4.2. The Laplace Equation. The differential operator

$$\nabla^2 := \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

is called the *Laplace operator* or *Laplacian*. In two variables, the Laplace equation is given by

$$y_{xx} + y_{yy} = 0.$$

Suppose that $y = uv$. As in previous examples, we have

$$-\frac{u_{xx}}{u} = \frac{v_{tt}}{v} = A,$$

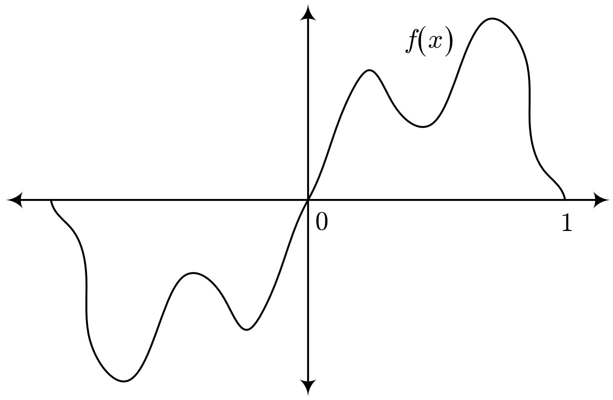


Figure 3.9: Extension of $f(x)$ to an odd function.

from which we obtain

$$u_{xx} + Au = 0, \qquad v_{tt} - Av = 0.$$

Note that $u(0) = 0 = u(\pi) \Rightarrow A = n^2$ for some $n = 1, 2, \dots$. Therefore, $u = \sin(nx)$.

Solving $v_{tt} - n^2 v = 0$ gives us $v = c_1 e^{nx} + c_2 e^{-nx}$. But there are lots of other interesting solutions.

4.3. The Heat Equation. Recall that the heat equation in one dimension is given by

$$y_{xx} = \frac{1}{k} y_t.$$

Suppose that $y = u(x)v(t)$ (although there are other solutions as well). Then

$$y_{xx} = u_{xx} v, \qquad y_t = u v_t.$$

Therefore,

$$u_{xx} v = \frac{1}{k} u v_t$$

and it follows that

$$\frac{u_{xx}}{u} = \frac{1}{k} \frac{v_t}{v} = A.$$

The equation together with the boundary conditions determines the possible values for A. Note that $u_{xx} = Au$ implies that there are possible values for A. Then $v_t = Akv$ determines v. For example,

$$u(0) = 0 = u(L) \Longrightarrow u = c_n \sin\Big(\frac{n\pi}{L} x\Big)$$

for $n = 1, 2, \dots$, i.e., $A = n^2\pi^2/L^2$. Therefore

$$v = b e^{-\frac{n^2\pi^2}{L^2} kt}.$$

Generalizing to three dimensions we have the following.

Example 3.37 (Heat equation). The heat equation is given by $u_t = k u_{xx}$, where u is a function of x and t. In three dimensions, the function is $u(x, y, z, t)$, and satisfies

$$u_t = k\left(u_{xx} + u_{yy} + u_{zz}\right) = k\nabla^2 u,$$

where ∇^2 is the Laplacian as defined in Section 4.2

Switching to spherical coordinates, we have

$$x = r\cos(\theta)\sin(\phi),$$
$$y = r\sin(\theta)\sin(\phi),$$
$$z = r\cos(\theta).$$

Therefore,

$$u_t = k\left(u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2\sin^2(\phi)}u_{\theta\theta} + \frac{1}{r^2}u_{\phi\phi} + \frac{\cos(\phi)}{r^2\sin(\phi)}u_\phi\right).$$

We look for solutions of the form

$$u(r,\theta,\phi,t) = T(t)\underbrace{R(r)P(\theta)\Phi(\phi)}_{Z(r,\theta,\phi)}.$$

Then $T'Z = kT\nabla^2 Z$. Therefore,

$$\frac{1}{k}\frac{T'}{T} = A = \frac{\nabla^2 Z}{Z}.$$

Hence, A is a constant. Substituting

$$Z = R\underbrace{P\Phi}_{Y}.$$

into $\nabla^2 Z = AZ$ gives

$$\begin{aligned}
R''Y + \frac{2}{r}R'Y + \frac{R}{r^2\sin^2(\phi)}Y_{\theta\theta} + \frac{R}{r^2}Y_{\theta\theta} + \frac{R\cos(\phi)}{r^2\sin(\phi)}Y_\phi &= ARY,\\
\left(r^2R'' + 2rR' - Ar^2R\right)Y &= -R\left(\frac{1}{\sin^2(\phi)}Y_{\theta\theta} + Y_{\phi\phi} + \frac{\cos(\phi)}{\sin(\phi)}Y_\phi\right),\\
\frac{r^2R'' + 2rR' - Ar^2R}{R} = \lambda &= -\frac{1}{Y}\left(\frac{1}{\sin^2(\phi)}Y_{\theta\theta} + Y_{\phi\phi} + \frac{\cos(\phi)}{\sin(\phi)}Y_\phi\right),\\
\frac{1}{\sin^2(\phi)}Y_{\theta\theta} + Y_{\phi\phi} + \frac{\cos(\phi)}{\sin(\phi)}Y_\phi &= -\lambda Y,\\
\frac{1}{\sin^2(\phi)}P''\Phi + \Phi''P + \frac{\cos(\phi)}{\sin(\phi)}\Phi'P &= -\lambda P\Phi,\\
\frac{1}{\sin^2(\phi)}P''\Phi &= -P\left(\Phi'' + \frac{\cos(\phi)}{\sin(\phi)}\Phi' + \lambda\Phi\right),\\
\frac{P''}{P} = B &= -\frac{\sin^2(\phi)}{\Phi}\left(\Phi'' + \frac{\cos(\phi)}{\sin(\phi)}\Phi' + \lambda\Phi\right).
\end{aligned}$$

Hence, B is a constant.

Consider the special case $B = 0$. Then

$$\Phi'' + \frac{\cos(\phi)}{\sin(\phi)}\Phi' + \lambda\Phi = 0,$$

i.e.,

$$\frac{d^2\Phi}{d\phi^2} + \frac{\cos(\phi)}{\sin(\phi)}\frac{d\Phi}{d\phi} + \lambda\Phi = 0.$$

Let $w = \cos(\phi)$. Then $0 \le \phi \le \pi \Rightarrow -1 \le w \le 1$. Then

$$\frac{d\Phi}{d\phi} = \frac{d\Phi}{dw}\frac{dw}{d\phi} = \frac{d\Phi}{dw}\big(-\sin(\phi)\big).$$

Therefore

$$\begin{aligned}\frac{d^2\Phi}{d\phi^2} &= \frac{d^2\Phi}{dw^2}\frac{dw}{d\phi}\big(-\sin(\phi)\big) + \frac{d\Phi}{dw}\big(-\cos(\phi)\big)\\ &= \frac{d^2\Phi}{dw^2}\sin^2(\phi) - \frac{d\Phi}{dw}\cos(\phi)\end{aligned}$$

and it follows that

$$\begin{aligned}\frac{d^2\Phi}{dw^2}\sin^2(\phi) - \frac{d\Phi}{dw}\cos(\phi) + \frac{\cos(\phi)}{\sin(\phi)}\frac{d\Phi}{dw}\big(-\sin(\phi)\big) + \lambda\Phi = 0,\\ \frac{d^2\Phi}{dw^2}\big(1-\cos^2(\phi)\big) - 2\cos(\phi)\frac{d\Phi}{dw} + \lambda\Phi = 0,\end{aligned}$$

finally resulting in

$$\big(1-w^2\big)\frac{d^2\Phi}{dw^2} - 2w\frac{d\Phi}{dw} + \lambda\Phi = 0,$$

which is a Legendre equation. ◇

4.4. The Schrödinger Equation. The Schrödinger equation is given by

$$i\Psi_t = -\frac{\hbar}{2m}\Psi_{xx},$$

where $\hbar$ is the *Reduced Planck's constant* defined by $\hbar = h/2\pi$ with h being Planck's constant, and m is mass. The function $\Phi(x,y)$ is complex-valued.

For a given time t,

$$\frac{\displaystyle\int_a^b \big|\Psi(x,t)\big|^2 dx}{\displaystyle\int_{-\infty}^{\infty} \big|\Psi(x,t)\big|^2 dx}$$

is the probability of finding the particle in $[a,b]$ at that time.

In three dimensions, the Schrödinger equation is

$$i\Psi_t = -\frac{\hbar}{2m}\left(\Psi_{xx}+\Psi_{yy}+\Psi_{zz}\right) = -\frac{\hbar}{2m}\nabla^2\Psi,$$

where ∇^2 is the Laplacian as defined in Example 4.2. For $B \subset \mathbb{R}^3$,

$$\frac{\displaystyle\int_B \big|\Psi(x,y,z,t)\big|^2 d\mathbf{v}}{\displaystyle\int_{\mathbb{R}^3} \big|\Psi(x,y,z,t)\big|^2 d\mathbf{v}}$$

is the probability of finding the particle within B at time t.

Adding an external force $\frac{1}{\hbar}V(x,y,z,t)$, the Schrödinger equation becomes

$$i\Psi_t = -\frac{\hbar}{2m}\nabla^2\Psi + \frac{1}{\hbar}V(x,y,z,t)\Psi.$$

Switching to spherical coordinates, we have

$$\begin{aligned}x &= r\sin(\theta)\cos(\phi),\\ y &= r\sin(\theta)\sin(\phi),\\ z &= r\cos(\phi).\end{aligned}$$

We assume that $V(x,y,z,t)$ depends only on r, i.e., independent of t, θ, and ϕ. We then look for solutions of the form

$$\Psi(r,\theta,\phi,t) = \underbrace{R(r)\Theta(\theta)\Phi(\phi)}_{u(x,y,z)}T(t).$$

Then the equation becomes

$$iuT' = -\frac{\hbar}{2m}T\nabla^2 u + \frac{1}{\hbar}VTu.$$

Therefore,

$$\frac{i\hbar}{T}T' = E = \frac{1}{u}\left(-\frac{\hbar^2}{2m}\nabla^2 u + Vu\right).$$

Hence, E is constant. It then follows that

$$i\hbar T' = Et, \qquad -\frac{\hbar^2}{2m}\nabla^2 u + Vu = Eu,$$

and from these we have

$$T(t) = Ce^{-iEt/\hbar}, \qquad -\frac{\hbar^2}{2m}\nabla^2 u + Vu = Eu.$$

Therefore,

$$-\frac{\hbar}{2m}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}\right)u + Vu = Eu.$$

With

$$u = R(r)\underbrace{\Theta(\theta)\Phi(\phi)}_{Y(\theta,\phi)},$$

we have

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{2mr^2}{\hbar^2}(E-V) = \lambda = -\frac{1}{Y}\left(\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2(\theta)}\frac{\partial^2 Y}{\partial\phi^2}\right).$$

The second equation is then

$$\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2(\theta)}\frac{\partial^2 Y}{\partial\phi^2} + \lambda Y = 0.$$

Separating again, there exists a constant v such that

$$\frac{d^2\Phi}{d\phi^2} + v\Phi = 0,$$

$$\frac{1}{\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right) + \left(\lambda - \frac{v}{\sin^2(\theta)}\right)\Theta = 0. \tag{$*$}$$

For Equation $(*)$, we have

$$\frac{1}{\sin(\theta)}\left(\cos(\theta)\frac{d\Theta}{d\theta} + \sin(\theta)\frac{d^2\Theta}{d\theta^2}\right) + \left(\lambda - \frac{v}{\sin^2(\theta)}\right)\Theta = 0.$$

Let $w = \cos(\theta)$. Then

$$\Theta(\theta) \equiv \underbrace{\Theta(\cos^{-1}(w))}_{P(w)}$$

and $0 \le \theta \le 2\pi \Leftrightarrow -1 \le w \le 1$. Therefore,

$$\frac{\cos(\theta)}{\sin(\theta)}\frac{dP}{dw}\frac{dw}{d\theta} + \frac{d}{d\theta}\left(\frac{dP}{dw}\frac{dw}{d\theta}\right) + \left(\lambda - \frac{v}{1-w^2}\right)P = 0,$$

$$\begin{aligned}\frac{\cos(\theta)}{\sin(\theta)}\frac{dP}{dw}\big(-\sin(\theta)\big) - \frac{d}{d\theta}\left(\frac{dP}{dw}\sin(\theta)\right) + \left(\lambda - \frac{v}{1-w^2}\right)P = 0,\\ -w\frac{dP}{dw} - \left(\frac{d^2P}{dw^2}\frac{dw}{d\theta}\sin(\theta) + \frac{dP}{dw}\cos(\theta)\right) + \left(\lambda - \frac{v}{1-w^2}\right)P = 0,\\ -w\frac{dP}{dw} + \frac{d^2P}{dw^2}\sin^2(\theta) - w\frac{dP}{dw} + \left(\lambda - \frac{v}{1-w^2}\right)P = 0.\end{aligned}$$

Therefore,

$$\big(1-w^2\big)\frac{d^2P}{dw^2} - 2w\frac{dP}{dw} + \left(\lambda - \frac{v}{1-w^2}\right)P = 0.$$

When $v = 0$, this becomes Legendre's equation.

5. Zeros of Solutions of Second Order Linear Differential Equations

PROPOSITION 3.38. *A nontrivial solution of $y'' + P(x)y' + Q(x)y = 0$ can have only finitely many zeros on a bounded interval.*

PROOF. The proof is given in `MATB44` using the Bolzano-Weierstrass Theorem. □

Let

$$I := e^{-\int \frac{P(x)}{2}\,dx}$$

so that

$$I' = -\frac{P(x)}{2}I.$$

Let $y = Iu$. Then

$$y' = I'u + Iu' = -\frac{P}{2}Iu + u'$$

and

$$\begin{aligned}y'' &= -\frac{P'}{2}Iu - \frac{P}{2}I'u - \frac{P}{2}Iu' + I'u' + Iu''\\ &= -\frac{P'}{2}Iu + \frac{P^2}{4}Iu - \frac{P}{2}Iu' - \frac{P}{2}Iu' + Iu''\\ &= -\frac{P'}{2}Iu + \frac{P^2}{4}Iu - PIu' + Iu''.\end{aligned}$$

Therefore,

$$y'' + P(x)y' + Q(x)y = 0,$$

becomes

$$\underbrace{-\frac{P'}{2}Iu + \frac{P^2}{4}Iu - \cancel{PIu'} + Iu''}_{y''}\underbrace{-\frac{P^2}{2}Iu + \cancel{PIu'}}_{P(x)y'} + \underbrace{QIu}_{Q(x)y} = 0,$$

so

$$u'' + \underbrace{\left(Q - \frac{P'}{2} + \frac{P^2}{4}\right)}_{\alpha(x)}u = 0,$$

since $I \neq 0$. Also, $y = Iu$ with $I \neq 0$, so $y = 0 \Leftrightarrow u = 0$. So the properties of the zeros of the solutions to the equation of the form $u'' + \alpha u = 0$ hold for the general cases as well. Using y instead of u, let us consider $y'' + \alpha y = 0$.

LEMMA 3.39 (Comparison lemma). *Suppose that $\alpha(x) \geq \beta(x)$ for all $x \in [a,b]$ but $\alpha(x) \not\equiv \beta(x)$. Suppose that*

$$y''(x) + \alpha(x)y = 0,$$
$$z''(x) + \beta(x)z = 0.$$

Then (strictly) between any two zeros of z, there exists a zero of y.

PROOF. By Proposition 3.38, z can have only finitely many zeros on $[a,b]$. Let s_0 and s_1 be consecutive zeros of z. Then z does not change sign in (s_0, s_1), as shown in Figure 3.10.

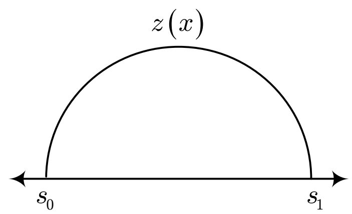


Figure 3.10: Two consecutive zeros of z between which z does not change sign.

Suppose that y has no zero in (s_0, s_1). Then y also does not change sign in (s_0, s_1). Replacing y by $-y$ and z by $-z$ does not change equations or the location of zeros. So assume that $z \geq 0$ and $y \geq 0$. Then

$$z \geq 0 \Longrightarrow \begin{cases} z'(s_0) \geq 0, \\ z'(s_1) \leq 0, \end{cases}$$

and we have

$$\begin{aligned} \frac{d}{dx}(yz' - y'z) &= y'z' + yz'' - y''z - y'z' \\ &= yz'' - zy'' \\ &= -\beta yz + \alpha yz \\ &= (\alpha - \beta)\, yz \\ &\geq 0. \end{aligned}$$

Therefore, $yz' - y'z$ is increasing. But

$$\alpha \neq \beta \Longrightarrow \underbrace{(yz' - zy')(s_0)}_{\underbrace{\underbrace{y(s_0)}_{\geq 0}\underbrace{z'(s_0)}_{\geq 0}}_{\geq 0}} < \underbrace{(yz' - y'z)(s_1)}_{\underbrace{\underbrace{y(s_1)}_{\geq 0}\underbrace{z'(s_1)}_{\leq 0}}_{\leq 0}}.$$

This is a contradiction. Therefore, y has a zero in (s_0, s_1). $\square$

COROLLARY 3.40. *Let α, β, y, and z be defined as in Lemma 3.39. Let $\#_D(f)$ be defined as the number of zeros of f on D. Then*

$$\#_{[a,b]}(z) \leq \#_{[a,b]}(y) + 1.$$

Corollary 3.41. *Let α, β, y, and z be defined as in Lemma 3.39. In addition, suppose that $y(a) = z(a)$ and $y'(a) = z'(a)$. Let $\#_D(f)$ be defined as the number of zeros of f on D and let $\mathcal{Z}^k_{[a,b]}(f)$ be defined as the kth zero of f on $[a,b]$. Then*

$$\#_{[a,b]}(y) \geq \#_{[a,b]}(z),$$
$$\mathcal{Z}^k_{[a,b]}(y) \leq \mathcal{Z}^k_{[a,b]}(z)$$

for $k = 1, 2, \ldots, \#_{[a,b]}(z)$.

Proof. If z is never zero on $(a, b]$, then there is nothing to prove. Therefore, let x_1 be the first zero of z. We now show that $y = 0$ some place on (a, x_1).

Replacing z by $-z$ if necessary, assume that $z \geq 0$ on (a, x_1). Then $z'(x_1) < 0$. If $y \neq 0$ on (a, x_1), assume that $y > 0$ on (a, x_1). Then

$$\begin{aligned}
\int_a^{x_1} (\alpha - \beta)\, yz\, dx &= \int_a^1 \frac{d}{dx}\,(yz' - y'z)\, dx \\
&= \left.(yz' - y'z)\right|_a^{x_1} \\
&= y(x_1)z'(x_1) - y'(x_1)z(x_1) - y(a)z'(a) + y'(a)z(a) \\
&= y(x_1)z'(x_1).
\end{aligned}$$

Note that $y(x_1) \geq 0$ but $z'(x_1) \leq 0$, so $y(x_1)z'(x_1) \leq 0$. But we integrated a positive function, so the integral is positive. This is a contradiction.

Therefore, $y = 0$ someplace to the left of x_1. If the second zero of z is at x_2, then y has another zero on (x_1, x_2), so the second zero of y is to the left of x_2, and so on. ☐

Suppose that $y'' + \alpha(x)y = 0$, where $\alpha(x) > 0$ for all $x \in [a, b]$. Let

$$L = \min\Big(\big\{\alpha(x) : x \in [a,b]\big\}\Big),$$
$$U = \max\Big(\big\{\alpha(x) : x \in [a,b]\big\}\Big).$$

Therefore, $L \leq \alpha(x) \leq U$. Suppose c and d are consecutive zeros of y on $[a, b]$. We will compute a lower and upper bound on $d - c$.

We compare first with $z'' + Lz = 0$. Note that y has a zero between any two zeros of a solution of z. One solution is $z = \sin\Big(\sqrt{L}\,(x - c)\Big)$. Since $z(c) = 0$, we also have $z\Big(c + \pi/\sqrt{L}\Big) = 0$. Therefore, $d \in \Big(c, \pi/\sqrt{L}\Big)$, i.e., $y(c) = 0$ and the next zero of y occurs before $c + \pi/\sqrt{L}$. Therefore, $d - c < \pi/\sqrt{L}$.

We now compare with $z'' + Uz = 0$. Any solution of z has a zero between any two solutions of y, i.e., any solution of z has a zero in (c, d). One solution is $z = \sin\Big(\sqrt{U}\,(x - c)\Big)$. Since $z(c) = 0$, we also have $z\Big(c + \pi/\sqrt{U}\Big) = 0$. Therefore, $c + \pi/\sqrt{U} \in (c, d)$, and so $d - c > \pi/\sqrt{U}$.

Theorem 3.42. *Suppose that $L \leq \alpha(x) \leq U$ on some interval $[a, b]$ (with $\alpha(x) \not\equiv L$ and $\alpha(x) \not\equiv U$) and let c and d be any two zeros of y on $[a, b]$. Then within $[a, b]$, we have*

$$\frac{\pi}{\sqrt{U}} < |d - c| < \frac{\pi}{\sqrt{L}}. \tag{3.19}$$

Corollary 3.43. *We have*

$$\frac{(b-a)\sqrt{L}}{\pi} - 1 < \#_{[a,b]}(y) < \frac{(b-a)\sqrt{U}}{\pi} + 1. \tag{3.20}$$

PROOF. Let m be the minimum distance between zeros. There are $\#_{[a,b]}(y) - 1$ subintervals formed between the zeros, so

$$\#_{[a,b]}(y) - 1 \le \frac{b-a}{m}.$$

(Even if we use the minimum distance, we cannot fit in more intervals than this.) But

$$\frac{b-a}{m} \le \frac{(b-a)\sqrt{U}}{\pi},$$

so it follows that

$$\#_{[a,b]}(y) < \frac{(b-a)\sqrt{U}}{\pi} + 1.$$

On the other hand, even spreading out the solutions as much as possible and leaving a gap of size (almost) $\pi/\sqrt{L}$ on each side, we need at least that

$$\big(\#_{[a,b]}(y) + 1\big)\frac{\pi}{\sqrt{L}} > b - a.$$

Therefore,

$$\#_{[a,b]}(y) > \frac{(b-a)\sqrt{L}}{\pi} - 1. \qquad \square$$

COROLLARY 3.44. *If $y(a) = 0$, then*

$$\frac{(b-a)\sqrt{L}}{\pi} - 1 < \#_{(a,b]}(y) < \frac{(b-a)\sqrt{U}}{\pi}. \tag{3.21}$$

PROOF. Since one zero of $[a,b]$ is used up at a, there is room for less than $(b-a)\sqrt{U}/\pi$ on $(a,b]$. The argument on the other bound is the same. $\square$

EXAMPLE 3.45. Give a lower and upper bound on the number of zeros of a solution of $y'' + \big(2\sin(x)\big)y = 0$ on $[0, 2\pi]$. $\diamond$

SOLUTION. Note that $1 \le 2 + \sin(x) \le 3$. Therefore, $L = 1$ and $U = 3$. By Equation (3.20), we have

$$\begin{aligned} \frac{2\pi\sqrt{1}}{\pi} - 1 <& \#_{[0,2\pi]}(y) < \frac{2\pi\sqrt{3}}{\pi} + 1, \\ 1 <& \#_{[0,2\pi]}(y) < \underbrace{2\sqrt{3}+1}_{\approx 4.4641}. \end{aligned}$$

So y has at least 2 but no more than 4 zeros on $[0, 2\pi]$. $\diamond$

EXAMPLE 3.46. Suppose $\alpha(x) < 0$ on $[a,b]$. Give a lower and upper bound on the number of zeros of a solution of $y'' + \alpha(x)y = 0$. $\diamond$

SOLUTION. We compare with the solution of $z'' = 0$. Then we know that $z = mx + b$. We use the solution $z \equiv 1$. Clearly, it has no zeros. Therefore,

$$\begin{aligned} \#_{[a,b]}(y) &\le \#_{[a,b]}(z) + 1, \\ \#_{[a,b]}(y) &\le 0 + 1, \\ \#_{[a,b]}(y) &\le 1. \end{aligned}$$

Therefore, y has at most one zero. $\diamond$

EXAMPLE 3.47. Suppose that $y'' + (xe^{-x} + 1)\,y = 0$ with $y(0) = 0$. Show that y has infinitely many zeros on $(0, \infty)$ and give lower and upper bounds on the position of the nth zero. $\diamond$

Solution. Let $f(x) := xe^{-x} + 1$. Then we know that $f(x) \geq 0$ for all $x \in [0, \infty)$. Since $f'(x) = e^{-x} - xe^{-x}$, the maximum occurs at $x = 1$. Therefore,

$$f(x) \leq \frac{1}{e} + 1$$

for all $x \in [0, \infty)$. Comparing with $z'' + z = 0$, we know that y has a zero between each zero of $\sin(x)$. Therefore, z has infinitely many zeros. Let z_1 and z_2 be two zeros of z. Then

$$\frac{\pi}{\sqrt{\frac{1}{e} + 1}} < |z_1 - z_2| < \pi.$$

Therefore, the nth zero occurs after $n\pi/\sqrt{1/e + 1}$ but before $n\pi$. ◇

Example 3.48. Find an upper and lower bound to the eigenvalues of

$$\begin{cases} y'' = -\lambda\,(x+1)\,y, \\ y(0) = 0, \\ y(1) = 0, \end{cases}$$

which is a Sturm-Liouville problem. ◇

Solution. Rearranging, we have $y'' + \lambda\,(x+1)\,y = 0$ with $a = 1$ and $b = 1$. Consider first $\lambda = 0$. Then the solution of $y'' = 0$ with $y(0) = 0$ is $y = cx$, which has no zeros on $(0, 1)$. Therefore, $\lambda = 0$ is before the first eigenvalue, and so all eigenvalues are positive.

Hence, suppose $\lambda > 0$. On $[0, 1]$, we have $\lambda \leq \lambda\,(x+1) \leq 2\lambda$. Therefore, $L = \lambda$ and $U = 2\lambda$. Let y_λ be a solution of $y'' = -\lambda\,(x+1)\,y$ with $y(0) = 0$. Then

$$\frac{(1-0)\sqrt{\lambda}}{\pi} - 1 < \#_{(0,1)}(y) < \frac{(1-0)\sqrt{2\lambda}}{\pi},$$
$$\frac{\sqrt{\lambda}}{\pi} - 1 < \#_{(0,1)}(y) < \frac{\sqrt{2\lambda}}{\pi}.$$

Consider $\lambda = \pi^2/2$. Then

$$\frac{1}{\sqrt{2}} - 1 < \#_{(0,1)}(y) < 1 \Longrightarrow \#_{(0,1)}(y_\lambda) = 0 \Longrightarrow \lambda_1 \geq \frac{\pi^2}{2},$$

where λ_1 is the first eigenvalue.

Now consider $\lambda = \pi^2$. Then

$$0 = 1 - 1 < \#_{(0,1)}(y_\lambda) < \sqrt{2} \Longrightarrow \#_{(0,1)}(y) = 1.$$

Therefore,

$$\frac{\pi^2}{2} \leq \lambda_1 < \pi^2 \leq \lambda_2,$$

where λ_2 is the second eigenvalue. ◇

6. Proof of the Properties of Sturm-Liouville Problems

We now consider the proof of Theorem 3.26 (p. 72).

PROOF OF THEOREM 3.26. We have

$$\begin{cases} Ly = -\lambda w(x) y, \\ B_a(y) = 0, \\ B_b(y) = 0. \end{cases}$$

Let $L = a_0 y'' + a_0' y' + a_2 y$ be self-adjoint and $w(x), a_0(x)$ never zero on $[a,b]$. For simplicity, consider the special case where the conditions $B_a(y) = 0 = B_b(y)$ are $y(a) = 0 = y(b)$. Recall that for $y'' + P(x)y' + Q(x)y = 0$, we can eliminate the y' term by substituting $y = uI$, where $I = e^{-\int \frac{P(x)}{2}\,dx}$. This yields

$$u'' + \left(Q - \frac{P'}{2} + \frac{P^2}{4}\right) u$$

and the equation becomes

$$u'' + \underbrace{\left(\frac{a_2}{a_0} - \left(\frac{a_0'}{2a_0}\right)' + \frac{1}{4}\left(\frac{a_0'}{a_0}\right)^2\right)}_{\equiv q(x)} u = -\lambda \underbrace{\frac{W}{a_0}}_{\equiv v(x)} I\ u,$$

where the zeros of u are precisely the zeros of y since I is never zero. Also, $y(a) = 0 \Rightarrow u(0) = 0$ and $y(b) = 0 \Rightarrow u(b) = 0$ and $v(x) \neq 0$ on $[a,b]$. Therefore, consider our new Sturm-Liouville problem

$$\begin{cases} y'' + q(x)u = -\lambda v(x) u, \\ u(a) = 0, \\ u(b) = 0. \end{cases}$$

It is still self-adjoint since $1' = 0$. We will consider $v(x) > 0$ (if $v(x) < 0$, then the properties we derive hold for $-\lambda$). Consider the solutions of

$$\begin{aligned} u'' + q(x) + \lambda v(x) u &= 0, \\ u(a) &= 0, \end{aligned}$$

ignoring for the moment whether or not they satisfy $u(b) = 0$. Let u_λ be a solution to this.

If $\lambda > \tilde{\lambda}$, then

$$q(x) + \lambda v(x) > q(x) + \tilde{\lambda} v(x),$$

so by Lemma 3.39, we have

$$\begin{aligned} \#_{[a,b]}\left(u_{\tilde{\lambda}}\right) &\le \#_{[a,b]}(u_\lambda), \\ \mathcal{Z}^k_{[a,b]}(u_\lambda) &\le \mathcal{Z}^k_{[a,b]}\left(u_{\tilde{\lambda}}\right), \end{aligned}$$

i.e., as λ increases, $\#_{[a,b]}(u_\lambda)$ is a non-decreasing function, and the kth zero moves leftward (decreasing) with increasing λ.

CLAIM 3.49. *If there exists an L (which may always be chosen so that $L < 0$) such that $\lambda < L$, then its corresponding* BVP *has no solution.*

PROOF. Find $M > 0$ such that $\left|q(x)/v(x)\right| \le M$ for all $x \in [a,b]$ and find $K > 0$ such that $v(x) \ge K$ for all $x \in [a,b]$. Then

$$-M \le \frac{q(x)}{v(x)} \le M \Longrightarrow -M + \lambda \le \frac{q(x)}{v(x)} + \lambda \le M + \lambda$$

$$\Longrightarrow (\lambda - M)\, v(x) \leq q(x) + \lambda v(x) \leq (M + \lambda)\, v(x).$$

For $\lambda < L$, we have $M + \lambda < 0$, so

$$v(x)\,(M + \lambda) \leq k\,(M + \lambda)$$

for such λ, i.e.,

$$q(x) + \lambda v(x) \leq k\,(M + \lambda)\,,$$

so

$$\#_{[a,b]}(u_\lambda) \leq \#_{[a,b]}(z),$$

where z any solution of $z'' + (M + \lambda)\, kz = 0$ with $z(a) = 0$. Let $-\tau^2 = (M + \lambda)\, k$. Then

$$z = c_1 e^{\tau x} + c_2 e^{-\tau x}$$

and

$$z = 0 \Longrightarrow c_1 e^{2\tau x} + c_2 = 0 \Longrightarrow e^{2\tau x} = -\frac{c_2}{c_1}.$$

This has at most one solution for x (when $c_2/c_1 < 0$), which must be at a. Therefore, u_λ also has no zeros other than a, and in particular $u_\lambda(b) \neq 0$. Therefore, λ is not an eigenvalue if $\lambda < L$. ■

Therefore, by Claim 3.49, there is a smallest eigenvalue.

Suppose that $\lambda > M$. Then $\lambda - M > 0$, so

$$q(x) + \lambda v(x) \geq (\lambda - M)\, v(x) \geq (\lambda - M)\, K.$$

Therefore, $\#_{[a,b]}(u_\lambda) \geq \#_{[a,b]}(z_\lambda)$, where z_λ is any solution of $z'' + (\lambda - M)\, kz = 0$ with $z(a) = 0$. We have

$$z_\lambda = C \sin\!\left(\sqrt{\lambda - M}\sqrt{K} x\right).$$

We see that $\lim_{\lambda\to\infty} \#_{[a,b]}(z_\lambda) = \infty$. Therefore, $\lim_{\lambda\to\infty} \#_{[a,b]}(u_\lambda) = \infty$.

As λ increases, the $(m+1)$st zero of u_λ moves to the left, and when it reaches b, we have found the mth eigenvalue. Therefore, the eigenvalues form as a sequence

$$\lambda_1 < \lambda_2 < \cdots < \lambda_m < \cdots$$

with the eigenfunction for λ_m having $m - 1$ zeros on (a, b), and $\lim_{m\to\infty} \lambda_m = \infty$. □

CHAPTER 4

Midterm Review

1. Laplace Transforms

Assume that

$$f \in \xi_\alpha = \Big\{ f : \big|f(x)\big| \le Ce^{\alpha x} \text{ for sufficiently large } x \Big\}.$$

Then

$$\mathcal{L}\big(f(x)\big)(s) = \hat{f}(s) := \int_0^\infty e^{-st} f(t)\, dt.$$

PROPERTY (Laplace transform).

(1) $\lim_{s\to\infty} \hat{f}(s) = 0$.

(2) $\mathcal{L}(f') = s\hat{f} - f(0)$.

(3) $\mathcal{L}\big(e^{ax} f(x)\big) = \hat{f}(s-a)$.

(4) $\mathcal{L}(xf) = -\frac{d}{ds}\hat{f}(s)$.

(5) $\mathcal{L}(x^n) = \Gamma(n+1)/s^{n+1}$, *where* $n \ge 0$ *and*

$$\Gamma(n) := \int_0^\infty x^{n-1} e^{-x}\, dx.$$

(6) $\mathcal{L}(f * g) = \hat{f}(s)\hat{g}(s)$, *where*

$$(f * g)(x) := \int_0^x f(x-t) g(t)\, dt.$$

(7) *(a)* $\mathcal{L}\big(u(x-a) f(x-a)\big) = e^{-as}\hat{f}(s)$.

(b) $\mathcal{L}\big(u(x-a) f(x)\big) = e^{-as}\mathcal{L}\big(f(x+a)\big)$, *where*

$$u(t) := \begin{cases} 0, & t < 0, \\ 1, & t \ge 0. \end{cases}$$

(8) $\mathcal{L}(\delta) = 1$, *where* δ *is the Dirac delta "function". More precisely,* $\lim_{\epsilon\to 0} \mathcal{L}(f_\epsilon) = 1$, *where*

$$f_\epsilon := \begin{cases} 1, & 0 \le x \le \frac{1}{\epsilon}, \\ 0, & \text{otherwise.} \end{cases}$$

2. Phase Portraits

Let $x' = F(x,y)$ and $y' = G(x,y)$ and let $\mathbf{V} = (F, G)$.

(1) A *critical/equilibrium/singular point* is one at which $x' = 0$ and $y' = 0$.

(2) An equilibrium point P is *stable* if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that any solution which comes within a distance of δ of P never thereafter gets farther than ϵ from P.

(3) An equilibrium point P is *asymptotically stable* if there exists an r such that $\lim_{t\to\infty} \mathbf{x}(t) = P$ for every solution $\mathbf{x}(t)$ which comes within a distance of r of P.

2.1. Linear Systems. Consider the linear system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \tag{$*$}$$

with $\det(\mathbf{A}) \neq 0$ and suppose that $\mathbf{0}$ is the only equilibrium point.

(1) Real eigenvalues
- Same sign
  - Positive values indicate an unstable node.
  - Negative values indicate an asymptotically stable node.
- Opposite signs indicate an unstable saddle.

(2) Complex eigenvalues $a \pm bi$
- $a < 0$ indicates an asymptotically stable spiral.
- $a = 0$ indicates a centre (stable).
- $a > 0$ indicates an unstable spiral.

2.2. Nonlinear Systems. Consider System $(*)$ with

$$\mathbf{A}(x, y) = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}$$

of which P is a critical point. The linear approximation of $\mathbf{A}$ at $P = (p, q)$ is $\mathbf{A}(p, q)$.

Definition (Liapunov function). The function $E(x, y)$ is a *Liapunov function* for the critical point at $\mathbf{0}$ if

(1) $E(x, y) > 0$ for $(x, y) \neq (0, 0)$ and $E(0, 0) = 0$.
(2) E is differentiable.
(3) for any solution $\mathbf{x}(t)$, there exists an $r > 0$ such that $\nabla E \cdot \mathbf{V}$ whenever $\|\mathbf{x}\| < r$.

◊

Theorem.

(a) If a Liapunov function exists, then $\mathbf{0}$ *is stable.*

(b) If there exists a Liapunov function such that

(2′) E is differentiable with continuous derivatives,

(3′) for all $\mathbf{x}(t)$ there exists an $r > 0$ such that $\nabla E \cdot \mathbf{V} < 0$ whenever $0 \leq |\mathbf{x}\| < r$,

then $\mathbf{0}$ *is asymptotically stable. Furthermore, if there exists a Liapunov function such that*

(3″) for all $\mathbf{x}(t)$ there exists an $r > 0$ such that $\nabla E \cdot \mathbf{V}$ whenever $0 < \|\mathbf{x}\| < r$,

then $\mathbf{0}$ *is unstable.*

Corollary. *Suppose that P is a critical point of System $(*)$. Then*

(1) If the real part of both eigenvalues of the corresponding linearized system is negative, then P is asymptotically stable.

(2) If the real part of at least one eigenvalue of the corresponding linearized system is positive, then P is unstable.

Theorem (Poincaré-Bendixon). *Let $\mathbf{x}' = \mathbf{V}(\mathbf{x})$ and $\mathbf{V} = (F, G)$. Let R be a closed bounded region in $\mathbb{R}^2$ with no critical points of $\mathbf{V}$. If $\mathbf{x}(t)$ is a solution which lies in R for all $t \geq t_0$, then either*

(1) $\mathbf{x}(t)$ *is a periodic solution, or*

(2) $\mathbf{x}(t)$ *spirals toward a periodic solution.*

To determine if the solutions never leave R, try computing $\mathbf{V} \cdot \mathbf{n}$, where $\mathbf{n}$ is the outward normal to R, around the boundary of R. If $\mathbf{V} \cdot \mathbf{n} \leq 0$ around the boundary, then the solution in R can't get out ($\mathbf{V}$ points into the region). The region R will always have an annular shape, since index theory implies that a periodic solution surrounds at least one critical point.

3. Index Theory

Definition (Winding number). Let $\mathbf{V} : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field and let γ be a simple closed counterclockwise curve. Suppose that there are no critical points on $\mathbf{V}$ on γ. Then $I_{\mathbf{V}}(\gamma)$ is the *winding number* of $\mathbf{V}(\gamma)$ around $\mathbf{0}$. It represents 2πth the change in angle that $\mathbf{V}(\mathbf{x})$ makes with the horizontal after $\mathbf{x}$ moves around γ. ♢

Suppose that P is a critical point of System $(*)$. Then

$$I\mathbf{V}(P) := I_{\mathbf{V}}(\gamma)$$

for any γ encircling P once counterclockwise but containing no other critical points of $\mathbf{V}$.

Theorem. *We have*

$$I_{\mathbf{V}}(P) = I_{\mathbf{W}}(P) = \begin{cases} 1, & \det(\mathbf{A}) > 0 \text{ (non-saddle)}, \\ -1, & \det(\mathbf{A}) < 0 \text{ (saddle)}, \end{cases}$$

where $\mathbf{W}$ *is the linear approximation to* $\mathbf{V}$ *of* P, *provided that* $\det(\mathbf{A}) \neq 0$, *where*

$$\mathbf{A}(x, y) = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}.$$

Theorem. *For any closed curve* γ, *we have*

$$I_{\mathbf{V}}(\gamma) = \sum_{P \in S} I_{\mathbf{V}}(P)\,,$$

where S *is the set of the critical points of* $\mathbf{V}$ *lying inside* γ.

Theorem. *If* γ *is a counterclockwise periodic solution, then* $I_{\mathbf{V}}(\gamma) = 1$.

Corollary. *If* γ *is a counterclockwise periodic solution, then*

$$\sum_{P \in S} I_{\mathbf{V}}(P) = 1,$$

where S *is the set of the critical points of* $\mathbf{V}$ *lying inside* γ. *In particular,* γ *encloses at least one critical point.*

Theorem. *If* $\lim_{t \to \pm\infty} \mathbf{x}(t)$ *exists, then it is an equilibrium point.*

CHAPTER 5

Review

Review Item 1. Consider

$$\begin{cases} Ly = 0, \\ B_a(y) = 0, \\ B_b(y) = 0, \end{cases} \tag{$*$}$$

where y_1 and y_2 are linearly independent solutions to $Ly = 0$. Let

$$\triangle := \begin{vmatrix} B_a(y_1) & B_b(y_1) \\ B_a(y_2) & B_b(y_2) \end{vmatrix}.$$

Then System $(*)$ has a nontrivial solution if and only if $\triangle = 0$. ♢

Review Item 2. Suppose that

$$\begin{cases} Ly = \lambda y, \\ B_a(y) = 0, \\ B_b(y) = 0, \end{cases} \tag{$**$}$$

if and only if $(L - \lambda I)\, y = 0$, where $y_1(\lambda)$ and $y_2(\lambda)$ are independent solutions to $(L - \lambda I)\, y = 0$ and let

$$\triangle(\lambda) = \begin{vmatrix} B_a\big(y_1(\lambda)\big) & B_b\big(y_1(\lambda)\big) \\ B_a\big(y_2(\lambda)\big) & B_b\big(y_2(\lambda)\big) \end{vmatrix}.$$

Then System $(**)$ has a nontrivial solution if and only if $\triangle(\lambda) = 0$. The various λ satisfying $\triangle(\lambda) = 0$ are called *eigenvalues* and the nontrivial solution ρ_λ is called an *eigenfunction*.

We might not be able to solve $\triangle(\lambda) = 0$ explicitly, but, regardless of that, we can find ρ_λ by

$$y = c_1 y_1(\lambda) + c_2 y_2(\lambda).$$

We solve $B_a(y) = 0$ to get $c_2 = g c_1$, so

$$y_1 = c_1 \big(\underbrace{y_1(\lambda) + g y_2(\lambda)}_{\rho_\lambda}\big).$$

Note that the solution of $B_b(y) = 0$ should be equivalent if λ does indeed satisfy $\triangle(\lambda) = 0$. ♢

Review Item 3. Let $V_B = \big\{ f : B_a(f) = 0 = B_b(f) \big\}$, let $w \geq 0$, and let

$$\langle f, g \rangle_w := \int_a^b w(x) f(x) g(x)\, dx.$$

Definition: The differential operator L is self-adjoint with respect to $\langle\ ,\ \rangle_w$ if and only if $\langle Lf, g \rangle_w = \langle f, Lg \rangle_w$ for all $f, g \in V_B$.

Theorem: The expression $a_0y'' + a_1y' + a_2y$ is self adjoint (with respect to 1) if and only if $a_1 = a_0'$.
If

$$Ly = a_0y'' + a_1y' + a_2y$$

and

$$w = e^{\int \frac{a_1 - a_0'}{a_0}\,dx},$$

then wL is self-adjoint.

◊

Review Item 4. Consider

$$\begin{cases} Ly = \lambda w y, \\ B_a(y) = 0, \\ B_b(y) = 0 \end{cases}$$

with L being self-adjoint (Sturm-Liouville). Then

(1) The eigenvalues form a sequence $\lambda_1 > \lambda_2 > \cdots > \lambda_n > \cdots$ with $\lim_{n\to\infty} \lambda_n = -\infty$.
(2) For each eigenvalue λ_n, there is an eigenfunction ρ_n that has exactly n zeros on (a, b).
(3) The eigenfunctions form an orthogonal basis with respect to the inner product $\langle\ ,\ \rangle_w$.
(4) The function g can be written as a (generalized) Fourier series

$$g(x) = \sum_{n=1}^{\infty} a_n \rho_n,$$

where

$$a_n := \frac{\langle g, \rho_n\rangle_w}{\langle \rho_n, \rho_n\rangle_w}.$$

◊

Review Item 5. Consider

$$\begin{cases} Ly = f, \\ B_a(y) = 0, \\ B_b(y) = 0, \end{cases}$$

where y_1 and y_2 are linearly independent solutions to $Ly = 0$. Let

$$\triangle := \begin{vmatrix} B_a(y_1) & B_b(y_1) \\ B_a(y_2) & B_b(y_2) \end{vmatrix}.$$

Case 1: $\triangle \neq 0$: We rechoose y_1 and y_2 such that $B_a(y_1) = 0 = B_b(y_2)$.
Case 2: $\triangle = 0$: We rechoose y_1 such that $B_a(y_1) = 0 = B_b(y_1)$.

Green's function is given by

$$g(x,t) := \begin{cases} \dfrac{y_2(t)y_1(x)}{p(t)W\big(y_1(t), y_2(t)\big)}, & x \le t, \\ \\ \dfrac{y_1(t)y_2(x)}{p(t)W\big(y_1(t), y_2(t)\big)}, & t \le x. \end{cases}$$

Note that if L is self-adjoint, then $p(t)W\big(y_1(t), y_2(t)\big)$ is a constant. Let

$$Gh := \int_a^b g(x,t)f(t)\,dt.$$

Then $LGh = g$, so $y = Gf$ solves the equation $Ly = f$, where

$$B_a(Gf) = B_a(y_1)\int_a^b \frac{y_2(t)f(t)}{p(t)W\big(y_1(t), y_2(t)\big)}\,dt,$$
$$B_b(Gf) = B_b(y_2)\int_a^b \frac{y_1(t)f(t)}{p(t)W\big(y_1(t), y_2(t)\big)}\,dt.$$

Therefore, $y = Gf$ also satisfies $B_a(y) = 0 = B_b(y)$ in System $(*)$. In System $(**)$, it satisfies $B_a(y) = 0$, but $B_b(y) = 0$ requires

$$\int_a^b \frac{y_1(t)f(t)}{p(t)W\big(y_1(t), y_2(t)\big)}\,dt = 0,$$

in which case the general solution is

$$y = cy_1(x) + \int_a^b g(x,t)f(t)\,dt.$$

$\Diamond$

Review Item 6. Consider $y'' + P(x)y' + Q(x)y = 0$. Let $y = Iu$, where

$$I := e^{-\int \frac{P(x)}{2}\,dx}.$$

We get $u'' + \alpha(x)u = 0$, where

$$\alpha(x) = Q - \frac{P'}{2} - \frac{P^2}{4}.$$

Comparison lemma: Suppose that $\alpha(x) \geq \beta(x)$ for all $x \in [a,b]$, with $\alpha(x) \neq \beta(x)$. Further suppose that

$$y'' + \alpha(x)y = 0,$$
$$z'' + \beta(x)z = 0.$$

Then between any two zeros of z, there exists a zero of y. If, in addition, $y(a) = z(a)$ and $y'(a) = z'(a)$, then $\mathcal{Z}^k_{[a,b]}(y) \leq \mathcal{Z}^k_{[a,b]}(z)$.

Theorem: If $L \leq \alpha(x) \leq U$ for all $x \in [a,b]$ and $\alpha(x)$ is not identically equal to a constant, then

(1) $\pi/\sqrt{U} < |z_k - z_{k+1}| < \pi/\sqrt{L}$, where z_k is the kth zero of y.
(2) $\frac{(b-a)\sqrt{L}}{\pi} - 1 < \#_{(a,b)}(y) < \frac{(b-a)\sqrt{U}}{\pi} - 1$.

If, in addition, $y(a) = 0$, then

$$\#_{(a,b)}(y) < \frac{(b-a)\sqrt{U}}{\pi}.$$

$\Diamond$

Review Item 7. We have

$$\hat{f}(s) = \mathcal{L}(f) := \int_0^\infty e^{-st}f(t)\,dt.$$

(1) $\mathcal{L}\big(f'(x)\big) = s\mathcal{L}\big(f(x)\big) - f(0)$
(2) If f and g are constants, then $\mathcal{L}(f) = \mathcal{L}(g) \Leftrightarrow f = g$.
(3) $\mathcal{L}\big(e^{ax}f(x)\big) = \mathcal{L}(f)\,(s-a)$
(4) (a) $\mathcal{L}\big(xf(x)\big) = -\frac{d}{ds}\hat{f}(s)$
    (b) $\mathcal{L}\big(x^n f(x)\big) = (-1)^n\frac{d^n}{ds^n}\hat{f}(s)$

(c) $\mathcal{L}\Big(\frac{f(x)}{x}\Big) = -\int_0^s \hat{f}(u)\,du$

(5) $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$, where

$$(f * g)(x) := \int_0^x f(t)g(x-t)\,dt.$$

(a) $f * g = g * f$

(b) $(f * g) * h = f * (g * h)$

(6) If f has exponential order, then $\lim_{s\to\infty} f(s) = 0$.

(7) Let

$$u(x) := \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Then $\mathcal{L}\big(u(x-x_0)f(x-x_0)\big) = e^{-sx_0}\hat{f}(s)$.

$\Diamond$

Review Item 8. Consider $\mathbf{X}' = \mathbf{A}\mathbf{X}$, where λ_1 and λ_2 are eigenvalues of $\mathbf{A}$.

(1) (a) If $\lambda_1 < \lambda_2 < 0$ (i.e., $\det(\mathbf{A}) > 0$ and $\operatorname{tr}(\mathbf{A}) < 0$), then the system has a stable node and the curves are tangent to the eigenvector corresponding to the eigenvalue of the smaller value.

(b) If $0 < \lambda_1 < \lambda_2$ (i.e., $\det(\mathbf{A}) > 0$ and $\operatorname{tr}(\mathbf{A}) > 0$), then the system has an unstable node, and the curves approach asymptotically to the eigenvector corresponding to the eigenvalue of the larger absolute value.

(2) If $\lambda_1 < 0 < \lambda_2$ (i.e., $\det(\mathbf{A}) < 0$), then the system has a saddle point.

(3) If λ_1 and λ_2 are complex with $\lambda = a + ib$, then
- $a < 0$ indicates a stable spiral.
- $a > 0$ indicates an unstable spiral.
- $a = 0$ indicates a centre.

$\Diamond$

Review Item 9. Consider

$$\begin{cases} \dfrac{dx}{dt} = F(x,y), \\ \dfrac{dy}{dt} = G(x,y) \end{cases}$$

and let $\mathbf{V} = (F, G)$. Then a point p is an *equilibrium point* if (and only if)

$$\left.\frac{dx}{dt}\right|_p = 0, \qquad \left.\frac{dy}{dt}\right|_p = 0.$$

The point p is called *stable* if for all $\epsilon > 0$, there exists a $\delta > 0$ such that any solution which comes within δ of p never thereafter gets farther than ϵ from p. The point p is called *asymptotically stable* if there exists an r such that every solution which comes within r of p approaches p as $t \to \infty$.

Theorem (Liapunov's Second Method). *Suppose that $(0,0)$ is an equilibrium point and there exists a differentiable function E such that*

(1) $E(x,y) > 0$ for $(x,y) \neq (0,0)$ and $E(0,0) = 0$.

(2) If, for any solution $\big(x(t), y(t)\big)$, there exists an $r > 0$ such that $\nabla E \cdot \mathbf{V} \leq 0$ whenever $x^2 + y^2 < r$, then $(0,0)$ is stable.

Furthermore, if the stronger condition is satisfied:

(2′) If, for any solution $\big(x(t), y(t)\big)$, there exists an $r > 0$ and $\alpha > 0$ such that $\nabla E \cdot \mathbf{V} \le -\alpha E$ for $x^2 + y^2 < r$,

then $(0,0)$ is asymptotically stable.

Corollary. *Let*

$$\mathbf{A} = \begin{bmatrix} \frac{\partial F}{\partial x}\Big|_p & \frac{\partial F}{\partial y}\Big|_p \\ \frac{\partial G}{\partial x}\Big|_p & \frac{\partial G}{\partial y}\Big|_p \end{bmatrix}.$$

If both eigenvalues of $\mathbf{A}$ are negative, then p is asymptotically stable. If at least one eigenvalue of $\mathbf{A}$ is positive, then p is unstable.

◊

Review Item 10.

Theorem (Poincaré-Bendixon). *Let R be a closed and bounded region and let $\big(x(t), y(t)\big)$ be a solution which lies in R for all $t \ge t_0$. if there are no equilibrium points in R, then either $\big(x(t), y(t)\big)$*

(1) is a periodic solution or

(2) it spirals towards a periodic solution.

To check that the solution stays in R for $t \ge 0$, examine $\mathbf{V} \cdot \mathbf{n}$, where $\mathbf{n}$ is the outward pointing normal vector to R. If $\mathbf{V} \cdot \mathbf{n} \le 0$ everywhere on ∂R, then it cannot get out.

Theorem (Liénard). *Let $f, h : \mathbb{R} \to \mathbb{R}$ and let $g(x) := \int_0^x f(t)\,dt$. Suppose that*

(1) f is continuous and even.

(2) there exists an $a > 0$ such that

- *$g(a) < 0$, where $0 < x < a$.*
- *$g(x) > 0$, where $x > a$.*
- *$f(x) > 0$, where $x > a$.*

(3) $\lim_{x\to\infty} g(x) = \infty$.

(4) h is odd and $h(x) > 0$ for $x > 0$.

Then $x'' + f(x)x' + h(x) = 0$ has a unique periodic solution and every other solution spirals towards it.

◊

APPENDIX A

The Gram-Schmidt Process

Given linearly independent vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots$, we want to form a new set of vectors $\mathbf{f}_1, \mathbf{f}_2, \ldots$ in which all vectors are mutually orthogonal. Let $\mathbf{f}_1 = \mathbf{e}_1$. We obtain $\mathbf{f}_2$ by removing the components of $\mathbf{e}_2$ in the $\mathbf{e}_2$ direction, as show in in Figure A.1. To do this, we have

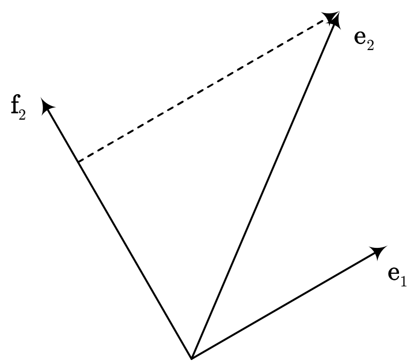


Figure A.1: The vector $\mathbf{f}$ is obtained by removing the component of $\mathbf{e}_2$ that is in the $\mathbf{e}_1$ direction, which is shown as a dashed line.

$$\mathbf{f}_2 = \mathbf{e}_2 - \frac{\langle \mathbf{e}_2, \mathbf{e}_1 \rangle}{\langle \mathbf{e}_1, \mathbf{e}_1 \rangle} \mathbf{e}_1.$$

To check that $\mathbf{f}_2 \perp \mathbf{e}_1$, we see that we indeed have

$$\langle \mathbf{f}_2, \mathbf{e}_1 \rangle = \langle \mathbf{e}_2, \mathbf{e}_1 \rangle - \frac{\langle \mathbf{e}_2, \mathbf{e}_1 \rangle}{\langle \mathbf{e}_1, \mathbf{e}_1 \rangle} \langle \mathbf{e}_1, \mathbf{e}_1 \rangle = \langle \mathbf{e}_2, \mathbf{e}_1 \rangle - \langle \mathbf{e}_2, \mathbf{e}_1 \rangle = 0.$$

Similarly, we have

$$\mathbf{f}_3 = \mathbf{e}_3 - \frac{\langle \mathbf{e}_3, \mathbf{e}_1 \rangle}{\langle \mathbf{e}_1, \mathbf{e}_1 \rangle} \mathbf{e}_1 - \frac{\langle \mathbf{e}_3, \mathbf{e}_2 \rangle}{\langle \mathbf{e}_2, \mathbf{e}_2 \rangle} \mathbf{e}_2.$$

To check that $\mathbf{f}_3 \perp \mathbf{e}_1$, we see that we indeed have

$$\langle \mathbf{f}_3, \mathbf{e}_1 \rangle = \langle \mathbf{e}_3, \mathbf{e}_1 \rangle - \frac{\langle \mathbf{e}_3, \mathbf{e}_1 \rangle}{\langle \mathbf{e}_1, \mathbf{e}_1 \rangle} \langle \mathbf{e}_1, \mathbf{e}_1 \rangle - \frac{\langle \mathbf{e}_3, \mathbf{e}_2 \rangle}{\langle \mathbf{e}_2, \mathbf{e}_2 \rangle} \underbrace{\langle \mathbf{e}_2, \mathbf{e}_1 \rangle}_{0} = 0.$$

Continuing, we produce a mutually orthogonal set, and we can normalize it later if desired.

Example A.1. Suppose that $\langle f, g \rangle := \int_0^1 f(x) g(x)\, dx$ and let

$$e_1 = 1, \qquad e_2 = x, \qquad e_3 = x^2.$$

Find a new set $\{f_1, f_2, f_3\}$ in which its elements are mutually orthogonal to those of $\{e_1, e_2, e_3\}$. ◇

SOLUTION. First, we let $f_1 = e_1 = 1$. Then

$$f_2 = e_2 - \frac{\langle e_1, e_2\rangle}{\langle e_1, e_1\rangle} = x - \frac{\displaystyle\int_0^1 x\,dx}{\displaystyle\int_0^1 dx} = x - \frac{1/2}{1}\cdot 1 = x - \frac{1}{2}.$$

For f_3, we have

$$\begin{aligned}
f_3 &= \overbrace{x^2}^{e_3} - \overbrace{\frac{\displaystyle\int_0^1 x^2\,dx}{\displaystyle\int_0^1 dx}\cdot 1}^{\frac{\langle e_3,e_1\rangle}{\langle e_1,e_1\rangle}e_1} - \overbrace{\left(\frac{\displaystyle\int_0^1 x^2\left(x-\frac{1}{2}\right)dx}{\displaystyle\int_0^1 \left(x-\frac{1}{2}\right)^2 dx}\right)\left(x-\frac{1}{2}\right)}^{\frac{\langle e_3,e_2\rangle}{\langle e_2,e_2\rangle}e_2}\\
&= x^2 - \frac{1}{3} - \frac{\frac{1}{4}-\frac{1}{6}}{\left.\frac{\left(x-\frac{1}{2}\right)^3}{3}\right|_0^1}\left(x-\frac{1}{2}\right)\\
&= x^2 - \frac{1}{3} - \frac{1}{12}\frac{1}{\left(\frac{\frac{1}{8}+\frac{1}{8}}{3}\right)}\left(x-\frac{1}{2}\right)\\
&= x^2 - \frac{1}{3} - \frac{1}{12}\frac{3}{1/4}\left(x-\frac{1}{2}\right)\\
&= x^2 - \frac{1}{3} - \frac{1}{12}(3\cdot 4)\left(x-\frac{1}{2}\right)\\
&= x^2 - \frac{1}{3} - \left(x-\frac{1}{2}\right)\\
&= x^2 - x + \frac{1}{2} - \frac{1}{3}\\
&= x^2 - x + \frac{1}{6}.
\end{aligned}$$

Therefore, elements in the set

$$\left\{1, x-\frac{1}{2}, x^2-x+\frac{1}{6}\right\}$$

are mutually orthogonal to $\{1, x, x^2\}$. ◇