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Limit sets of Teichmüller geodesics with minimal nonuniquely ergodic vertical foliation, II

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Abstract. Given a sequence of curves on a surface, we provide conditions which ensure that (1) the sequence is an infinite quasi-geodesic in the curve complex, (2) the limit in the Gromov boundary is represented by a nonuniquely ergodic ending lamination, and (3) the sequence divides into a finite set of subsequences, each of which projectively converges to one of the ergodic measures on the ending lamination. The conditions are sufficiently robust, allowing us to construct sequences on a closed surface of genus g for which the space of measures has the maximal dimension $3g - 3$, for example.

We also study the limit sets in the Thurston boundary of Teichmüller geodesic rays defined by quadratic differentials whose vertical foliations are obtained from the constructions mentioned above. We prove that such examples exist for which the limit is a cycle in the 1-skeleton of the simplex of projective classes of measures visiting every vertex.

Contents

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1. Introduction

This paper builds on the work of the second and fourth author with Anna Lenzhen, [22], in which the authors construct a sequence of curves in the five-punctured sphere S with the following properties (see Section 2 for definitions). First, the sequence is a quasi-geodesic ray in the curve complex of S, and hence converges to some ending lamination ν. Second, ν is nonuniquely ergodic, and the sequence naturally splits into two subsequences, each of which converges to one of the ergodic measures on ν in the space of projective measured laminations. Third, for any choice of measure $\bar{\nu}$ on ν and base point X in Teichmüller space, the Teichmüller ray based at X and defined by the quadratic differential with vertical foliation $\bar{\nu}$, accumulates on the entire simplex of measures on ν in the Thurston compactification. The construction in [22] was actually a family of sequences depending on certain parameters.

In this paper we extract the key features of the sequences produced in the above construction as a set of *local* properties for any sequence of curves $\{\gamma_k\}_{k=0}^{\infty}$ on any surface, which we denote $\mathcal{P}$; see Section 3 and Definition 3.1 as well as Section 7 for examples. Here, "local" is more precisely *m-local* for some $2 \leq m \leq \xi(S)$ (where $\xi(S) = \dim_{\mathbb{C}}(\mathrm{Teich}(S))$), and means that the conditions in $\mathcal{P}$ involve relations between curves contained subsets of the form $\{\gamma_k, \dots, \gamma_{k+2m}\}$ for $k \geq 0$. We refer to the number m as the *subsequence counter*. Most of the conditions in $\mathcal{P}$ are stated in terms of intersection numbers, though they also include information about twisting which is recorded in an auxiliary sequence $\{e_k\}_{k=0}^{\infty} \subset \mathbb{N}$.

Theorem 1.1. *For appropriate choices of parameters in $\mathcal{P}$, any sequence $\{\gamma_k\}_{k=0}^{\infty}$ in $\mathcal{C}(S)$ satisfying $\mathcal{P}$ will be the vertices of a quasi-geodesic in $\mathcal{C}(S)$ and hence will limit to an ending lamination ν in $\partial\mathcal{C}(S) \cong \mathcal{EL}(S)$.*

If $\mu = \gamma_0 \cup \cdots \cup \gamma_{m-1}$, then for any $k \geq m$, the subsequence counter, we have

$$d_{\gamma_k}(\mu, \nu) \stackrel{+}{\asymp} e_k.$$

On the other hand, there is a constant $R > 0$ with the property that for any proper subsurface $W \neq \gamma_k$ for any $k \in \mathbb{N}$ we have

$$d_W(\mu, \nu) < R.$$

See Propositions 4.4 and 4.5 for precise statements. Here d_W is the projection coefficient for W and d_γ the projection coefficient for (the annular neighborhood of) γ; see Section 2.4.

Although the conditions in $\mathcal{P}$ only provide local information about intersection numbers, we can deduce estimates on intersection numbers between any two curves in the sequence from this; see Theorem 5.1. From these estimates, we are able to promote the convergence $\gamma_k \to \nu$ in $\overline{\mathcal{C}(S)}$ into precise information about convergence in $\mathcal{PML}(S)$. To state this, we note that the local condition depends on the subsequence counter m. There are m subsequences $\{\gamma_i^h\}_{i=0}^{\infty}$, for $h = 0, \dots, m-1$, defined by $\gamma_i^h = \gamma_{im+h}$.

Theorem 1.2. *For appropriate choices of parameters in $\mathcal{P}$, and any sequence $\{\gamma_k\}_{k=0}^{\infty}$ in $\mathcal{C}(S)$ satisfying $\mathcal{P}$, the ending lamination $\nu \in \mathcal{EL}(S)$ from Theorem 1.1 is nonuniquely ergodic. Moreover, if m is the subsequence counter, then the dimension of the space of measures on ν is precisely m, and the m subsequences $\{\gamma_i^h\}_{i=0}^{\infty}$ converge to m ergodic measures $\bar{\nu}^h$ on ν, for $h = 0, \dots, m-1$, spanning the space of measures.*

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For precise statements, see Theorems 5.10, 6.1, and 6.5.

We note that for any nonuniquely ergodic lamination ν, the space of measures is always the cone on the *simplex of measures on* ν, denoted $\Delta(\nu)$, which is projectively well-defined. The vertices of $\Delta(\nu)$ are the *ergodic measures*, and the dimension of the space of measures is at most $\xi(S)$: This follows from the fact that the Thurston symplectic form on the $2\xi(S)$-dimensional space $\mathcal{ML}(S)$ must restrict to zero on the cone on $\Delta(\nu)$ since it is bounded above by the geometric intersection number, [35, Section 3.2], and consequently must be at most half-dimensional (see also [28, Section 1] and the reference to [20, 40]). We note that the subsequence counter m can also be at most $\xi(S)$, and the explicit constructions in Section 7 are quite flexible and provide examples with this maximal dimension, as well as examples with smaller dimensions.

As an application of these theorems, together with the main result of the first and third author in [5] and Theorem 1.1, we have:

Corollary 1.3. *Suppose that ν is as in Theorem* 1.1. *Any Weil–Petersson geodesic ray with forward ending lamination ν is recurrent to a compact subset of the moduli space.*

Here, the *ending lamination* of a Weil–Petersson geodesic ray is given as in [3, 4]. The corollary, which follows directly from [5, Theorem 4.1] after observing that ν satisfies the condition of *nonannular bounded combinatorics* (see Proposition 4.5), provides greater insight into the class of Weil–Petersson ending laminations that violate *Masur's criterion*. In particular, these nonuniquely ergodic laminations determine *recurrent* Weil–Petersson geodesic rays, by contrast to the setting of Teichmüller geodesics where Masur's criterion [30] guarantees a Teichmüller geodesic with such a vertical foliation diverges.

For any lamination ν coming from a sequence $\{\gamma_k\}_{k=0}^{\infty}$ satisfying $\mathcal{P}$, as well as some additional conditions (see (8.8) in Section 8 and condition $\mathcal{P}$ (iv) in Section 9), we analyze the limit set of a Teichmüller geodesic ray defined by a quadratic differential with vertical foliation $\bar{\nu}$ supported on ν. To describe our result about the limiting behavior of this geodesic ray, we denote the simplex of the projective classes of measures supported on the lamination by $\Delta(\nu)$ in the space of projective measured foliations, viewed as the Thurston boundary of Teichmüller space.

Theorem 1.4. *Suppose that ν is the limiting lamination of a sequence $\{\gamma_k\}_{k=0}^{\infty}$ satisfying the conditions $\mathcal{P}$, $\mathcal{P}$* (iv)*, and* (8.8). *Let*

$$\bar{\nu} = \sum_{h=0}^{m-1} x_h \bar{\nu}^h,$$

where $x_h > 0$ for $h = 0, \dots, m-1$, and let $r: [0, \infty) \to \mathrm{Teich}(S)$ be a Teichmüller geodesic ray with vertical measured lamination $\bar{\nu}$. Then the limit set of r in the Thurston boundary is the simple closed curve in the simplex $\Delta(\nu)$ of measures on ν that is the concatenation of edges

$$\big[[\bar{\nu}^0], [\bar{\nu}^1]\big] \cup \big[[\bar{\nu}^1, \bar{\nu}^2]\big] \cup \dots \cup \big[[\bar{\nu}^{m-1}], [\bar{\nu}^0]\big].$$

When $m \geq 3$, the theorem shows that there are Teichmüller geodesics whose limit set does not contain any point in the interior of $\Delta(\nu)$. In addition, it answers the following question raised by Jonathan Chaika.

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Question 1.5. Is the limit set of each Teichmüller geodesic ray simply connected?

For $m \geq 3$, the theorem shows that answer to this question is no. Namely, Teichmüller geodesic rays with vertical measured lamination as above provide examples of geodesics with limit set being a topological circle, and hence not simply connected.

The results of this paper (as well as those of [22]) were inspired by work of Masur in [29], Lenzhen [23], and Gabai [14]. In [29] Masur showed that if ν is a uniquely ergodic foliation, then any Teichmüller ray defined by a quadratic differential with vertical foliation supported on ν limits to $[\nu]$ in the Thurston compactification. Lenzhen [23] gave the first examples of Teichmüller rays which do not converge in the Thurston compactification. Lenzhen's rays were defined by quadratic differentials with non-minimal vertical foliations, and in both [22] and [8], nonconvergent rays defined by quadratic differentials with minimal vertical foliations were constructed. The methods in these two papers are quite different, and as mentioned above, the approach taken in this paper is more closely related to that of [22]. We also note the results of this paper, as well as [8,22,23], are in sharp contrast to the work of Hackobyan and Saric in [17] where it is shown that Teichmüller rays in the *universal Teichmüller space* always converge in the corresponding Thurston compactification.

Our example of nonuniquely ergodic laminations obtained from a sequence of curves are similar to those produced by Gabai in [14]. On the other hand, our construction provides additional information, especially important are the estimates on intersection numbers and subsurface projections, that allow us to study the limiting behavior of the associated Teichmüller rays. For more on the history and results about the existence and constructions of nonuniquely ergodic laminations and the study of limit sets of Teichmüller geodesics with such vertical laminations we refer the reader to the introduction of [22].

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2. Background

We use the following notation throughout this paper.

Notation 2.1. Let $K \geq 1$ and $C \geq 0$ and let $f, g\colon X \to \mathbb{R}$ be two functions. We write

- $f \overset{+}{\asymp}_C g$ if $f(x) - C \leq g(x) \leq f(x) + C$ for all $x \in X$,
- $f \overset{*}{\asymp}_K g$ if $\frac{1}{K} f(x) \leq g(x) \leq K f(x)$ for all $x \in X$,
- $f \asymp_{K,C} g$ if $\frac{1}{K}(f(x) - C) \leq g(x) \leq K f(x) + C$ for all $x \in X$,
- $f \overset{*}{\prec}_K g$ if $f(x) \leq K g(x)$ for all $x \in X$,
- $f \overset{+}{\prec}_C g$ if $f(x) \leq g(x) + C$ for all $x \in X$,
- $f \prec_{K,C} g$ if $f(x) \leq K g(x) + C$ for all $x \in X$.

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When the constants are known from the text we drop them from the notations. Finally, we also write $f = O(g)$ if $f \overset{*}{\prec} g$.

Let $S = S_{g,b}$ be an orientable surface of finite type with genus g and b holes (a hole can be either a puncture or a boundary component). Define the complexity of S by $\xi(S) = 3g-3+b$. The main surface we will consider will have $\xi > 1$ and all holes will be punctures. However, we will also be interested in subsurfaces and covers of the main surface, which can also have $\xi \leq 1$. For surfaces S with $\xi(S) \geq 1$, we will equip it with a reference metric, which is any complete, hyperbolic metric of finite area with geodesic boundary (if any).

2.1. Curve complexes. For any surface Y, $\xi(Y) \geq 1$, the curve complex of Y, denoted by $\mathcal{C}(Y)$, is a flag complex whose vertices are the isotopy classes of simple closed curves on Y that are essential, meaning non-null homotopic and nonperipheral. For $\xi(Y) > 1$, a set of $k+1$ distinct isotopy classes of curves defines a k-simplex if any pair can be represented by disjoint curves. For $\xi(Y) = 1$ (Y is $S_{0,4}$ or $S_{1,1}$), the definition is modified as follows: a set of $k+1$ distinct isotopy classes defines a k-simplex if the curves can be represented intersecting twice (for $Y = S_{0,2}$) or once (for $Y = S_{1,1}$).

The only surface Y with $\xi(Y) < 1$ of interest for us is a compact annulus with two boundary components. These arise as follows. For any essential simple closed curve α on our main surface S, let Y_α denote the annular cover of S to which α lifts. The reference hyperbolic metric on S lifts and provides a compactification of this cover by a compact annulus with boundary (which is independent of the metric). The curve complex of α, denoted $\mathcal{C}(Y_\alpha)$, or simply $\mathcal{C}(\alpha)$, has vertex set being the properly embedded, essential arcs in Y_α, up to isotopy fixing the boundary pointwise. A set of isotopy classes of arcs spans a simplex if any pair can be realized with disjoint interiors.

Distances between vertices in $\mathcal{C}(Y)$ (for any Y) will be measured in the 1-skeleton, so the higher-dimensional simplices are mostly irrelevant. Masur and Minsky [31] proved that for any Y, there is a $\delta > 0$ so that $\mathcal{C}(Y)$ is δ-hyperbolic.

For surfaces Y with $\xi(Y) \geq 1$, we also consider the arc and curve complex $\mathcal{AC}(Y)$, defined in a similar way to $\mathcal{C}(Y)$. Here vertices are isotopy classes of essential simple closed curves and essential, properly embedded arcs (isotopies need not fix the boundary pointwise), with simplices defined again in terms of disjoint representatives. Arc and curve complexes are quasi-isometric to curve complexes, and so are also δ-hyperbolic.

Multicurves (respectively, multiarcs) are disjoint unions of pairwise nonisotopic essential simple closed curves (respectively, simple closed curves and properly embedded arcs). Up to isotopy a multicurve (respectively, multiarc) determines, and is determined by, a simplex in $\mathcal{C}(S)$ (respectively, $\mathcal{AC}(S)$). A marking μ is a pants decomposition base(μ), called the base of μ, together with a transversal curve β_α, for each $\alpha \in$ base(μ), which is a curve minimally intersecting α and disjoint from base(μ) $- \alpha$. A partial marking μ is similarly defined, but not every curve in the pants decomposition base(μ) is required to have a transversal curve.

For more details on curve complexes, arc and curve complexes, and markings, we refer the reader to [31].

Remark 2.2. When the number $\xi(S)$ is at least 1, it is equal to the number of curves in a pants decomposition. When all the holes of S are punctures, $\xi(S)$ is also the complex dimension of Teichmüller space of S.

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2.2. Laminations and foliations. A lamination will mean a geodesic lamination (with respect to the reference metric if no other metric is specified), and a measured lamination is a geodesic lamination ν, called the support, with an invariant transverse measure $\bar{\nu}$. We will often refer to a measured lamination just by the measure $\bar{\nu}$ (as this determines the support ν). The space of all measured laminations will be denoted $\mathcal{ML}(S)$, and for any two metrics, the resulting spaces of measured laminations are canonically identified. By taking geodesic representatives, simple closed curves and multicurves determine geodesic laminations. Weighted simple closed curves and multicurves determine measured laminations are dense in $\mathcal{ML}(S)$, and the geometric intersection number extends to a continuous, bi-homogeneous function

$$i\colon \mathcal{ML}(S) \times \mathcal{ML}(S) \to \mathbb{R}.$$

By a measured foliation on S we will mean a singular measured foliation with prong singularities of negative index (and at punctures, filling in the puncture produces a k-prong singularity with $k \geq 1$). When convenient, a measured foliation may be considered only defined up to measure equivalence, and the space of measure equivalence classes of measured foliations is denoted $\mathcal{MF}(S)$. The spaces $\mathcal{MF}(S)$ and $\mathcal{ML}(S)$ are canonically identified, and we will frequently not distinguish between measured laminations and measured foliations. A foliation or lamination is uniquely ergodic if it supports a unique (up to scaling) transverse measure, or equivalently, if the first return map to (the double of) any transversal is uniquely ergodic. Otherwise it is nonuniquely ergodic. We write $\mathcal{PML}(S)$ and $\mathcal{PMF}(S)$ for the quotient spaces, identifying measured laminations or foliations that differ by scaling the measure. See [7, 13, 25, 35, 39] for complete definitions, detailed discussion, and equivalence of $\mathcal{MF}(S)$ and $\mathcal{ML}(S)$.

2.3. Gromov boundary of the curve complex. A lamination ν on S is called an ending lamination if it is minimal (every leaf is dense) and filling (every simple closed geodesic on the surface nontrivially, transversely intersect ν). Every ending lamination admits a transverse measure, and we let $\mathcal{EL}(S)$ denote the space of all ending laminations. This is the quotient space of the subspace of $\mathcal{ML}(S)$ consisting of measured laminations supported on ending laminations, by the map which forgets the measures. The following theorem of Klarreich [21] identifies the Gromov boundary of $\mathcal{C}(S)$ with $\mathcal{EL}(S)$.

Theorem 2.3 (Boundary of the curve complex). *There is a homeomorphism Φ from the Gromov boundary of $\mathcal{C}(S)$ equipped with its standard topology to $\mathcal{EL}(S)$.*

Let $\{\gamma_k\}_{k=0}^{\infty}$ be a sequence of curves in $\mathcal{C}_0(S)$ that converges to a point x in the Gromov boundary of $\mathcal{C}(S)$. Regarding each γ_k as a projective measured lamination, any accumulation point of the sequence $\{\gamma_k\}_{k=0}^{\infty}$ in $\mathcal{PML}(S)$ is supported on $\Phi(x)$.

We will use this theorem throughout to identify points in $\partial\mathcal{C}(S)$ with ending laminations in $\mathcal{EL}(S)$.

2.4. Subsurface coefficients. An *essential subsurface* Y of a surface Z with $\xi(Y) \geq 1$ is a closed, connected, embedded subsurface whose boundary components are either essential curves in Z or boundary component of Z, and whose punctures are punctures of Z. All such subsurfaces are considered up to isotopy, and we often choose representatives that are components of complements of small neighborhoods of simple closed geodesics, which therefore

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have minimal, transverse intersection with any lamination. The only essential subsurfaces Y of Z with $\xi(Y) < 1$ are not actually subsurfaces at all, but rather such a Y is the compactified annular covers Y_α of Z associated to a simple closed curve α in Z. We sometimes confuse an annular neighborhood of α with the cover Y_α (hence the reference to it as a subsurface) when convenient. We will always write $Y \subseteq Z$ to denote an essential subsurface, even when it is not, strictly speaking, a subset of Z.

Let $Y \subseteq Z$ be an essential nonannular subsurface and λ a lamination (possibly a multicurve) and we define the subsurface projection of λ to Y. Represent Y as a component of the complement of a very small neighborhood of geodesic multicurve. If $\lambda \cap Y = \emptyset$, then define $\pi_Y(\lambda) = \emptyset$. Otherwise, $\pi_Y(\lambda)$ is the union of all curves which are (i) simple closed curve components of $Y \cap \lambda$ or (ii) essential components of $\partial N(a \cup \partial Y)$, where $a \subset \lambda \cap Y$ is any arc, and $N(a \cup \partial Y)$ is a regular neighborhood of the union. If Y_α is an essential annular subsurface, then $\pi_{Y_\alpha}(\lambda)$, or simply $\pi_\alpha(\lambda)$, is defined as follows. For any component of the preimage of λ in the annular cover corresponding to α, the closure is an arc in Y_α, and we take the union of all such arcs that are essential (that is, the arcs that connect the two boundary components).

For a marking μ (or a partial marking), if $Y = Y_\alpha$ is an annulus with core curve $\alpha \in \operatorname{base}(\mu)$, then $\pi_Y(\mu) = \pi_\alpha(\beta_\alpha)$, where β_α is the transverse curve for α in μ. Otherwise, $\pi_Y(\mu) = \pi_Y(\operatorname{base}(\mu))$. For any lamination or partial marking λ and any essential subsurface Y, $\pi_Y(\lambda)$ is a subset of diameter at most 2.

Let μ, μ' be laminations, multiarcs, or partial markings on Z and $Y \subset Z$ an essential subsurface. The *Y-subsurface coefficient* of μ and μ' is defined by

$$d_Y(\mu, \mu') := \operatorname{diam}_{\mathcal{C}(Y)}(\pi_Y(\mu) \cup \pi_Y(\mu')).$$

Remark 2.4. The subsurface coefficient is sometimes alternatively defined as the (minimal) distance between $\pi_Y(\mu)$ and $\pi_Y(\mu')$. Since the diameter of the projection of any marking or lamination is bounded by 2, these definitions differ by at most 4. The definition we have chosen satisfies a triangle inequality (when the projections involved are nonempty), which is particular useful for our purposes.

The following lemma provides an upper bound for a subsurface coefficient in terms of intersection numbers.

Lemma 2.5 ([32, Section 2]). *Given curves $\alpha, \beta \in \mathcal{C}(S)$, for any essential subsurface $Y \subseteq S$ we have*

$$d_Y(\alpha, \beta) \leq 2i(\alpha, \beta) + 1.$$

When Y is an annular subsurface, the above bound holds with multiplicative factor 1.

Remark 2.6. The bound in the above lemma can be improved to $\prec \log i(\alpha, \beta)$ for $\xi(Y) \geq 1$, but the bound given is sufficient for our purposes.

The following result equivalent to [9, Corollary D] provides for a comparison between the logarithm of intersection number and sum of subsurfaces coefficients. For a pair of markings μ, μ', the intersection number $i(\mu, \mu')$ is defined to be the sum of the intersection numbers of the curves in μ with those in μ'.

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Theorem 2.7. *Given $A > 0$ sufficiently large, there are constants so that for any two multi-curves, multi-arcs or markings μ and μ' we have*

$$\log i(\mu, \mu') \asymp \sum_{\substack{W \subseteq Y \\ \text{nonannular}}} \{d_W(\mu, \mu')\}_A + \sum_{\substack{W \subseteq Y \\ \text{annular}}} \log\{d_W(\mu, \mu')\}_A,$$

where W is so that $\mu, \mu' \pitchfork W$.

In this theorem, $\{\cdot\}_A$ is a cut-off function defined by $\{x\}_A = x$ if $x \geq A$, and $\{x\}_A = 0$ if $x < A$.

Notation 2.8. Given a lamination or a partial marking μ and subsurface Y, we say that μ and Y overlap, writing $\mu \pitchfork Y$ if $\pi_Y(\mu) \neq \emptyset$. For any marking μ and any subsurface Y, we have $\mu \pitchfork Y$. Given two subsurfaces Y and Z, if $\partial Y \pitchfork Z$ and $\partial Z \pitchfork Y$, then we say that Y and Z overlap, and write $Y \pitchfork Z$.

The inequality first proved by J. Behrstock [1] relates subsurface coefficients for overlapping subsurfaces.

Theorem 2.9 (Behrstock inequality). *There is a constant $B_0 > 0$ so that given a partial marking or lamination μ and subsurfaces Y and Z satisfying $Y \pitchfork Z$ we have*

$$\min\{d_Y(\partial Z, \mu), d_Z(\partial Y, \mu)\} \leq B_0$$

whenever $\mu \pitchfork Y$ and $\mu \pitchfork Z$.

Remark 2.10. As shown in [26], the constant B_0 can be taken to be 10. In fact, if one projection is at least 10, then the other is ≤ 4.

The following theorem is a straightforward consequence of the Bounded Geodesic Image Theorem [32, Theorem 3.1].

Theorem 2.11 (Bounded geodesic image). *Given $k \geq 1$ and $c \geq 0$, there is a constant $G > 0$ with the following property. Let $Y \subsetneq S$ be a subsurface. Let $\{\gamma_k\}_{i=0}^{\infty}$ be a 1-Lipschitz (k, c)-quasi-geodesic in $\mathcal{C}(S)$ so that $\gamma_k \pitchfork Y$ for all i. Then* $\operatorname{diam}_Y(\{\gamma_k\}_{i=0}^{\infty}) \leq G$.

2.5. Teichmüller theory. Throughout the paper, we assume that S is a surface and that any holes of S are punctures. The Teichmüller space of S, $\mathrm{Teich}(S)$, is the space of equivalence classes of marked complex structures $[f: S \to X]$, where f is an orientation preserving homeomorphism to a finite-type Riemann surface X, where $(f: S \to X) \sim (g: S \to Y)$ if $f \circ g^{-1}$ is isotopic to a conformal map. We often abuse notation, and simply refer to X as a point in Teichmüller space, with the equivalence class of marking implicit. We equip $\mathrm{Teich}(S)$ with the Teichmüller metric, whose geodesics are defined in terms of quadratic differentials.

Let X be a finite-type Riemann surface and let $T^{(1,0)*}X$ be the holomorphic cotangent bundle of X. A quadratic differential q is a nonzero, integrable, holomorphic section of the bundle $T^{(1,0)*}X \otimes T^{(1,0)*}X$. In local coordinates q has the form $q(z)dz^2$, where $q(z)$ is holomorphic function. Changing to a different coordinate w, q changes by the square of the

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derivative, and is thus given by $q(z(w))(\frac{\partial w}{\partial z})^2 dw^2$. The integrability condition is only relevant when X has punctures, in which case it guarantees that q has at worst simple poles at the punctures.

In local coordinates away from zeros of q the quadratic differential q determines the 1-form $\sqrt{q(z)dz^2}$. Integrating this 1-form determines *a natural coordinate* $\zeta = \xi + i\eta$. Then the trajectories of $d\xi \equiv 0$ and $d\eta \equiv 0$, respectively, determine the horizontal and vertical foliations of q on X. Integrating $|d\xi|$ and $|d\eta|$ determines transverse measures on vertical and horizontal foliations, respectively. These extend to measured foliations on the entire surface S with singularities at the zeros. Using the identification $\mathcal{MF}(S) \cong \mathcal{ML}(S)$, we often refer to the vertical and horizontal measured laminations of q.

Now given a quadratic differential q on X, the associated Teichmüller geodesic is determined by the family of Riemann surfaces X_t defined by local coordinates $\zeta_t = e^t\xi + e^{-t}\eta$, where $\zeta = \xi + i\eta$ is a natural coordinate of q at X and $t \in \mathbb{R}$. Every Teichmüller geodesic ray based at X is determined by a quadratic differential q on X. See [15] for details on Teichmüller space and quadratic differentials.

2.6. The Thurston compactification. Given a point $[f\colon S \to X]$ in $\mathrm{Teich}(S)$ and a curve α, the hyperbolic length of α at $[f\colon S \to X]$ is defined to be hyperbolic length of the geodesic homotopic to $f(\alpha)$ in X. Again abusing notation and denoting the point in $\mathrm{Teich}(S)$ by X, we write the hyperbolic length simply as $\mathrm{Hyp}_X(\alpha)$. The hyperbolic length function extends to a continuous function

$$\mathrm{Hyp}_{(\cdot)}(\cdot)\colon \mathrm{Teich}(S) \times \mathcal{ML}(S) \to \mathbb{R}.$$

The Thurston compactification, $\overline{\mathrm{Teich}(S)} = \mathrm{Teich}(S) \cup \mathcal{PML}(S)$, is constructed so that a sequence $\{X_n\} \subseteq \mathrm{Teich}(S)$ converges to $[\bar{\nu}] \in \mathcal{PML}(S)$ if and only if

$$\lim_{n\to\infty} \frac{\mathrm{Hyp}_{X_n}(\alpha)}{\mathrm{Hyp}_{X_n}(\beta)} = \frac{i(\bar{\nu},\alpha)}{i(\bar{\nu},\beta)}$$

for all simple closed curves α, β with $i(\bar{\nu},\beta) \neq 0$. See [2, 13] for more details on the Thurston compactification.

2.7. Some hyperbolic geometry. Here we list a few important hyperbolic geometry estimates. For a hyperbolic metric $X \in \mathrm{Teich}(S)$ and a simple closed curve α, in addition to the length $\mathrm{Hyp}_X(\alpha)$, we also have the quantity $w_X(\alpha)$, the *width of* α *in* X. This is the width of a maximal embedded tubular neighborhood of α in the hyperbolic metric X – that is, $w_X(\alpha)$ is the maximal w so that the open $w/2$-neighborhood of α is an annular neighborhood of α. The Collar Lemma (see e.g. [6, Section 4.1]) provides a lower bound on the width:

Lemma 2.12. *For any simple closed curve α, we have*

$$w_X(\alpha) \geq 2\sinh^{-1}\left(\frac{1}{\sinh(\mathrm{Hyp}_X(\alpha)/2)}\right).$$

Consequently,

$$w_X(\alpha) \stackrel{+}{\asymp} 2\log\left(\frac{1}{\mathrm{Hyp}_X(\alpha)}\right). \tag{2.1}$$

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The second statement comes from the first, together with an easy area argument. The implicit additive error depends only on the topology of S.

We also let $\epsilon_0 > 0$ be the Margulis constant, which has the property that any two hyperbolic geodesics of length at most ϵ_0 must be embedded and disjoint.

2.8. Short markings. For $L > 0$ sufficiently large, an L-bounded length marking at $X \in \mathrm{Teich}(S)$ (or L-short marking) is a marking with the property that any curve in base(μ) has hyperbolic length less than L, and so that for each $\alpha \in \mathrm{base}(\mu)$, the transversal curve to α has smallest possible length in X. Choosing ϵ sufficiently large (larger than the Bers constant of the surface) the distance between any two points in Teichmüller space can be estimated up to additive and multiplicative error in terms of the subsurface coefficients of the short markings at those points, together with the lengths of their base curves; see [37].

3. Sequences of curves

Over the course of the next three sections we will provide general conditions on a sequence of curves which guarantee that any accumulation point in $\mathcal{PML}(S)$ of this sequence is a nonuniquely ergodic ending lamination. In [14, Section 9], Gabai describes a construction of minimal filling nonuniquely ergodic geodesic laminations. The construction is topological in nature. Our construction in this paper and that of [22] can be considered as quantifications of Gabai's construction where the estimates for intersection numbers are computed explicitly. These estimates allow us to provide more detailed information about the limits in $\mathcal{PML}(S)$ as well as limiting behavior of associated Teichmüller geodesics.

In this section we state conditions a sequence of curves can satisfy, starting with an example, and describe a useful way of mentally organizing them. The conditions are motivated by the examples in [22], and so we recall that construction to provide the reader concrete examples to keep in mind. A more robust construction that illustrates more general phenomena is detailed in Section 7.

Throughout the rest of this paper $\{e_k\}_{k=0}^{\infty}$ is an increasing sequence of integers satisfying

$$e_{k+1} \geq a e_k \quad \text{for any } k \geq 0, \tag{3.1}$$

where $a > 1$. Consequently, for all $l < k$, we have $e_k \geq a^{k-l} e_l$.

3.1. Motivating example. The motivating examples are sequences of curves in $S_{0,5}$, the five-punctured sphere. We view this surface as the double of a pentagon minus its vertices over its boundary. This description provides an obvious order five rotational symmetry ρ obtained by rotating the pentagon counter-clockwise by an angle $4\pi/5$. Let γ_0 be a curve which is the boundary of a small neighborhood of one of the sides of the pentagon and let $\gamma = \rho^2(\gamma_0)$ (see Figure 1). Write $\mathcal{D} = \mathcal{D}_\gamma$ for the positive Dehn twist about γ.

Now define γ_k to be the image of γ_0 under a composition of powers of $\mathcal{D}$ and ρ by the following formula:

$$\gamma_k = \mathcal{D}^{e_2} \rho \mathcal{D}^{e_3} \rho \cdots \mathcal{D}^{e_k} \rho \mathcal{D}^{e_{k+1}} \rho(\gamma_0).$$

The first five curves, $\gamma_0, \ldots, \gamma_4$, in the sequence are shown in Figure 1.

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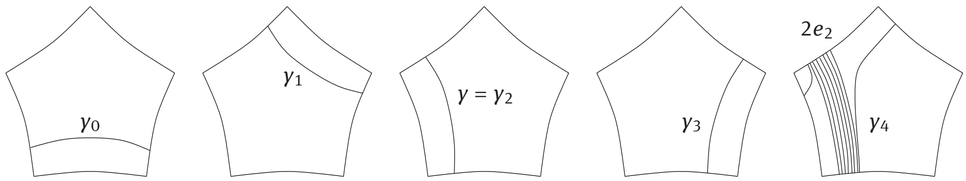

Figure 1. The curves $\gamma_0, \dots, \gamma_4$ in $S_{0,4}$. Any five consecutive curves $\gamma_{k-2}, \dots, \gamma_{k+2}$ differ from those shown here by a homeomorphism, and replacing e_2 by e_k.

Observe that for any $k \geq 3$, the four consecutive curves $\gamma_{k-2}, \dots, \gamma_{k+1}$ are just the image of $\gamma_0, \dots, \gamma_3$ under the homeomorphism

$$\Phi_{k-1} = \mathcal{D}^{e_2}\rho \cdots \mathcal{D}^{e_{k-1}}\rho.$$

Furthermore, the next curve in the sequence, γ_{k+2}, is the image of $\mathcal{D}^{e_k}\rho(\gamma_3)$. In particular, up to homeomorphism, any five consecutive curves $\gamma_{k-2}, \dots, \gamma_{k+2}$ in the sequence appear as in Figure 1 with e_2 replaced by e_k.

3.2. Intersection conditions. We now describe the general conditions, and verify that the above sequence of curves satisfies them. To begin, we fix positive integers $b_1 \leq b \leq b_2$. We will also assume that $e_0 > E + G$ (and hence by (3.1) $e_k > a^k(E + G)$ for all k), where G is the constant from Theorem 2.11 and E is the constant in Theorem 4.1 below. For the examples in $S_{0,5}$ described above, we will have $b = b_1 = b_2 = 2$.

In the next definition, $\mathcal{D}_\gamma$ is the Dehn twist in a curve γ.

Definition 3.1. Suppose that $m \leq \xi(S)$, and assume that b, b_1, b_2, a, and $\{e_k\}_{k=0}^{\infty}$ are as above. We say that a sequence of curves $\{\gamma_k\}_{k=0}^{\infty}$ on S satisfies $\mathcal{P}$ if the following properties hold for all $k \geq 0$:

(i) $\gamma_k, \dots, \gamma_{k+m-1}$ are pair-wise disjoint and distinct,

(ii) $\gamma_k, \dots, \gamma_{k+2m-1}$ fill the surface S,

(iii) $\gamma_{k+m} = \mathcal{D}_{\gamma_k}^{e_k}(\gamma'_{k+m})$, where γ'_{k+m} is a curve such that

$$i(\gamma'_{k+m}, \gamma_j) \begin{cases} \in [b_1, b_2] & \text{for } j \in \{k-m, \dots, k-1\}, \\ = b & \text{for } j = k, \\ = 0 & \text{for } j \in \{k+1, \dots, k+m-1\}, \end{cases}$$

(here we ignore any situation with $j < 0$).

We will wish to impose some additional constraints on the constant a (specifically, we will require it to be chosen so that (5.4) holds), and so in the notation we sometimes express the dependence on a writing $\mathcal{P} = \mathcal{P}(a)$. Of course, $\mathcal{P}$ depends on the choice of constants $b_1 \leq b \leq b_2$ and the sequence $\{e_k\}$, but we will impose no further constraints on the b constants, and the conditions on $\{e_k\}$ depend on a.

Here we verify that the sequence of curves on $S_{0,5}$ described above satisfies these conditions with $m = 2$. Note that the conditions are all "local", meaning that they involve a consecutive sequence of at most $2m + 1$ curves – for our example, that is a sequence of at most five

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consecutive curves. As noted above, any five consecutive curves $\gamma_{k-2}, \dots, \gamma_{k+2}$ differ from those in Figure 1 by applying the homeomorphism $\Phi_{k-1} = \mathcal{D}^{e_2}\rho \cdots \mathcal{D}^{e_{k-1}}\rho$, and changing e_2 to e_k. From this, it is straight forward to verify that this sequence satisfies these conditions.

Since $m = 2$, condition (i) says that two consecutive curves are disjoint, while condition (ii) says that four consecutive curves fill $S_{0,5}$. Note that (i) is clearly true for γ_0, γ_1 and (ii) for $\gamma_0, \dots, \gamma_3$. Since any two or four consecutive curves differ from these by a homeomorphism, conditions (i) and (ii) hold for all k.

Finally, note that $\gamma_4 = \mathcal{D}_{\gamma}^{e_2}\rho(\gamma_3)$, and so setting $\gamma_4' = \rho(\gamma_3)$ and observing that $\gamma = \gamma_2$, (iii) clearly holds for $k = 2$ by inspection of Figure 1. The case for general k follows from this figure as well, after applying Φ_{k-1}. Specifically, γ_{k+2} is obtained from $\rho(\gamma_3)$ by applying $\Phi_{k-1}\mathcal{D}_{\gamma_2}^{e_k}$, or equivalently, setting $\gamma_{k+2}' = \Phi_{k-1}(\rho(\gamma_3))$,

$$\gamma_{k+1} = \Phi_{k-1}\mathcal{D}_{\gamma_2}^{e_k}\Phi_{k-1}^{-1}(\Phi_{k-1}(\rho(\gamma_3))) = \mathcal{D}_{\Phi_{k-1}(\gamma_2)}^{e_k}(\gamma_{k+2}') = \mathcal{D}_{\gamma_k}^{e_k}(\gamma_{k+2}').$$

Since $\gamma_{k-2}, \gamma_{k-1}, \gamma_k, \gamma_{k+1}, \gamma_{k+2}', \gamma_{k+2}$ are the images of $\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4', \gamma_4$, respectively, under Φ_{k-1}, condition (iii) follows for general k by inspection of Figure 1.

Returning to the general case, we elaborate a bit on the properties in $\mathcal{P}$. First we make a simple observation.

Lemma 3.2. *For every $j, k \geq 0$ with $j \in \{k - m + 1, \dots, k\}$, we have*

$$i(\gamma_{k+m}, \gamma_j) \in [b_1, b_2],$$

moreover $i(\gamma_{k+m}, \gamma_k) = b$.

Proof. Since $\gamma_{k+m} = \mathcal{D}_{\gamma_k}^{e_k}(\gamma_{k+m}')$ and $\mathcal{D}_{\gamma_k}(\gamma_j) = \gamma_j$ (because $i(\gamma_j, \gamma_k) = 0$), it follows that

$$i(\gamma_{k+m}, \gamma_j) = i(\mathcal{D}_{\gamma_k}^{-e_k}(\gamma_{k+m}), \mathcal{D}_{\gamma_k}^{-e_k}(\gamma_j)) = i(\gamma_{k+m}', \gamma_j) \in [b_1, b_2]$$

proving the first statement. For the special case $j = k$, $i(\gamma_{k+m}', \gamma_k) = b$, and the second statement follows. □

3.3. Visualizing the conditions of $\mathcal{P}$. The conditions imposed in $\mathcal{P}$ involve intervals of length m and $2m$, as well as mod m congruence conditions. It is useful to view the tail of the sequence starting at any curve γ_i (for example, when $i = 0$ this is the entire sequence), in the following form:

$$\begin{array}{l} \gamma_i \longrightarrow \gamma_{i+1} \longrightarrow \cdots \longrightarrow \gamma_{i+m-1} \\ \longrightarrow \gamma_{i+m} \longrightarrow \gamma_{i+m+1} \longrightarrow \cdots \longrightarrow \gamma_{i+2m-1} \\ \longrightarrow \gamma_{i+2m} \longrightarrow \gamma_{i+2m+1} \longrightarrow \cdots. \end{array} \tag{3.2}$$

From the first condition of $\mathcal{P}$, all curves in any row are pairwise disjoint. Lemma 3.2 tells us that γ_i intersects the curve directly below it b times and it intersects everything in the row

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directly below it between b_1 and b_2 times. The second condition in $\mathcal{P}$ tells us that any two consecutive rows fill S. The third condition (part of which is used in the proof of Lemma 3.2), can be thought of as saying that going straight down two rows from γ_i to γ_{i+2m} gives a curve that "almost" differs by the power of the Dehn twist $\mathcal{D}_{\gamma_{i+m}}^{e_{i+m}}$. To understand this interpretation, note that γ'_{i+2m} and γ_{i+2m} differ precisely by this power of a twist, while on the other hand, each of γ'_{i+2m} and γ_i have intersection number at most b_2 with the filling set $\gamma_i, \dots, \gamma_{i+2m-1}$ (which we view as saying that γ_i and γ'_{i+2m} are "similar").

4. Curve complex quasi-geodesics

The purpose of this section is to provide general conditions (Theorem 4.1) on a sequence of subsurfaces in terms of subsurface coefficients of consecutive elements which guarantee that their boundaries define a quasi-geodesic in the curve complex of the surface. Appealing to Theorem 2.3, we deduce that such sequences determine an ending lamination. We end by proving that a sequence of curves satisfying $\mathcal{P}$ are core curves of annuli satisfying the conditions of Theorem 4.1, and hence are vertices of a quasi-geodesic in $\mathcal{C}(S)$ defining an ending lamination $\nu \in \mathcal{EL}(S)$.

Variations of this result appeared in [26], [11], [34], [22], and [5] for example. Here our conditions only involve the intersection pattern and projection coefficients of fixed number of consecutive subsurfaces along the sequence. In this sense these are local conditions.

Theorem 4.1 (Local to global). *Given a surface S and $2 \le m \le \xi(S)$, there are constants $E > C > 0$ with the following properties. Let $\{Y_k\}_{k=0}^{\infty}$ be a sequence of subsurfaces of S. Suppose that for each integer $k \ge 0$,*

(1) *the multi-curves $\partial Y_k, \dots, \partial Y_{k+m-1}$ are pairwise disjoint,*

(2) *$Y_k \pitchfork Y_j$ for all $j \in \{k+m, \dots, k+2m-1\}$,*

(3) *$d_{Y_k}(\partial Y_j, \partial Y_{j'}) > E$ for any $j \in \{k+m, \dots, k+2m-1\}$, $j' \in \{k-2m+1, \dots, k-m\}$.*

Then for every j, j', k with $j \ge k+m$ and $j' \le k-m$ we have

$$Y_k \pitchfork Y_j \text{ and } Y_k \pitchfork Y_{j'} \tag{4.1}$$

and

$$d_{Y_k}(\partial Y_j, \partial Y_{j'}) \ge d_{Y_k}(\partial Y_{k-m}, \partial Y_{k+m}) - C. \tag{4.2}$$

Furthermore, suppose that for some $n \ge 1$ and all $k \ge 0$,

(4) *the multi-curves $\partial Y_k, \dots, \partial Y_{k+2n-1}$ fill S.*

Then for any two indices $k, j \ge 0$ with $|k-j| \ge 2n$ we have

$$d_S(\partial Y_j, \partial Y_k) \ge \frac{|k-j|}{4n} - \left(\frac{m}{2n} + 1\right). \tag{4.3}$$

In the hypotheses (as well as the conclusions) of this theorem, we ignore any condition in which there is a negative index.

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Proof. Set the constants

$$C = 2B_0 + 4 + G \quad \text{and} \quad E = C + B_0 + G + 4.$$

Here B_0 is the constant from Theorem 2.9 (Behrstock inequality) and G is the constant from Theorem 2.11 (Bounded geodesic image theorem) for a geodesic (i.e. $k = 1, c = 0$). We prove (4.1) and (4.2) simultaneously by a double induction on $(j - k, k - j')$.

For the base of induction, suppose that $m \leq k - j' \leq 2m - 1$ and $m \leq j - k \leq 2m - 1$. Statement (4.1) follows from (2). To prove (4.2) note that by (1) $\partial Y_{k+m}, \ldots, \partial Y_j$ are pairwise disjoint and have non-empty projections to Y_k. Consequently, the distance in Y_k between any two of these boundaries is at most 2, and so

$$\operatorname{diam}_{Y_k}(\{\partial Y_l\}_{l=k+m}^{j}) \leq 2.$$

Similarly, $\operatorname{diam}_{Y_k}(\{\partial Y_l\}_{l=j'}^{k-m}) \leq 2$. By the triangle inequality we have

$$\begin{aligned} d_{Y_k}(\partial Y_j, \partial Y_{j'}) &\geq d_{Y_k}(\partial Y_{k-m}, \partial Y_{k+m}) - d_{Y_k}(\partial Y_j, \partial Y_{k+m}) - d_{Y_i}(\partial Y_{k-m}, \partial Y_{j'}) \\ &\geq d_{Y_k}(\partial Y_{k-m}, \partial Y_{k+m}) - 4 \geq d_{Y_k}(\partial Y_{k-m}, \partial Y_{k+m}) - C, \end{aligned}$$

which is the bound (4.2).

Suppose that (4.1) and (4.2) hold for all $m \leq k - j' \leq 2m - 1$ and $m \leq j - k \leq N$, for some $N \geq 2m - 1$. We suppose $j - k = N + 1$ and we must prove both (4.1) and (4.2) for $(j - k, k - j')$.

From the base of induction we already have $Y_k \pitchfork Y_{j'}$. To complete the proof of (4.1), we prove $Y_k \pitchfork Y_j$. Since $m = (k + m) - k \leq 2m - 1$ and $m \leq j - (k + m) = N + 1 - m \leq N$, from the inductive hypothesis we have

$$Y_k \pitchfork Y_{k+m} \quad \text{and} \quad Y_j \pitchfork Y_{k+m}$$

and

$$d_{Y_{k+m}}(\partial Y_k, \partial Y_j) \geq d_{Y_{k+m}}(\partial Y_k, \partial Y_{k+2m}) - C \geq E - C \geq 4.$$

Consequently, $i(\partial Y_k, \partial Y_j) \neq 0$ and $Y_k \pitchfork Y_j$ as required.

We now turn to the proof of (4.2). Since $Y_k \pitchfork Y_j$ and $Y_k \pitchfork Y_{j'}$, by (2) we may write the following triangle inequality:

$$d_{Y_k}(\partial Y_{j'}, \partial Y_j) \geq d_{Y_k}(\partial Y_{k-m}, \partial Y_{k+m}) - d_{Y_k}(\partial Y_{k-m}, \partial Y_{j'}) - d_{Y_k}(\partial Y_j, \partial Y_{k+m}). \tag{4.4}$$

Since $m \leq j - (k + m) = N + 1 - m \leq N$, from the inductive hypothesis we have

$$d_{Y_{k+m}}(\partial Y_k, \partial Y_j) \geq d_{Y_{k+m}}(\partial Y_k, \partial Y_{k+2m}) - C \geq E - C \geq B_0.$$

By Theorem 2.9, $d_{Y_k}(\partial Y_{k+m}, Y_j) \leq B_0$. On the other hand, as in the proof of the base case of induction, since $m \leq k - j' \leq 2m - 1$ we have

$$d_{Y_k}(\partial Y_{k-m}, \partial Y_{j'}) \leq 2.$$

Combining these two inequalities with (4.4), we obtain

$$\begin{aligned} d_{Y_k}(\partial Y_{j'}, \partial Y_j) &\geq d_{Y_k}(\partial Y_{k-m}, \partial Y_{k+m}) - B_0 - 2 \\ &\geq d_{Y_k}(\partial Y_{k-m}, \partial Y_{k+m}) - C. \end{aligned}$$

This completes the first half of the double induction.

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We now know that statements (4.1) and (4.2) hold for all j, j', k with $m \leq k - j' \leq 2m - 1$ and all $j - k \geq m$. We assume that they hold for $m \leq k - j' \leq N$ and $j - k \geq m$ for some $N \geq 2m - 1$, and prove that they hold for $k - j' = N + 1$. The proof of (4.1) is completely analogous to the proof in the first part of the induction, and we omit it. The proof of (4.2) is also similar, but requires one additional step so we give the proof.

We may again write the triangle inequality (4.4). Since $m \leq (k - m) - j' = N + 1 - m \leq N$, by the inductive hypothesis we have

$$d_{Y_{k-m}}(\partial Y_k, \partial Y_{j'}) \geq E - C \geq B_0,$$

and so Theorem 2.9 again implies $d_{Y_k}(\partial Y_{k-m}, \partial Y_{j'}) \leq B_0$. If $j - k \leq 2m - 1$, then as above $d_{Y_k}(\partial Y_{k+m}, \partial Y_j) \leq 2$. Otherwise, by induction we have

$$d_{Y_{k+m}}(\partial Y_k, \partial Y_j) \geq E - C \geq B_0$$

and Theorem 2.9 once again implies $d_{Y_k}(\partial Y_{k+m}, \partial Y_j) \leq B_0$. Combining these inequalities with (4.4), we have

$$\begin{aligned} d_{Y_k}(\partial Y_{j'}, \partial Y_j) &\geq d_{Y_k}(\partial Y_{k-m}, \partial Y_{k+m}) - B_0 - \max\{2, B_0\} \\ &\geq d_{Y_k}(\partial Y_{k-m}, \partial Y_{k+m}) - C. \end{aligned}$$

This completes the proof of (4.2), and hence the double induction is finished.

Now further assuming (4), we prove (4.3). Note that we must have $n \geq m$. Without loss of generality we assume that $j < k$, so that $k - j \geq 2n \geq 2m$. For the rest of the proof, for any $s, r \in \mathbb{Z}$, $s \leq r$, we write $[s, r] = \{t \in \mathbb{Z} \mid s \leq t \leq r\}$.

Suppose that δ is any multi-curve. Let $\mathcal{I}(\delta) = \{s \in [j, k] \mid i(\delta, \partial Y_s) \neq 0\}$.

Claim 4.2. *Suppose that $s', r' \in [j, k] \setminus \mathcal{I}(\delta)$. Then $|r' - s'| \leq 4n - 2$.*

Observe that by the claim, $[j, k] \setminus \mathcal{I}(\delta)$ contains fewer than $4n$ integers.

Proof. Without loss of generality, we assume that $s' < r'$, and suppose for a contradiction that $r' - s' \geq 4n - 1$. By (4), $\partial Y_{s'+n}, \ldots, \partial Y_{s'+3n-1}$ fills S, and so there exists t with $s' + n \leq t \leq s' + 3n - 1$ and $t \in \mathcal{I}(\delta)$.

Now observe that $s' + m \leq s' + n \leq t$ and $t \leq s' + 3n - 1 \leq r' - n \leq r' - m$; by the first part of the theorem we know that

$$d_{Y_t}(\partial Y_{s'}, \partial Y_{r'}) \geq E - C > 4.$$

On the other hand, since $i(\delta, \partial Y_{s'}) = 0 = i(\delta, \partial Y_{r'})$, and since $t \in \mathcal{I}(\delta)$ implies $\pi_{Y_t}(\delta) \neq \emptyset$, the triangle inequality implies

$$d_{Y_t}(\partial Y_{s'}, \partial Y_{r'}) \leq d_{Y_t}(\partial Y_{s'}, \delta) + d_{Y_t}(\delta, \partial Y_{r'}) \leq 2 + 2 = 4,$$

a contradiction. □

Let η be a geodesic in $\mathcal{C}(S)$ connecting ∂Y_j to ∂Y_k. For any $l \in \{j + m, \ldots, k - m\}$, by (4.2) we have that

$$d_{Y_l}(\partial Y_j, \partial Y_k) \geq E - C > G.$$

Thus Theorem 2.11 guarantees that there is a curve $\delta_l \in \eta$ disjoint from Y_l. Choose one such $\delta_l \in \eta$ for each $l \in [j + m, k - m]$. By the previous claim there are at most $4n$ integers $l' \in [j + m, k - m]$ such that $i(\delta_l, \partial Y_{l'}) = 0$, and hence $l \mapsto \delta_l$ is at most $4n$-to-1.

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Therefore, η contains at least $\frac{k-j-2m+1}{4n} > \frac{k-j}{4n} - \frac{m}{2n}$ curves. It follows that

$$d_S(\partial Y_j, \partial Y_k) \geq \frac{k-j}{4n} - \left(\frac{m}{2n} + 1\right)$$

proving (4.3). This completes the proof of the theorem. □

Theorem 4.3. *Let $\{Y_k\}_{k=0}^{\infty}$ be an infinite sequence of subsurfaces satisfying conditions* (1)–(4) *in Theorem* 4.1. *Then there exists a unique $\nu \in \mathcal{EL}(S)$ so that any accumulation point of $\{\partial Y_k\}_{k=0}^{\infty}$ in $\mathcal{PML}(S)$ is supported on ν.*

Proof. By Theorem 4.1, inequality (4.3), the sequence $\{\partial Y_k\}_{k=0}^{\infty}$ is (multi-curve) quasi-geodesic in $\mathcal{C}(S)$. Furthermore, $\mathcal{C}(S)$ is δ-hyperbolic. Thus the sequence converges to a point in the Gromov boundary of $\mathcal{C}(S)$. Theorem 2.3 completes the proof. □

We complete this section by showing that $\mathcal{P}$ is sufficient to imply the hypotheses of Theorem 4.1. Given a curve α and an annular subsurface Y_β with core curve β, we note that $\alpha \pitchfork Y_\beta$ if and only if $i(\alpha, \beta) \neq 0$. Consequently, to remind the reader of the relation to Theorem 4.1, we write $\alpha \pitchfork \beta$ to mean $i(\alpha, \beta) \neq 0$.

Proposition 4.4. *All curves in a sequence $\{\gamma_k\}_{k=0}^{\infty}$ satisfying $\mathcal{P}(a)$ with $a > 2$ and $e_0 \geq E$ are the core curves of annuli satisfying conditions* (1)–(4) *of Theorem* 4.1 *with $n = m$. Consequently, $\{\gamma_k\}_{k=0}^{\infty}$ is a* 1*-Lipschitz, $(4m, \frac{3}{2})$-quasi-geodesic in $\mathcal{C}(S)$ and there exists $\nu \in \mathcal{EL}(S)$ so that any accumulation point of $\{\gamma_k\}_{k=0}^{\infty}$ in $\mathcal{PML}(S)$ is supported on ν.*

Proof. Condition (i) of $\mathcal{P}$ is the same as condition (1) of Theorem 4.1, while (ii) is just condition (4) with $n = m$. Condition (2) follows from Lemma 3.2. Finally, to see that condition (3) is satisfied, we note that $d_{\gamma_k}(\gamma_{k-m}, \gamma_{k+m}) \geq e_k > a^k E > 2E > E$ for all $k \geq m$. Furthermore, for $k - 2m + 1 \leq j \leq k - m$, $\gamma_j \pitchfork \gamma_k$ by Lemma 3.2, and similarly $\gamma_{j'} \pitchfork \gamma_k$, for $k + m \leq j' \leq k + 2m - 1$. For j and j' in these intervals, we obtain $i(\gamma_j, \gamma_{k-m}) = 0$ and $i(\gamma_{j'}, \gamma_{k+m}) = 0$. Therefore, by the triangle inequality, we have $d_{\gamma_k}(\gamma_j, \gamma_{j'}) \geq a^k E - 2 > E$, as required by (3). □

4.1. Subsurface coefficient bounds. We will need estimates on all subsurface coefficients for a sequence satisfying $\mathcal{P}$. This follows from what we have done so far, together with similar arguments.

Proposition 4.5. *Given a sequence $\{\gamma_k\}_{k=0}^{\infty}$ satisfying $\mathcal{P}(a)$ with $a > 2$ and $e_0 \geq E$, then there exists an $R > 0$ with the following properties:*

(1) *If i, j, k satisfy $j \leq i - m$ and $i + m \leq k$, then $\gamma_i \pitchfork \gamma_k$, $\gamma_i \pitchfork \gamma_j$, and*

$$d_{\gamma_i}(\gamma_j, \gamma_k) \stackrel{+}{\asymp}_R e_i \quad \text{and} \quad d_{\gamma_i}(\gamma_j, \nu) \stackrel{+}{\asymp}_R e_i. \tag{4.5}$$

(2) *If $W \subsetneq S$ is a proper subsurface, $W \neq \gamma_i$ for any i, then for any j, k with $\gamma_j \pitchfork W$ and $\gamma_k \pitchfork W$,*

$$d_W(\gamma_j, \gamma_k) < R \quad \text{and} \quad d_W(\gamma_j, \nu) < R. \tag{4.6}$$

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Let μ be a marking on S. Then there is a constant $R(\mu)$ so that:

- *For any k sufficiently large and $i \le k - m$ we have*

$$d_{\gamma_i}(\mu, \gamma_k) \stackrel{+}{\asymp}_{R(\mu)} e_i \quad \text{and} \quad d_{\gamma_i}(\mu, \nu) \stackrel{+}{\asymp}_{R(\mu)} e_i. \tag{4.7}$$

- *For any proper subsurface $W \ne \gamma_i$ for any i we have*

$$d_W(\mu, \gamma_k) < R(\mu) \quad \text{and} \quad d_W(\mu, \nu) < R(\mu). \tag{4.8}$$

Proof. We begin with the proofs of (4.5) and (4.6). First note that since any accumulation point of $\{\gamma_k\}$ in $\mathcal{PML}(S)$ is supported on ν, any Hausdorff accumulation point of $\{\gamma_k\}$ contains ν. Thus, for any fixed, proper subsurface $W \subsetneq S$ and all sufficiently large k we have $\pi_W(\nu) \subseteq \pi_W(\gamma_k)$. Furthermore, since ν is an ending lamination, $\pi_W(\nu) \ne \emptyset$, and hence $d_W(\gamma_k, \nu) \le 1$, for k sufficiently large. Therefore, for each of (4.5) and (4.6), the statement on the left implies the one on the right after increasing the constant by at most 1. Thus it suffices to prove the two statements on the left.

We begin with (4.5). From the conditions in $\mathcal{P}$, we have $d_{\gamma_i}(\gamma_{i-m}, \gamma_{i+m}) = e_i$. By Theorem 4.1 (which is applicable according to Proposition 4.4), $\{\gamma_l\}_{l=i+m}^{k}$ is a 1-Lipschitz $(4m, 3/2)$-quasi-geodesic such that every curve has nonempty projection to γ_i. Therefore, by Theorem 2.11 and the triangle inequality we have

$$|d_{\gamma_i}(\gamma_{i-m}, \gamma_k) - d_{\gamma_i}(\gamma_{i-m}, \gamma_{i+m})| \le d_{\gamma_i}(\gamma_{i+m}, \gamma_k) \le G.$$

Note that G depends only on m. Similar reasoning implies

$$|d_{\gamma_i}(\gamma_j, \gamma_k) - d_{\gamma_i}(\gamma_{i-m}, \gamma_k)| \le d_{\gamma_i}(\gamma_j, \gamma_{i-m}) \le G.$$

Combining these, we have

$$\begin{aligned} |d_{\gamma_i}(\gamma_j, \gamma_k) - d_{\gamma_i}(\gamma_{i-m}, \gamma_{i+m})| &= |d_{\gamma_i}(\gamma_j, \gamma_k) - d_{\gamma_i}(\gamma_{i-m}, \gamma_k) \\ &\qquad + d_{\gamma_i}(\gamma_{i-m}, \gamma_k) - d_{\gamma_i}(\gamma_{i-m}, \gamma_{i+m})| \\ &\le 2G. \end{aligned}$$

It follows that $d_{\gamma_i}(\gamma_j, \gamma_k) \stackrel{+}{\asymp}_{2G} e_i$. For $R \ge 2G$, (4.5) holds.

We now move on to the inequalities in (4.6), and without loss of generality assume that $j \le k$. If $k \le j + 2m - 1$, then the conditions in $\mathcal{P}$ together with Lemma 3.2 imply $i(\gamma_j, \gamma_k) \le b_2$, so by Lemma 2.5, $d_W(\gamma_j, \gamma_k) \le 2b_2 + 1$.

Next, suppose that $k = j + 2m$. Let γ'_k be the element guaranteed by $\mathcal{P}$, so that

$$\gamma_k = \mathcal{D}_{\gamma_{k-m}}^{e_{k-m}}(\gamma'_k).$$

There are two cases to consider depending on whether $\gamma'_k \not\pitchfork W$ or $\gamma'_k \pitchfork W$. If $\gamma'_k \not\pitchfork W$, then since $\gamma_k = \mathcal{D}_{\gamma_{k-m}}^{e_{k-m}}(\gamma'_k) \pitchfork W$, we must have $\gamma_{k-m} \pitchfork W$. Now observe that

$$j \le k - m = j + m \le j + 2m - 1 \quad \text{and} \quad k - m \le k \le k - m + 2m - 1.$$

It follows from the previous paragraph that

$$d_W(\gamma_j, \gamma_{k-m}) \le 2b_2 + 1 \quad \text{and} \quad d_W(\gamma_{k-m}, \gamma_k) \le 2b_2 + 1,$$

hence

$$d_W(\gamma_j, \gamma_k) \le 4b_2 + 2.$$

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Now suppose that $\gamma'_k \pitchfork W$. If $\gamma_{k-m} \pitchfork W$, then just as in the first case we have

$$d_W(\gamma_j, \gamma_k) \leq 4b_2 + 2.$$

Suppose then that $\gamma_{k-m} \not\pitchfork W$. If W is not an annulus, then $\pi_W(\gamma_k) = \pi_W(\gamma'_k)$ since $\mathcal{D}_{\gamma_{k-m}}$ is supported outside W. Therefore

$$d_W(\gamma_j, \gamma_k) = d_W(\gamma_j, \gamma'_k) \leq 2b_2 + 1$$

since $i(\gamma_j, \gamma'_k) \leq b_2$. If W is an annulus, because $W \neq \gamma_{k-m}$ and $\gamma_{k-m} \not\pitchfork W$, it easily follows that

$$d_W(\gamma_j, \gamma_k) \leq d_W(\gamma_j, \gamma'_k) + d_W(\gamma'_k, \gamma_k) \leq (2b_2 + 1) + 1$$

(see e.g. [12]). Therefore, we have shown that if $k \leq j + 2m$, we have

$$d_W(\gamma_j, \gamma_k) \leq 4b_2 + 2. \tag{4.9}$$

Now we suppose that $k > j + 2m$. Setting $\delta = \partial W$, as in the proof of Theorem 4.1 we let $\mathcal{I}(\delta) = \{s \in [j,k] \mid i(\delta, \gamma_s) \neq 0\}$. Similarly, we let $\mathcal{I}(W) = \{s \in [j,k] \mid \gamma_s \pitchfork W\}$, and observe that $\mathcal{I}(\delta) \subseteq \mathcal{I}(W)$.

Note that $j, k \in \mathcal{I}(W)$, and we let $s \leq r$ be such that $[j,s], [r,k] \subseteq \mathcal{I}(W)$ are maximal subintervals of $\mathcal{I}(W)$ containing j and k, respectively (if $\mathcal{I}(W) = [j,k]$, we can arbitrarily choose $j \leq s < k$ and $r = s+1$ for the argument below). By our choice of r and s, it follows that $s+1, r-1 \notin \mathcal{I}(W)$, and so Claim 4.2 implies $r - 1 - (s+1) \leq 4m - 2$ and hence $r - s \leq 4m$.

Note that since any $2m$ consecutive curves fill S, either $r - s \leq 2m$, or else there exists $s', r' \in \mathcal{I}(W)$ such that $s < s' \leq r' < r$ and $r - r', r' - s', s' - s \leq 2m$. For example, consider the extremal case that $r - s = 4m$. Then

$$s' = \max \mathcal{I}(W) \cap [s, s+2m] \quad \text{and} \quad r' = \min \mathcal{I}(W) \cap [s+2m, r]$$

have the desired properties. Indeed, $s' - s, r - r'$ are clearly less than $2m$. If $r' - s' > 2m$, then since any $2m$ consecutive curves fill S, there must be some $s' < u < r'$ in $\mathcal{I}(W)$, contradicting the choice of either s' or r'. The general case is similar.

By the triangle inequality and (4.9) we have

$$d_W(\gamma_s, \gamma_r) \leq d_W(\gamma_s, \gamma_{s'}) + d_W(\gamma_{s'}, \gamma_{r'}) + d_W(\gamma_{r'}, \gamma_r) \leq 12b_2 + 6. \tag{4.10}$$

Since $\{\gamma_l\}_{l=j}^{s}$ and $\{\gamma_l\}_{l=r}^{k}$ are 1-Lipschitz $(4m, 3/2)$-quasi-geodesics with $\gamma_l \pitchfork W$ for all $l \in [j,s] \cup [r,k]$, we can apply Theorem 2.11, and so the triangle inequality and (4.10) give us

$$d_W(\gamma_j, \gamma_k) \leq d_W(\gamma_j, \gamma_s) + d_W(\gamma_s, \gamma_r) + d_W(\gamma_r, \gamma_k) \leq 2G + 12b_2 + 6.$$

So the inequality on the left of (4.6) holds for any $R \geq 2G + 12b_2 + 6$. This completes the proof of the first four estimates.

Given a marking μ, note that the intersection number of any curve in μ and any of the curves in the set of filling curves $\gamma_0, \ldots, \gamma_{2m-1}$ is bounded. Then the estimates in (4.7) follow from the ones in (4.5) and Lemma 2.5 respectively. Similarly the estimates in (4.6) follow from the ones in (4.8). $\square$

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5. Measures supported on laminations

In this section we begin by proving intersection number estimates for a sequence of curves satisfying $\mathcal{P}$. Using these estimates, we decompose the sequence into m subsequences and prove that these converge in $\mathcal{PML}(S)$. In the next section, we will show that these m limits are precisely the vertices of the simplex of measures on the single topological lamination ν from Proposition 4.4.

5.1. Intersection number estimates. Here we estimate the intersection numbers of curves in the sequence of curves $\{\gamma_k\}_{k=0}^{\infty}$ satisfying $\mathcal{P}$. The estimates will be in terms of the constant b and sequence $\{e_k\}$ fixed above. Specifically, given $i, k \in \mathbb{N}$ with $k \geq i$, define

$$A(i,k) := \prod_{\substack{i+m \leq j < k \text{ and} \\ j \equiv k \bmod m}} b e_j. \tag{5.1}$$

When the set of indices of the product is the empty set, we define the product to be 1. It is useful to observe that for $k \geq i + 2m$,

$$A(i,k) = b e_{k-m} A(i, k-m).$$

It is also useful to arrange the indices as in (3.2) in the following form:

$$\begin{array}{cccc} i & i+1 & \cdots & i+m-1 \\ i+m & i+m+1 & \cdots & i+2m-1 \\ i+2m & i+2m+1 & \cdots & \end{array} \tag{5.2}$$

Then $A(i,k)$ is 1 exactly when k is in the first or second row. If k is below these rows, then the product defining $A(i,k)$ is over all indices j directly above k, up to and including the entry in the second row.

We now state the main estimate on intersection numbers.

Theorem 5.1. *Suppose that $\{\gamma_k\}_{k=0}^{\infty}$ is a sequence on a surface S satisfying $\mathcal{P}(a)$. For a is sufficiently large, there is a constant $\kappa = \kappa(a) > 1$, so that for each i, k with $k \geq i + m$ we have*

$$i(\gamma_i, \gamma_k) \overset{*}{\asymp}_{\kappa} A(i,k). \tag{5.3}$$

Recall that for $i \leq k < i + m$, $i(\gamma_i, \gamma_k) = 0$. Combining this with the theorem gives estimates on all intersection numbers $i(\gamma_i, \gamma_k)$, up to a uniform multiplicative error.

Throughout all that follows, we will assume that a sequence of curves $\{\gamma_k\}_{k=0}^{\infty}$ satisfies $\mathcal{P} = \mathcal{P}(a)$ for $a > 1$.

Outline of the proof. The proof is rather complicated involving multiple induction arguments, so we sketch the approach before diving into the details. The upper bound on $i(\gamma_i, \gamma_k)$ is proved first, and is valid for any $a > 1$. We start by recursively defining a function $K(i,k)$ for all nonnegative integers $i \leq k$. By induction, we will prove that

$$i(\gamma_i, \gamma_k) \leq K(i,k) A(i,k).$$

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By a second induction, we will bound $K(i,k) \le K_1 = K_1(a)$, with the bound $K_1(a)$ a decreasing function of a. Next, we will recursively define a function $K'(i,k) = K'(i,k,a)$. By another induction, we prove that

$$i(\gamma_i, \gamma_k) \ge K'(i,k)A(i,k).$$

For a sufficiently large, we prove $K'(i,k,a) \ge K_2 = K_2(a) > 0$. Setting $\kappa = \max\{K_1, \frac{1}{K_2}\}$ will prove the theorem.

Upper bound. Recall from $\mathcal{P}$ (Definition 3.1) that for any $k \ge 2m$, the set of curves $\{\gamma_l\}_{l=k-2m}^{k-1}$ fill the surface, and the curve γ'_k intersects each of these at most b_2 times. Consequently, all complementary components of $S \setminus (\gamma_{k-2m} \cup \cdots \cup \gamma_{k-1})$ are either disks or once-punctured disks containing at most $2mb_2$ pairwise disjoint arcs of γ'_k. In examples we may have many fewer than $2mb_2$ such arcs, and it is useful to keep track of this constant on its own. Consequently, we set

$$B \le 2mb_2$$

to be the maximum number of arcs in any complementary component (over all configurations in minimal position).

We are now ready for a recursive definition which will be used in the bounds on intersection numbers (it is useful again to picture the indices as in (5.2)):

$$K(i,k) = \begin{cases} 0 & \text{for } i \le k < i+m, \\ b_2 & \text{for } i+m \le k < i+2m, \\ K(i,k-m) + 2B\sum_{l=k-2m}^{k-1} \frac{A(i,l)}{A(i,k)} K(i,l) & \text{for } i+2m \le k. \end{cases}$$

Lemma 5.2. *For all $i \le k$, we have $i(\gamma_i, \gamma_k) \le K(i,k)A(i,k)$.*

The proof takes advantage of the following well-known estimate on the intersection of two curves after applying a power of a Dehn twist on one proved in [13, Exposé 4, Appendix A], see also [19, Section 4, Lemma 4.2].

Proposition 5.3 (Intersection number after Dehn twist). *Let δ, δ', and β be curves in $\mathcal{C}(S)$. Then for any integer e*

$$|i(\mathcal{D}_\beta^e \delta, \delta') - |e| i(\beta, \delta) i(\beta, \delta')| \le i(\delta, \delta').$$

As above, $\mathcal{D}_\beta$ is a Dehn twist in β. This proposition has the following general application to intersection numbers of curves with the curves in our sequence.

Proposition 5.4. *For any curve δ and any $k \ge 2m$, we have*

$$|i(\delta, \gamma_k) - be_{k-m} i(\delta, \gamma_{k-m})| \le 2B \sum_{l=k-2m}^{k-1} i(\delta, \gamma_l).$$

Proof. Since $\gamma_k = \mathcal{D}_{\gamma_{k-m}}^{e_{k-m}}(\gamma'_k)$, Proposition 5.3 implies

$$|i(\delta, \gamma_k) - be_{k-m} i(\delta, \gamma_{k-m})| \le i(\delta, \gamma'_k).$$

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Assume all curves intersect minimally transversely and that there are no triple points of intersection. From the definition of B, all complementary components of $S \setminus (\gamma_{k-2m} \cup \cdots \cup \gamma_{k-1})$ contain at most B pairwise disjoint arcs of γ'_k. Therefore, between any two consecutive intersection points of δ with $\gamma_{k-2m} \cup \cdots \cup \gamma_{k-1}$, there are at most $2B$ intersections points with γ'_k (any two arcs in a disk component can intersect at most once, and in a once-punctured disk component can intersect in at most two points). Therefore,

$$i(\delta, \gamma'_k) \leq 2B \sum_{l=k-2m}^{k-1} i(\delta, \gamma_l).$$

Combining this with the above inequality proves the proposition. □

Proof of Lemma 5.2. Fix i. The proof is by induction on k. For $i \leq k < i + m$,

$$i(\gamma_i, \gamma_k) = 0, \quad K(i,k) = 0, \quad A(i,k) = 1,$$

so the lemma follows. Similarly, for $i + m \leq k < i + 2m$, $i(\gamma_i, \gamma_k) \leq b_2$, $K(i,k) = b_2$, and $A(i,k) = 1$, so again the lemma follows. Now suppose that $k \geq i + 2m$, and assuming that $i(\gamma_i, \gamma_l) \leq K(i,l)A(i,l)$ for all $i \leq l < k$, we must prove $i(\gamma_i, \gamma_k) \leq K(i,k)A(i,k)$.

Applying Proposition 5.4 to the case $\delta = \gamma_i$, we have

$$|i(\gamma_i, \gamma_k) - be_{k-m} i(\gamma_i, \gamma_{k-m})| \leq 2B \sum_{l=k-2m}^{k-1} i(\gamma_i, \gamma_l).$$

Therefore, we have

$$i(\gamma_i, \gamma_k) \leq be_{k-m} i(\gamma_i, \gamma_{k-m}) + 2B \sum_{l=k-2m}^{k-1} i(\gamma_i, \gamma_l).$$

Applying the inductive hypothesis and the definitions of A and K to this inequality, we obtain

$$\begin{aligned}
i(\gamma_i, \gamma_k) &\leq be_{k-m} i(\gamma_i, \gamma_{k-m}) + 2B \sum_{l=k-2m}^{k-1} i(\gamma_i, \gamma_l) \\
&\leq be_{k-m} K(i, k-m) A(i, k-m) + 2B \sum_{l=k-2m}^{k-1} K(i,l)A(i,l) \\
&= A(i,k)K(i,k-m) + A(i,k)2B \sum_{l=k-2m}^{k-1} \frac{A(i,l)}{A(i,k)} K(i,l) \\
&= A(i,k)\left(K(i,k-m) + 2B \sum_{l=k-2m}^{k-1} \frac{A(i,l)}{A(i,k)} K(i,l) \right) \\
&= A(i,k)K(i,k),
\end{aligned}$$

as required. □

Next we prove that $K(i,k)$ is uniformly bounded, and in particular:

Proposition 5.5. *There exists $K_1 = K_1(a) > 0$ so that for all $i \leq k$, $K(i,k) \leq K_1$ and in particular, $i(\gamma_i, \gamma_k) \leq K_1 A(i,k)$. As a function of a, $K_1(a)$ is decreasing.*

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For the proof of this proposition, we will need the following bound.

Lemma 5.6. *For all $i \le l < k$, we have*

$$\frac{A(i,l)}{A(i,k)} \le a^{1-\left\lfloor \frac{k-i}{m} \right\rfloor}.$$

Proof. If $k < i + 2m$, then $A(i,l), A(i,k) = 1$ and $a^{1-\left\lfloor \frac{k-i}{m} \right\rfloor} \ge 1$, so the inequality follows.

Now assume that $k \ge i + 2m$. By definition, we have

$$\frac{A(i,l)}{A(i,k)} = \frac{\displaystyle\prod_{\substack{i+m \le j' < l \text{ and} \\ j' \equiv l \bmod m}} be_{j'}}{\displaystyle\prod_{\substack{i+m \le j < k \text{ and} \\ j \equiv k \bmod m}} be_j}$$

(where $A(i,l)$ is 1 if $l < i + 2m$). Observe that the denominator has

$$r = \left\lfloor \frac{k-(i+m)}{m} \right\rfloor = \left\lfloor \frac{k-i}{m} \right\rfloor - 1 > 0$$

terms in the product, indexed by $j \in \{k-m, k-2m, \ldots, k-rm\}$, while the numerator has

$$s = \max\left\{0, \left\lfloor \frac{l-i}{m} \right\rfloor - 1\right\} \ge 0$$

terms, indexed by $j' \in \{l-m, l-2m, \ldots, l-sm\}$ (possibly the empty set). Since $l < k$, we have $s \le r$. Moreover, we have $k - pm > l - pm$, and thus $e_{k-pm} > ae_{l-pm}$ by (3.1), for all $p = 1, \ldots, s$. Since (3.1) also implies $e_j > a$ for all $j \ge 1$, combining these bounds with the equation above gives

$$\frac{A(i,l)}{A(i,k)} = \prod_{p=1}^{s} \frac{e_{l-pm}}{e_{k-pm}} \prod_{p=s+1}^{r} \frac{1}{e_{k-pm}} < \prod_{p=1}^{s} a^{-1} \prod_{p=s+1}^{r} a^{-1} = a^{-r} = a^{1-\left\lfloor \frac{k-i}{m} \right\rfloor},$$

as required. □

As an application, of Lemma 5.6, we prove

Lemma 5.7. *For all $i \le k$ we have*

$$K(i,k) \le b_2 \prod_{i+m \le j < k} \left(1 + 4mBa^{1-\left\lfloor \frac{j-i+1}{m} \right\rfloor}\right).$$

As above, the empty product is declared to be 1.

Proof. The proof is by induction on k. Since $K(i,k) \le b_2$ for $i \le k < i + 2m$, the lemma clearly holds for all such k. Now assume that $k \ge i + 2m$, and assume that the lemma holds for all integers less than k and at least i. Let l_0 be such that $k - 2m \le l_0 \le k - 1$ and

$$K(i,l_0) = \max\{K(i,l) \mid k - 2m \le l \le k - 1\}.$$

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From this, the definition of $K(i,k)$, and from Lemma 5.6 we have

$$
\begin{aligned}
K(i,k) &= K(i,k-m) + 2B \sum_{l=k-2m}^{k-1} \frac{A(i,l)}{A(i,k)} K(i,l) \\
&\leq K(i,l_0)\left(1 + 2B \sum_{l=k-2m}^{k-1} a^{1-\lfloor \frac{k-i}{m} \rfloor}\right) \\
&= K(i,l_0)\Big(1 + 4mBa^{1-\lfloor \frac{k-i}{m} \rfloor}\Big).
\end{aligned}
$$

Since $l_0 < k$, the proposed bound on $K(i,l_0)$ holds by the inductive assumption. Next, observe that the proposed upper bound is an increasing function of k. Indeed, the required bound for $K(i,k)$ is obtained from the one for $K(i,k-1)$ by multiplying by a number greater than or equal to 1. By this monotonicity, the above bound implies

$$
\begin{aligned}
K(i,k) &\leq K(i,l_0)\Big(1 + 4mBa^{1-\lfloor \frac{k-i}{m} \rfloor}\Big) \\
&\leq \left(b_2 \prod_{i+m \leq j < k-1} \Big(1 + 4mBa^{1-\lfloor \frac{j-i+1}{m} \rfloor}\Big)\right)\Big(1 + 4mBa^{1-\lfloor \frac{k-i}{m} \rfloor}\Big) \\
&= b_2 \prod_{i+m \leq j < k} \Big(1 + 4mBa^{1-\lfloor \frac{j-i+1}{m} \rfloor}\Big).
\end{aligned}
$$

This completes the proof. □

Proof of Proposition 5.5. The upper bound on $K(i,k)$ in Lemma 5.7 is itself bounded above by the infinite product

$$
K_1(a) = b_2 \prod_{j=i+m}^{\infty} \Big(1 + 4mBa^{1-\lfloor \frac{j-i+1}{m} \rfloor}\Big) = b_2 \prod_{l=0}^{\infty}\Big(1 + 4mBa^{-\lfloor \frac{l+1}{m} \rfloor}\Big),
$$

where we have substituted $l = j - i - m$. We will be done if we prove that this product is convergent, for all $a > 1$, since the product then clearly defines a decreasing function of a.

The infinite product converges if and only if the infinite series obtained by taking logarithms does. Since $\log(1+x) \leq x$, we have

$$
\begin{aligned}
\log\left(b_2 \prod_{l=0}^{\infty}\Big(1 + 4mBa^{-\lfloor \frac{l+1}{m} \rfloor}\Big)\right) &= \log(b_2) + \sum_{l=0}^{\infty} \log\Big(1 + 4mBa^{-\lfloor \frac{l+1}{m} \rfloor}\Big) \\
&\leq \log(b_2) + 4mB \sum_{l=0}^{\infty} a^{-\lfloor \frac{l+1}{m} \rfloor}.
\end{aligned}
$$

The last expression is essentially a geometric series, and hence converges for all $a > 1$, completing the proof. □

Lower bound. Let b_1 be the constant in $\mathcal{P}$ (Definition 3.1). We assume $a > 1$ is sufficiently large so that

$$
C = 8mBK_1 \sum_{j=1}^{\infty} a^{-j} < b_1 \tag{5.4}
$$

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(which is possible since $K_1 = K_1(a)$ is decreasing by Proposition 5.5). For all $k \geq i+m$, define the function $K'(i,k)$ by the following recursive formula for all $k \geq i+m$:

$$K'(i,k) = \begin{cases} C & \text{for } i+m \leq k < i+2m, \\ K'(i,k-m) - 2B\sum_{l=k-2m}^{k-1} \frac{A(i,l)}{A(i,k)} K(i,l) & \text{for } i+2m \leq k. \end{cases}$$

Lemma 5.8. *For all $k \geq i+m$, we have $i(\gamma_i,\gamma_k) \geq K'(i,k)A(i,k)$.*

Proof. Fix an integer $i \geq 0$. The proof is by induction on k. For the base case, we let $i+m \leq k < i+2m$. Then $A(i,k) = 1$ and $K'(i,k) = C < b_1$, while $i(\gamma_i,\gamma_k) \geq b_1$, and hence $i(\gamma_i,\gamma_k) \geq K'(i,k)A(i,k)$. We assume therefore that $k \geq i+2m$ and that the lemma is true for all $i+m \leq l < k$.

Applying Proposition 5.4 to the curve $\delta = \gamma_i$, together with Lemma 5.2 and the inductive hypothesis we have

$$\begin{aligned} i(\gamma_i,\gamma_k) &\geq e_{k-m} b\, i(\gamma_i,\gamma_{k-m}) - 2B \sum_{l=k-2m}^{k-1} i(\gamma_i,\gamma_l) \\ &\geq e_{k-m} b K'(i,k-m)A(i,k-m) - 2B \sum_{l=k-2m}^{k-1} K(i,l)A(i,l) \\ &= A(i,k)\left(K'(i,k-m) - 2B \sum_{l=k-2m}^{k-1} \frac{A(i,l)}{A(i,k)} K(i,l)\right) \\ &= A(i,k)K'(i,k), \end{aligned}$$

as required. □

Lemma 5.9. *Set $K_2 = C/2 > 0$. Then whenever $k \geq i+m$, $K'(i,k) \geq K_2$.*

Proof. If $i+m \leq k < i+2m$, then $K'(i,k) = C > C/2 = K_2 > 0$. Suppose now that $k \geq i+2m$, and let $k = p+sm$, where s and p are positive integers with $i+m \leq p < i+2m$ and $p \equiv k \bmod m$. Note that

$$\left\lfloor \frac{k-i}{m} \right\rfloor = \left\lfloor \frac{p+sm-i}{m} \right\rfloor = s + \left\lfloor \frac{p-i}{m} \right\rfloor = s+1.$$

By Lemma 5.6, it follows that for all $l < k$, we have $\frac{A(i,l)}{A(i,k)} \leq a^{-s}$. Then from the definition of K' and Proposition 5.5 we have

$$\begin{aligned} K'(i,k) &= K'(i,k-m) - 2B \sum_{l=k-2m}^{k-1} \frac{A(i,l)}{A(i,k)} K(i,l) \\ &\geq K'(i,k-m) - 2B \sum_{l=k-2m}^{k-1} a^{-s} K_1 \\ &\geq K'(i,k-m) - 2B(2m)a^{-s}K_1 \\ &= K'(i,k-m) - 4mBK_1 a^{-s}. \end{aligned}$$

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Iterating this inequality s times implies

$$K'(i,k) \geq K'(i,p) - 4mBK_1 \sum_{q=1}^{s} a^{-q}.$$

Since $i + m \leq p < i + 2m$, $K'(i,p) = C = 8mBK_1 \sum_{j=1}^{\infty} a^{-j}$ and hence

$$K'(i,k) \geq 4mBK_1 \left(2 \sum_{j=1}^{\infty} a^{-j} - \sum_{q=1}^{s} a^{-q} \right) \geq 4mBK_1 \sum_{j=1}^{\infty} a^{-j} = \frac{C}{2} = K_2.$$

This completes the proof. □

Proof of Theorem 5.1. For $a > 1$ satisfying (5.4), we have proved that for all $k \geq i + m$,

$$K_2 A(i,k) \leq i(\gamma_i, \gamma_k) \leq K_1 A(i,k).$$

Since $K_1, K_2 > 0$, setting $\kappa = \max\{K_1, \frac{1}{K_2}\}$ finishes the proof. □

Convention. From this point forward, we will assume that $\mathcal{P} = \mathcal{P}(a)$ always has $a > 1$ sufficiently large so that (5.4) is satisfied, and consequently the intersection numbers of curves in any sequence $\{\gamma_k\}_{k=0}^{\infty}$ satisfies (5.3) in Theorem 5.1. For concreteness, we note that from equation (5.4), $a \geq 16 > 2$ (though in fact, it is much larger).

5.2. Convergence in $\mathcal{ML}(S)$. Consider again a sequence of curves $\{\gamma_k\}_{k=0}^{\infty}$ which satisfies the conditions of Theorem 5.1. Let $\nu \in \mathcal{EL}(S)$ be the lamination from Proposition 4.4. In this subsection we will prove this sequence naturally splits into m convergent subsequences in $\mathcal{PML}(S)$.

For each $h = 0, \ldots, m-1$ and $i \in \mathbb{N}$ let

$$c_i^h = A(0, im + h) = \prod_{j=1}^{i-1} be_{jm+h}, \tag{5.5}$$

where A is defined in (5.1).

For each $h = 0, 1, \ldots, m-1$, define the subsequence γ_i^h of the sequence $\{\gamma_k\}_{k=0}^{\infty}$ by

$$\gamma_i^h = \gamma_{im+h}. \tag{5.6}$$

The main result of this section is the following theorem.

Theorem 5.10. *Suppose that $\{\gamma_k\}_{k=0}^{\infty}$ satisfies $\mathcal{P}$. Then for each $h = 0, 1, \ldots, m-1$, there exists a transverse measure $\bar{\nu}^h$ on ν so that*

$$\lim_{i \to \infty} \frac{\gamma_i^h}{c_i^h} = \bar{\nu}^h$$

in $\mathcal{ML}(S)$, where γ_i^h and c_i^h are as above.

We will need the following generalization of Theorem 5.1.

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Lemma 5.11. *For any curve δ, there exists $\kappa(\delta) > 0$ and $N(\delta) > 0$ so that for all $k \geq N(\delta)$,*

$$i(\delta, \gamma_k) \overset{*}{\asymp}_{\kappa(\delta)} A(0,k).$$

Remark 5.12. Note that in Theorem 5.1, we estimate $i(\gamma_i, \gamma_k)$ with a uniform multiplicative constant κ that works for any two curves γ_i and γ_k, but the comparison is with $A(i,k)$ rather than $A(0,k)$. On the other hand, the ratio of $A(0,k)$ and $A(i,k)$ is bounded by a constant depending on i, and not k, so the lemma for $\delta = \gamma_i$ is an immediate consequence of that theorem.

Proof. First we note that by Theorem 5.1, we have

$$i(\gamma_i, \gamma_k) \overset{*}{\asymp}_{\kappa} A(i,k).$$

From the definition of A, and the fact that $\{e_j\}_{j=0}^{\infty}$ is an increasing sequence, it follows that for each $i = 0, \dots, 2m-1$, and all $k \geq i$, we have the bound

$$1 \leq \frac{A(0,k)}{A(i,k)} \leq b^2 e_{2m} e_{3m}.$$

Setting $\kappa_0 = \kappa b^2 e_{2m} e_{3m}$, for each $i = 0, \dots, 2m-1$, we have

$$i(\gamma_i, \gamma_k) \overset{*}{\asymp}_{\kappa_0} A(0,k). \tag{5.7}$$

Next, let $d = 2m\kappa_0$. Note that since $\gamma_0, \dots, \gamma_{2m-1}$ fills S, the set of measured laminations

$$\Delta = \left\{ \bar{\lambda} \;\middle|\; \sum_{j=0}^{2m-1} i(\gamma_j, \bar{\lambda}) \overset{*}{\asymp}_d 1 \right\} \subset \mathcal{ML}(S)$$

is compact. From (5.7) we have

$$\left\{ \frac{\gamma_k}{A(0,k)} \right\}_{k=3m}^{\infty} \subset \Delta.$$

Let $\nu \in \mathcal{EL}(S)$ be the lamination from Proposition 4.4. Since ν is an ending lamination, the set of measures $\bar{\nu} \in \Delta$ supported on ν is a compact subset. By the continuity of the intersection number i, there exists $c(\delta) > 0$ so that $i(\delta, \bar{\nu}) \overset{*}{\asymp}_{c(\delta)} 1$ for all such $\bar{\nu}$.

Let $K(\delta) \subset \mathcal{ML}(S)$ be a compact neighborhood which contains the set of measures $\bar{\nu}$ which are supported on ν and are in Δ. By the continuity of the intersection number i again, we can take $K(\delta)$ sufficiently small so that there exists $\kappa(\delta) > 0$ such that

$$i(\delta, \bar{\lambda}) \overset{*}{\asymp}_{\kappa(\delta)} 1 \quad \text{for all } \bar{\lambda} \in K(\delta).$$

Since every accumulation point of $\{\frac{\gamma_k}{A(0,k)}\}_{k=3m}^{\infty}$ is a measure $\bar{\nu} \in \Delta$ supported on ν, it follows that there exists $N(\delta)$ so that

$$\frac{\gamma_k}{A(0,k)} \in K(\delta)$$

for all $k \geq N(\delta)$. Consequently, for all $k \geq N(\delta)$, we have $i(\delta, \gamma_k) \overset{*}{\asymp}_{\kappa(\delta)} A(0,k)$, which completes the proof. □

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Using the estimates from Lemma 5.11, we prove the next lemma. Theorem 5.10 will then follow easily.

Lemma 5.13. *For any curve δ and any $h = 0, \dots, m-1$, the sequence $\{i(\delta, \frac{\gamma_i^h}{c_i^h})\}_{i=0}^{\infty}$ converges.*

Proof. By Proposition 5.4 we have

$$|i(\delta, \gamma_{im+h}) - e_{(i-1)m+h} b\, i(\delta, \gamma_{(i-1)m+h})| \leq 2B \sum_{l=(i-2)m+h}^{im+h-1} i(\delta, \gamma_l).$$

Dividing both sides by $c_i^h = A(0, im+h) = be_{(i-1)m+h} A(0, (i-1)m+h)$, and letting $\kappa(\delta)$ be the constant from Lemma 5.11, it follows that for all $h = 0, \dots, m-1$, and i sufficiently large

$$\begin{aligned}
\left| i\left(\delta, \frac{\gamma_{im+h}}{c_i^h}\right) - i\left(\delta, \frac{\gamma_{(i-1)m+h}}{c_{i-1}^h}\right) \right| &\leq \frac{2B}{A(0, im+h)} \left(\sum_{l=(i-2)m+h}^{im+h-1} i(\delta, \gamma_l) \right) \\
&\leq \frac{2B}{A(0, im+h)} \left(\sum_{l=(i-2)m+h}^{im+h-1} \kappa(\delta) A(0, l) \right) \\
&= \sum_{l=(i-2)m+h}^{im+h-1} 2B\kappa(\delta) \frac{A(0, l)}{A(0, im+h)}.
\end{aligned}$$

Lemma 5.6 implies that the expressions in the final sum admit the following bounds:

$$\frac{A(0, l)}{A(0, im+h)} \leq a^{1 - \lfloor \frac{im+h-0}{m} \rfloor} = a^{1-i}.$$

Since $\gamma_i^h = \gamma_{im+h}$, we have

$$\left| i\left(\delta, \frac{\gamma_i^h}{c_i^h}\right) - i\left(\delta, \frac{\gamma_{i-1}^h}{c_{i-1}^h}\right) \right| \leq 4mB\kappa(\delta) a^{1-i}.$$

Consequently, for all $i > j$ sufficiently large, applying this inequality and the triangle inequality we have

$$\left| i\left(\delta, \frac{\gamma_i^h}{c_i^h}\right) - i\left(\delta, \frac{\gamma_j^h}{c_j^h}\right) \right| \leq 4mB\kappa(\delta) \sum_{l=j+1}^{i} a^{1-l}.$$

By taking i and j sufficiently large, the (partial) sum of the geometric series on the right can be made arbitrarily small. In particular, $\{i(\delta, \gamma_i^h / c_i^h)\}$ is a Cauchy sequence, hence converges. □

Proof of Theorem 5.10. Fix $h \in \{0, \dots, m-1\}$. Note that since the intersection numbers $\{i(\delta, \gamma_i^h / c_i^h)\}_{i=0}^{\infty}$ converge for all simple closed curves δ, it follows that $\{\gamma_i^h / c_i^h\}_{i=0}^{\infty}$ converges to some lamination $\bar{\nu}^h$ in the space of measured laminations $\mathcal{ML}(S)$ (since $\mathcal{ML}(S)$ is a closed subset of $\mathbb{R}^{\mathcal{C}(S)}$). By Proposition 4.4, $\bar{\nu}^h$ is supported on ν. □

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6. Ergodic measures

We continue to assume throughout the rest of this section that $\{\gamma_k\}_{k=0}^{\infty}$ satisfies $\mathcal{P}$ and that $\{\gamma_i^h/c_i^h\}_{i=0}^{\infty}$ for $h = 0, \dots, m-1$ are the subsequences defined in the previous section limiting to $\bar{\nu}^h$ supported on ν by Theorem 5.10 for each $h = 0, \dots, m-1$. We say that $\bar{\nu}^h$ and $\bar{\nu}^{h'}$ are *not absolutely continuous* if neither is absolutely continuous with respect to the other one. Note that this is weaker than requiring that the measures be mutually singular.

Recall from the introduction that the space of measures supported on ν is the cone on the simplex of measure $\Delta(\nu)$. We denote (choices of) the ergodic measures representing the vertices by $\bar{\mu}^0, \dots, \bar{\mu}^{d-1}$, where $0 \leq d \leq \xi(S)$ is the dimension of the space of measure on ν. The ergodic measures are mutually singular since the generic points are disjoint. It follows that if we write $\bar{\nu}^h$ and $\bar{\nu}^{h'}$ as nonnegative linear combinations of $\bar{\mu}^0, \dots, \bar{\mu}^{d-1}$, then $\bar{\nu}^h$ and $\bar{\nu}^{h'}$ are not absolutely continuous if and only if there exists $\bar{\mu}^j, \bar{\mu}^{j'}$ so that $\bar{\mu}^j$ has positive coefficient for $\bar{\nu}^h$ and zero coefficient for $\bar{\nu}^{h'}$, while $\bar{\mu}^{j'}$ has positive coefficient for $\bar{\nu}^{h'}$ and zero coefficient for $\bar{\nu}^h$.

The aim of this section is to show that $d = m$, and in particular, ν is nonuniquely ergodic. In fact, we will prove that up to scaling and reindexing we have $\bar{\mu}^h = \bar{\nu}^h$.

Using the estimates on the intersection numbers from Theorem 5.1, we first show that the measures $\bar{\nu}^h$ for $h = 0, \dots, m-1$, are pairwise not absolutely continuous.

Theorem 6.1. *Let $h, h' \in \{0, \dots, m-1\}$ and $h \neq h'$. Then*

$$\lim_{i\to\infty} \frac{i(\gamma_i^h, \bar{\nu}^h)}{i(\gamma_i^h, \bar{\nu}^{h'})} = \infty \quad \textit{and} \quad \lim_{i\to\infty} \frac{i(\gamma_i^{h'}, \bar{\nu}^{h'})}{i(\gamma_i^{h'}, \bar{\nu}^h)} = \infty.$$

In particular, the measures $\bar{\nu}^h$ and $\bar{\nu}^{h'}$ are not absolutely continuous with respect to each other.

The last statement is a consequence of the two limits, for if $\bar{\nu}^h$ and $\bar{\nu}^{h'}$ were positive linear combinations of the same set of ergodic measures, then these ratios would have to be bounded.

Proof. For $h \neq h'$, we will calculate that

$$i(\gamma_0^h, \gamma_{i+1}^h) i(\gamma_i^h, \bar{\nu}^h) \stackrel{*}{\asymp} 1 \quad \text{and} \quad \lim_{i\to\infty} i(\gamma_0^h, \gamma_{i+1}^h) i(\gamma_i^h, \bar{\nu}^{h'}) = 0. \tag{6.1}$$

Dividing the first equation by the second and taking limit (and doing the same with the roles of h and h' reversed) gives the desired limiting behavior.

To treat the two estimates in (6.1) simultaneously, we suppose for the time being that $h, h' \in \{0, \dots, m-1\}$, but we do not assume $h \neq h'$. From Theorem 5.10 together with (5.5) and (5.6) we have

$$\bar{\nu}^h = \lim_{k\to\infty} \frac{\gamma_k^h}{c_k^h} = \lim_{k\to\infty} \frac{\gamma_{km+h}}{A(0, km+h)}.$$

Combining this with (5.1), (5.6), and the estimate in Theorem 5.1, we see that for any i we may take k sufficiently large so that

$$\begin{aligned} i(\gamma_0^h, \gamma_{i+1}^h) i(\gamma_i^h, \bar{\nu}^{h'}) &\stackrel{*}{\asymp} i(\gamma_h, \gamma_{(i+1)m+h}) i\Big(\gamma_{im+h}, \frac{\gamma_{km+h'}}{A(0, km+h')}\Big) \\ &\stackrel{*}{\asymp} \frac{A(h, (i+1)m+h) A(im+h, km+h')}{A(0, km+h')}. \end{aligned} \tag{6.2}$$

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We will simplify the expression on the right, but the precise formula depends on whether $h' \geq h$ or $h' < h$. From the definition (5.1), the right-hand side of (6.2) can be written as

$$\frac{\prod_{r=1}^{i} be_{rm+h} \prod_{r=j_0}^{k-1} be_{rm+h'}}{\prod_{r=1}^{k-1} be_{rm+h'}} = \frac{\prod_{r=1}^{i} be_{rm+h}}{\prod_{r=1}^{j_0-1} be_{rm+h'}},$$

where $j_0 = i+1$ if $h' \geq h$ and $j_0 = i+2$ if $h' < h$. Therefore, from (6.2) we can write

$$i(\gamma_0^h, \gamma_{i+1}^h) i(\gamma_i^h, \bar{\nu}^{h'}) \overset{*}{\asymp} \begin{cases} \prod_{r=1}^{i} \frac{e_{rm+h}}{e_{rm+h'}}, & h' \geq h, \\ \frac{1}{be_{m+h'}} \prod_{r=1}^{i} \frac{e_{rm+h}}{e_{(r+1)m+h'}}, & h' < h. \end{cases}$$

Now observe that when $h' = h$, this becomes

$$i(\gamma_0^h, \gamma_{i+1}^h) i(\gamma_i^h, \bar{\nu}^h) \overset{*}{\asymp} 1,$$

proving the first of the two required equations. So, suppose $h \neq h'$. Then each of the i terms in the product is bounded above by a^{-1} since the index for the denominator is greater than that of the numerator, and $e_l \geq a e_{l-1}$ for all $l \geq 1$. Thus we have

$$i(\gamma_0^h, \gamma_{i+1}^h) i(\gamma_i^h, \bar{\nu}^{h'}) \overset{*}{\prec} a^{-i},$$

where when $h' < h$, we have absorbed the constant $be_{m+h'}$ into the multiplicative error since $m + h' < 2m$. Letting i tend to infinity, we arrive at the second of our required estimates, and have thus completed the proof. □

We immediately obtain the following:

Corollary 6.2. *The lamination ν is nonuniquely ergodic.*

In fact, Theorem 6.1 implies the main desired result of this section in a special case. To prove this, we first prove a lemma which will be useful in the general case as well.

Lemma 6.3. *If $m \geq d$, then $m = d$, the measures $\bar{\nu}^0, \ldots, \bar{\nu}^{m-1}$ are distinct and ergodic, and these can be taken as the vertices of $\Delta(\nu)$.*

Proof. Recall that $\bar{\mu}^0, \ldots, \bar{\mu}^{d-1}$ are ergodic measures spanning the (d-dimensional) space of measures on ν. For each $0 \leq h < m$, write

$$\bar{\nu}^h = \sum_{j=0}^{d-1} c_j^h \bar{\mu}^j,$$

where $c_j^h \geq 0$ for all j, h. Then for each i, h, and h', we have

$$i(\gamma_i^h, \bar{\nu}^{h'}) = \sum_{j=0}^{d-1} c_j^{h'} i(\gamma_i^h, \bar{\mu}^j).$$

Next, fix h and let $j_h \in \{0, \ldots, m-1\}$ be such that $c_{j_h}^h \neq 0$ and so that there exists a subsequence of γ_i^h, so that if $0 \leq j < m-1$ and $c_j^h \neq 0$, then

$$i(\gamma_i^h, \bar{\mu}^{j_h}) \geq i(\gamma_i^h, \bar{\mu}^j). \tag{6.3}$$

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Now suppose that for some $h' \neq h$, $c_{j_h}^{h'} \neq 0$. On the subsequence of $\{\gamma_i^h\}$ above where (6.3) holds, Theorem 6.1 implies

$$\begin{aligned} \infty &= \lim_{i\to\infty} \frac{\sum_j c_j^h i(\gamma_i^h, \bar\mu^j)}{\sum_j c_j^{h'} i(\gamma_i^h, \bar\mu^j)} \le \limsup_{i\to\infty} \frac{\sum_j c_j^h i(\gamma_i^h, \bar\mu^j)}{c_{j_h}^{h'} i(\gamma_i^h, \bar\mu^{j_h})} \\ &= \limsup_{i\to\infty} \sum_j \frac{c_j^h}{c_{j_h}^{h'}} \frac{i(\gamma_i^h, \bar\mu^j)}{i(\gamma_i^h, \bar\mu^{j_h})} \le \sum_j \frac{c_j^h}{c_{j_h}^{h'}} < \infty. \end{aligned}$$

This contradiction shows that $c_{j_h}^{h'} = 0$ for all $h' \neq h$. Since $c_{j_h}^h \neq 0$, it follows that $h \mapsto j_h$ defines an injective function $\{0, \ldots, m-1\} \to \{0, \ldots, d-1\}$. Since $m \ge d$, this function is a bijection, $m = d$, and $\bar\nu^h = c_{j_h}^h \bar\mu^{j_h}$. Since $\bar\mu^0, \ldots, \bar\mu^{d-1}$ are distinct ergodic measures spanning the simplex of measures on ν, the lemma follows. □

Corollary 6.4. *If $m = \xi(S)$, then the measures $\bar\nu^0, \ldots, \bar\nu^{m-1}$ are distinct and ergodic and can be taken as the vertices of $\Delta(\nu)$.*

Proof. Since the dimension of the space of ergodic measures d is at most $\xi(S)$, it follows that $m \ge d$, and hence Lemma 6.3 implies the result. □

6.1. The general case. In [24] Lenzhen and Masur prove that for any nonuniquely ergodic lamination ν the ergodic measures are "reflected" in the geometric limit of a Teichmüller geodesic whose vertical foliation is topologically equivalent to ν. We will use this to prove the following generalization of Corollary 6.4 we need.

Theorem 6.5. *Suppose that $\{\gamma_l\}_{l=0}^\infty$ satisfies $\mathcal{P}$ and that $\{\gamma_k^h\}_{k=0}^\infty$, $h = 0, \ldots, m-1$, is the partition into m subsequences with $\lim_{k\to\infty} \gamma_k^h = \bar\nu^h$, all supported on ν. Then the measures $\bar\nu^0, \ldots, \bar\nu^{m-1}$ are distinct and ergodic and can be taken as the vertices of $\Delta(\nu)$.*

Let $\bar\mu^0, \ldots, \bar\mu^{d-1}$ be the ergodic measures on ν and set

$$\bar\mu = \sum_{j=0}^{d-1} \bar\mu^j \quad \text{and} \quad \bar\gamma = \sum_{j=0}^{m-1} \gamma_j = \sum_{h=0}^{m-1} \gamma_0^h.$$

Here we are viewing the curves in the sum on the right as measured laminations with transverse counting measure on each curve. We choose a normalization for the measures $\bar\mu^j$ so that $i(\bar\gamma, \bar\mu) = 1$. According to [16], there is a unique complex structure on S from a marked Riemann surface $S \to X$ and unit area holomorphic quadratic differential q on X with at most simple poles at the punctures, so that the vertical foliation $|dx|$ is $\bar\mu$ and the horizontal foliation $|dy|$ is $\bar\gamma$. Area in the q-metric is computed by integrating $d\bar\mu|dy|$. We will also be interested in the measure obtained by integrating $d\bar\mu_j|dy|$ for each $j = 0, \ldots, d-1$, which we denote by Area_j. Of course, $\text{Area} = \sum_j \text{Area}_j$.

Next let g denote the Teichmüller geodesic defined by q. We will write

$$g(t) = [f_t \colon X \to X(t)],$$

where $X(t)$ is the terminal Riemann surface, or

$$g(t) = [f_t \colon (X, q) \to (X(t), q(t))],$$

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where $q(t)$ is the terminal quadratic differential. Note that since ν is a nonuniquely ergodic lamination by Masur's criterion [30] the geodesic g is divergent in the moduli space. The vertical and horizontal measure of a curve γ is denoted $v_{q(t)}(\gamma)$ and $h_{q(t)}(\gamma)$, which are precisely the intersection numbers with the horizontal and vertical foliations of $q(t)$, respectively. These are given by

$$v_{q(t)}(\gamma) = e^{-t} i(\gamma, |dy|) = e^{-t} i(\gamma, \bar{\gamma}) \quad \text{and} \quad h_{q(t)}(\gamma) = e^{t} i(\gamma, |dx|) = e^{t} i(\gamma, \bar{\mu}).$$

From this it follows that the natural area measure from $q(t)$ is the push forward of the area measure from q. Likewise, this area naturally decomposes as the push forward of the measures Area_j, for $j = 0, \ldots, d-1$. Consequently, we will often confuse a subset of X and its image in $X(t)$ and will simply write Area and Area_j in either X or $X(t)$.

Given $\epsilon > \epsilon' > 0$, an (ϵ', ϵ)-*thick subsurface* of $(X(t), q(t))$ is a compact surface Y and a continuous map $Y \to X(t)$, injective on the interior of Y with the following properties.

(1) The boundary of Y is sent to a union of $q(t)$-geodesics, each with extremal length less than ϵ' in $X(t)$.

(2) If Y is not an annulus, then every nonperipheral curve in Y has $q(t)$-length at least ϵ and Y has no peripheral Euclidean cylinders.

(3) If Y is an annulus, then it is a maximal Euclidean cylinder.

Remark 6.6. We will be interested in the case that $\epsilon' \ll \epsilon$. In this case, ∂Y has a large collar neighborhood in Y, which does not contain a Euclidean cylinder (i.e. a large modulus expanding annulus; see [36]). Consequently, ∂Y will have short hyperbolic and extremal length.

As an abuse of notation, we will write $Y \subset X$, although Y is only embedded on its interior. An (ϵ', ϵ)-*decomposition of* $(X(t), q(t))$ is a union of (ϵ', ϵ)-thick subsurfaces

$$Y_1(t), \ldots, Y_r(t) \subset X(t)$$

with pairwise disjoint interiors. We note that $X(t)$ need not be the union of these subsurfaces. For example, suppose that $(X(t), q(t))$ is obtained from two flat tori by cutting both open along a very short segment, and gluing them together along the exposed boundary component. If the area of one torus is very close to 1 and the other very close to 0, then an (ϵ', ϵ)-decomposition would consist of the larger slit torus, $Y(t)$, while $X(t) - Y(t)$ would be the (interior of the) smaller slit torus.

The key results from [24] we will need are summarized in the following theorem.

Theorem 6.7 (Lenzhen–Masur). *With the assumptions on the Teichmüller geodesic g above, there exist constants $\epsilon > 0$ and $B > 0$ with the following properties. Given any sequence of times $t_k \to \infty$, there exist a subsequence (still denoted $\{t_k\}$), a sequence of subsurfaces $Y_0(t_k), \ldots, Y_{d-1}(t_k)$ in $X(t_k)$, and a sequence $\epsilon_k \to 0$, so that for all $k \geq 1$:*

(1) *$Y_0(t_k), \ldots, Y_{d-1}(t_k)$ is an (ϵ_k, ϵ)-thick decomposition,*

(2) *$\mathrm{Area}_j(Y_j^0(t_k)) > B$ for all $0 \leq j \leq d-1$ and for any component $Y_j^0(t_k) \subset Y_j(t_k)$,*

(3) *$\mathrm{Area}_j(Y_i(t_k)) < \epsilon_k$ for all $0 \leq i, j \leq d-1$ with $i \neq j$,*

(4) *$\mathrm{Area}(X(t_k) - (Y_0(t_k) \cup \cdots \cup Y_{d-1}(t_k)) < \epsilon_k$.*

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The bulk of this theorem comes from [24, Proposition 1]. More precisely, in [24, proof of Proposition 1], the authors produce a sequence of subsurface $\{Y(t_k)\}$ whose components give an (ϵ_k, ϵ)-thick decomposition so that each component has area uniformly bounded away from zero, so that the areas of the complements tend to zero. For each ergodic measure $\bar{\mu}^j$ the authors then find subsurfaces $Y_i(t_k)$ so that $\mathrm{Area}_j(Y_i(t_k)) \to 0$ as $k \to \infty$ if $i \neq j$ (see [24, inequality (16)] and its proof). This proves (1), (3), and (4). Since $\mathrm{Area} = \sum_j \mathrm{Area}_j$, condition (2) follows as well.

To apply this construction, we will need the following lemma. First, for a curve γ and $t \geq 0$, let $\mathrm{cyl}_t(\gamma) \subset X(t)$ denote the (possibly degenerate) maximal Euclidean cylinder foliated by $q(t)$-geodesic representatives of γ. We note that $\mathrm{cyl}_t(\gamma) = f_t(\mathrm{cyl}_0(\gamma))$.

Lemma 6.8. *Given any sequence $t_k \to \infty$, let $Y_0(t_k), \dots, Y_{d-1}(t_k) \subset X(t_k)$ denote the (ϵ_k, ϵ)-thick decomposition from Theorem* 6.7 *(obtained after passing to a subsequence). Then for all k sufficiently large, each $Y_j(t_k)$ contains a curve from the sequence $\{\gamma_l\}$ as a nonperipheral curve, or else contains a component which is a cylinder with core curve in the sequence $\{\gamma_l\}$.*

We postpone the proof of this lemma temporarily and use it to easily prove the main result of this section.

Proof of Theorem 6.5. Let $t_k \to \infty$ be any sequence and let $Y_0(t_k), \dots, Y_{d-1}(t_k)$ be the (ϵ_k, ϵ)-thick decomposition obtained from Theorem 6.7 after passing to a subsequence. Let k be large enough so that the conclusion of Lemma 6.8 holds. For each $j \in \{0, \dots, d-1\}$ let γ_{l_j} be one of the curves in our sequence so that γ_{l_j} is either a nonperipheral curve in $Y_j(t_k)$, or else $Y_j(t_k)$ contains a cylinder component with core curve γ_{l_j}. Since $Y_0(t_k), \dots, Y_{d-1}(t_k)$ have disjoint interiors, it follows that $\gamma_{l_0}, \dots, \gamma_{l_{d-1}}$ are pairwise disjoint, pairwise nonisotopic curves. By Theorem 5.1, for example, the difference in indices of disjoint curves in our sequence is at most m, and consequently $\{\gamma_{l_0}, \dots, \gamma_{l_{d-1}}\}$ consists of at most m curves. That is, $m \geq d$. By Lemma 6.3, $d = m$, and $\bar{\nu}^0, \dots, \bar{\nu}^{m-1}$ are ergodic measures spanning the space of all measures on ν, proving the theorem. □

6.2. Areas and extremal lengths. The proof of Lemma 6.8 basically follows from the results of [36], together with the estimates on intersection numbers described at the beginning of this section and subsurface coefficient bounds in Section 4.1. Let

$$g(t) = [f_t : (X, q) \to (X(t), q(t)))]$$

be the Teichmüller geodesic described above with vertical foliation $\bar{\mu} = \sum \bar{\mu}_i$, the sum of the ergodic measures on ν, and horizontal foliation $|dy| = \bar{\gamma}$.

Suppose that $Y \to X(t)$ is a map of a connected surface into $X(t)$ which is an embedding on the interior, sends the boundary to $q(t)$-geodesics, and has no peripheral Euclidean cylinders unless Y is itself a Euclidean cylinder (in which case we assume it is maximal). As in the case of thick subsurfaces, we write $Y \subset X(t)$, though we are not assuming that Y is thick. Suppose that $Y \subset X(t)$ is a subsurface so that the leaves of the vertical and horizontal foliations intersect Y in arcs. This is the case for $Y = \mathrm{cyl}_t(\gamma_k)$ for all k sufficiently large, as well as any Y for which $\mathrm{Ext}_{X(t)}(\partial Y)$ is small when t is large, and these will be the main cases of interest for us.

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As in [36], the surface Y decomposes into a union of *horizontal strips*

$$Y = H_1(Y) \cup \cdots \cup H_r(Y)$$

and *vertical strips*

$$Y = V_1(Y) \cup \cdots \cup V_{r'}(Y).$$

Each horizontal strip $H_i(Y)$ is the image of map $f_i^H \colon [0,1] \times [0,1] \to Y$ which is injective on the interior, sends $[0,1] \times \{s\}$ to an arc of a horizontal leaf with endpoints on ∂Y. Furthermore, the images of the interiors of $f_1^H, \dots, f_r^H$ are required to be pairwise disjoint. Let $\ell_i^H = f_i^H([0,1] \times \{\frac{1}{2}\})$ be a "core arc" of the strip. Vertical strips are defined similarly (and satisfy the analogous properties for the vertical foliation) as are the core arcs $\ell_1^V, \dots, \ell_{r'}^V$.

Remark 6.9. This is a slight variation on the strip decompositions in [36].

The *width* of a horizontal strip $H_i(Y)$, denoted $w(H_i(Y))$, is the vertical variation of any (or equivalently, every) arc $H_i(\{s\} \times [0,1])$. The width of a vertical strip, $w(V_i(Y))$, is similarly defined in terms of the horizontal variation. An elementary, but important property of these strips is the following.

Proposition 6.10. *Let $Y \subset X(t)$ be as above. If*

$$Y = H_1(Y) \cup \cdots \cup H_r(Y) = V_1(Y) \cup \cdots \cup V_{r'}(Y)$$

is a decomposition into maximal horizontal and vertical strips, then

$$v_{q(t)}(\partial Y) = 2\sum_{i=1}^{r} w(H_i(Y)) \quad \textit{and} \quad h_{q(t)}(\partial Y) = 2\sum_{i=1}^{r'} w(V_i(Y)).$$

The area of Y can be estimated from this by the inequalities

$$\sum_{ij} w(H_i(Y))w(V_j(Y))(i(\ell_i^H, \ell_j^V) - 2) \le \mathrm{Area}(Y) \le \sum_{i,j} w(H_i(Y))w(V_j(Y))(i(\ell_i^H, \ell_j^V) + 2). \tag{6.4}$$

To see this, we note that the area of Y is the sum of the areas of the horizontal (or vertical) strips. Every time $V_j(Y)$ crosses $H_i(Y)$, it does so in a rectangle, which contains a unique point of intersection $\ell_i^H \cap \ell_j^V$, *except*, near the ends of $H_i(Y)$, where we might not see an entire rectangle (and consequently we may or may not see a point of $\ell_i^H \cap \ell_j^V$). We may also have an intersection point in $\ell_i^H \cap \ell_j^V$ that does not come in a complete rectangle (but only part of a rectangle). Adding and subtracting 2 to the intersection number accounts for the ends of $H_i(Y)$, and summing gives the bounds.

If Y is nonannular, then note that

$$\sum i(\ell_i^H, \ell_j^V) + 2 \prec i(\pi_Y(\bar{\gamma}), \pi_Y(\nu)).$$

To see this, we note that the horizontal foliation (for example) is $\bar{\gamma}$ and $\pi_Y(\bar{\gamma})$ is basically obtained from the arcs ℓ_i^H by surgering with arcs from the boundary (see also [36, Lemma 3.8]). Combining this inequality with the upper bound in (6.4) and Proposition 6.10, we obtain

$$\mathrm{Area}(Y) \prec h_{q(t)}(\partial Y) v_{q(t)}(\partial Y) i(\pi_Y(\bar{\gamma}), \pi_Y(\nu)). \tag{6.5}$$

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Now suppose that $Y = \mathrm{cyl}_t(\gamma)$ is a maximal Euclidean cylinder with core curve γ. Then there is a decomposition into strips with just one horizontal strip $H(Y)$ and one vertical strip $V(Y)$ and core arcs ℓ^H and ℓ^V, respectively. In this case, the intersection number $i(\ell^H, \ell^V)$ is just $d_Y(\bar{\gamma}, \nu)$ up to an additive constant (of at most 4 – again, see [36, Lemma 3.8]). Therefore, the bounds in (6.4) together with Proposition 6.10 implies

$$\frac{4\,\mathrm{Area}(\mathrm{cyl}_0(\gamma))}{h_{q(t)}(\gamma)v_{q(t)}(\gamma)} = \frac{4\,\mathrm{Area}(\mathrm{cyl}_0(\gamma))}{i(\gamma,\bar{\gamma})i(\gamma,\bar{\mu})} \overset{+}{\asymp} d_\gamma(\bar{\gamma},\nu). \tag{6.6}$$

In particular, if $d_\gamma(\bar{\gamma}, \nu)$ is large, then

$$\mathrm{Area}(\mathrm{cyl}_0(\gamma_k^h)) \overset{*}{\asymp} h_{q(t)}(\gamma)v_{q(t)}(\gamma)d_\gamma(\bar{\gamma},\nu) = i(\gamma,\bar{\gamma})i(\gamma,\bar{\mu})\gamma(\bar{\gamma},\nu).$$

The balance time of γ along the Teichmüller geodesic g is the unique $t \in \mathbb{R}$ so that

$$v_{q(t)}(\gamma) = h_{q(t)}(\gamma).$$

Consider $Y = \mathrm{cyl}_{t(\gamma)}(\gamma)$ at the balance time of γ, together with the horizontal and vertical strips $H(Y)$ and $V(Y)$, respectively. In this situation, the rectangles of intersections between $H(Y)$ and $V(Y)$ are actually squares. We can estimate the modulus of Y, which is the ratio of the length to the circumference using these squares. Specifically, we note that the circumference of Y is precisely the length of the diagonal of a square, while the length of Y is approximately half the number of squares, times the length of a diagonal. Since the number of squares is $|\ell^H \cap \ell^V| \overset{+}{\asymp} d_\gamma(\bar{\gamma},\nu)$, we see that the modulus is $2d_\gamma(\bar{\gamma},\nu)$, up to a uniform additive error. When $d_\gamma(\bar{\gamma},\nu)$ is sufficiently large, the reciprocal of this modulus provides an upper bound for the extremal length

$$\mathrm{Ext}_{t(\gamma)}(\gamma) \overset{*}{\prec} \frac{1}{d_\gamma(\bar{\gamma},\nu)}.$$

We note that this estimate was under the assumption that $\mathrm{cyl}_0(\gamma)$ was a nondegenerate annulus. In fact, if $d_\gamma(\bar{\gamma},\nu)$ is sufficiently large (e.g. at least 5), then $\mathrm{cyl}_0(\gamma)$ is indeed nondegenerate.

Proof of Lemma 6.8. Suppose that $t_k \to \infty$ is a sequence of times, $Y(t_k) \subset X(t_k)$ is a sequence of subsurfaces with $q(t)$-geodesic boundary, embedded on the interior and having no peripheral Euclidean cylinders, unless Y is itself a Euclidean cylinder in which case we assume it is a maximal Euclidean cylinder. We further assume that $\mathrm{Ext}_{X(t_k)}(\partial Y(t_k)) \to 0$. We pass to a subsequence, also denoted $\{t_k\}$, and assume that either $Y(t_k)$ is nonannular and no nonperipheral curve lies in the sequence $\{\gamma_l\}$, or that $Y(t_k)$ is a cylinder whose core is not a curve from our sequence $\{\gamma_l\}$. To prove the lemma, it suffices to prove that $\mathrm{Area}(Y(t_k)) \to 0$, for this implies that such subsurfaces $Y(t_k)$ cannot be a component of any $Y_j(t_k)$ from Theorem 6.7.

Decompose the sequence into an annular subsequence and nonannular subsequence, and we consider each case separately. For the nonannular subsurfaces, we bound the area of $Y(t_k)$ using inequality (6.5). Specifically, we note that since no γ_l is homotopic to a nonperipheral curve in $Y(t_k)$, Proposition 4.5 provides a uniform bound for $d_W(\bar{\gamma},\nu)$ for all subsurfaces $W \subset Y(t_k)$. By Theorem 2.7, follows that $i(\pi_Y(\bar{\gamma}), \pi_Y(\nu))$ is uniformly bounded. Since the extremal length of $\partial Y(t_k)$ is tending to zero, so is the $q(t_k)$-length, and so also the horizontal

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and vertical variations:

$$\lim_{k\to\infty} v_{q(t_k)}(\partial Y(t_k)) = 0 \quad \text{and} \quad \lim_{k\to\infty} h_{q(t_k)}(\partial Y(t_k)) = 0.$$

Combining this with (6.5) proves $\mathrm{Area}(Y(t_k)) \to 0$, as required.

The annular case is similar: Again by Proposition 4.5 since the core curve α_k of $Y(t_k)$ is not any curve from the sequence $\{\gamma_l\}$, we have that $d_{\alpha_k}(\bar{\gamma}, \bar{\nu})$ is uniformly bounded, while the horizontal and vertical variations of α_k tend to zero (since the extremal length, and hence $q(t_k)$-length, tends to 0). Appealing to (6.6) proves that $\mathrm{Area}(Y(t_k)) \to 0$ as $k \to \infty$ in this case, too. □

7. Constructions

In this section we provide examples of sequences of curves satisfying $\mathcal{P}$, and hence to which the results of Sections 3–6 apply.

7.1. Basic setup. Consider a surface S and m pairwise disjoint, nonisotopic curves $\gamma_0, \dots, \gamma_{m-1}$. For each k, let $\Gamma_k = (\gamma_0 \cup \dots \cup \gamma_{m-1}) - \gamma_k$, and let X_k be the component of S cut along Γ_k containing γ_k. For each k we assume the following:

(1) ∂X_k contains both γ_{k+1} and γ_{k-1} (with indices taken modulo m),

(2) we have chosen $f_k \colon S \to S$ a *fixed* homeomorphism which is the identity on $S \setminus X_k$, and pseudo-Anosov on X_k,

(3) the composition of f_k and the Dehn twist $\mathcal{D}^r_{\gamma_k}$, denoted $\mathcal{D}^r_{\gamma_k} f_k$, has translation distance at least 16 on the arc and curve graph $\mathcal{AC}(X_k)$ for any $r \in \mathbb{Z}$,

(4) there is some $b > 0$ so that $i(\gamma_k, f_k(\gamma_k)) = b$, independent of k.

For $0 \le k, h \le m-1$, let $\mathcal{J}(k,h)$ be the interval from k to h, mod m. This means that if $k < h$, then $\mathcal{J}(k,h) = \{k, k+1, \dots, h\}$ is the interval in $\mathbb{Z}$ from k to h, while if $h < k$, then

$$\mathcal{J}(k,h) = \{k, k+1, \dots, m-1, 0, \dots, h\}.$$

If $k = h$, then $\mathcal{J}(k,h) = \{k\} = \{h\}$.

For any $0 \le k, h \le m-1$, set

$$X_{k,h} = \bigcup_{l \in \mathcal{J}(k,h)} X_l.$$

If $k = h$, note that $X_{k,h} = X_k = X_h$. In general, $X_{k,h}$ is the component of S cut along $\Gamma_{k,h} = \gamma_{h+1} \cup \dots \cup \gamma_{k-1}$ containing all the curves $\gamma_k, \dots, \gamma_h$. That there is such a component follows inductively from the fact that $\gamma_{l\pm 1} \subseteq \partial X_l$, with indices taken mod m.

We also define

$$F_{k,h} = f_k \circ f_{k+1} \circ \dots \circ f_h,$$

where we are composing f_l over $l \in \mathcal{J}(k,h)$. Because f_l is supported on X_l, it follows that for all $0 \le k, h \le m-1$,

$$\gamma_k, \dots, \gamma_h, F_{k,h}(\gamma_h) \subset X_{k,h}.$$

In fact, the first and last curves in this sequence fill $X_{k,h}$.

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Lemma 7.1. *For each* $0 \le k, h \le m-1$, $\{\gamma_k, F_{k,h}(\gamma_h)\}$ *fills* $X_{k,h}$. *In particular, we have* $i(\gamma_l, F_{k,h}(\gamma_h)) \neq 0$ *for all* $l \in \mathcal{J}(k,h)$.

Remark 7.2. In the case $k = h+1 \pmod m$, we note that $X_{h+1,h} = S$ and the lemma states that

$$\{\gamma_{h+1}, F_{h+1,h}(\gamma_h)\} = \{\gamma_k, f_k f_{k+1} \cdots f_h(\gamma_h)\}$$

fills S. We also observe that for all $j \in \mathcal{J}(k,h)$, $X_{k,j} \subset X_{k,h}$. It follows that $\gamma_k, \gamma_{k+1}, \ldots, \gamma_h$ and $F_{k,k}(\gamma_k), \ldots, F_{k,h}(\gamma_h)$ are contained in $X_{k,h}$.

In the following proof, we write $\pi_{k,h}(\delta)$ for the arc-projection to $\mathcal{A}C(X_{k,h})$ of a curve δ. This is just the isotopy class of arcs/curves of δ intersected with $X_{k,h}$. Likewise, $d_{k,h}(\delta, \delta')$ is the distance between $\pi_{h,k}(\delta)$ and $\pi_{h,k}(\delta')$ in $\mathcal{A}C(X_{k,h})$. We similarly define π_k and d_k for the case $k = h$.

Proof. The last statement follows from the first assertion since, for all $l \in \mathcal{J}(k,h)$, $i(\gamma_l, \gamma_k) = 0$, and so assuming $\{\gamma_k, F_{k,h}(\gamma_h)\}$ fills, we must have $i(\gamma_l, F_{k,h}(\gamma_h)) \neq 0$.

The conditions on the curves and homeomorphisms are symmetric under cyclic permutation of the indices, so it suffices to prove the lemma for $h = m-1$ and $0 \le k \le h$ (which is slightly simpler notationally). We write $j = h-k$ and must prove that $\{\gamma_{h-j}, F_{h-j,h}(\gamma_h)\}$ fills $X_{h-j,h}$. We prove this by induction on j.

The base case is $j = 0$, in which case we are reduced to proving that $\{\gamma_h, f_h(\gamma_h)\}$ fills X_h. This follows from the fact that f_h has translation distance at least 16 on $\mathcal{A}C(X_h)$, and hence $d_h(\gamma_h, f_h(\gamma_h)) \ge 16$.

Suppose that for some $0 < j \le h$, $\{\gamma_{h+1-j}, F_{h+1-j,h}(\gamma_h)\}$ fill $X_{h+1-j,h}$, and we must prove that $\{\gamma_{h-j}, F_{h-j,h}(\gamma_h)\}$ fills $X_{h-j,h}$.

Note that since $\gamma_{h-j+1} \subset \partial X_{h-j}$, and $i(\gamma_{h-j+1}, F_{h+1-j,h}(\gamma_h)) \neq 0$ (because they fill $X_{h+1-j,h}$), it follows that $F_{h+1-j,h}(\gamma_h)$ has nontrivial projection to X_{h-j}. On the other hand, because γ_{h-j} is disjoint from $X_{h+1-j,h}$ (it is in fact a boundary component), it follows that $i(\gamma_{h-j}, F_{h+1-j,h}(\gamma_h)) = 0$, hence $d_{h-j}(\gamma_{h-j}, F_{h+1-j,h}) = 1$. Since f_{h-j} translates by at least 16 on $\mathcal{A}C(X_{h-j})$, it follows that

$$\begin{aligned} d_{h-j}(F_{h-j,h}(\gamma_h), \gamma_{h-j}) &= d_{h-j}(f_{h-j}(F_{h+1-j,h}(\gamma_h)), \gamma_{h-j}) \\ &\ge d_{h-j}(f_{h-j}(F_{h+1-j,h}(\gamma_h)), F_{h+1-j,h}(\gamma_h)) \\ &\qquad - d_{h-j}(F_{h+1-j,h}(\gamma_h), \gamma_{h-j}) \\ &\ge 16 - 1 = 15. \end{aligned}$$

Now suppose that $\{\gamma_{h-j}, F_{h-j,h}(\gamma_h)\}$ does not fill $X_{h-j,h}$. Let δ be an essential curve in $X_{h-j,h}$ which is disjoint from both γ_{h-j} and $F_{h-j,h}(\gamma_h)$. Observe that δ cannot intersect the subsurface X_{h-j} essentially, for otherwise

$$d_{h-j}(\gamma_{h-j}, F_{h-j,h}(\gamma_h)) \le d_{h-j}(\gamma_{h-j}, \delta) + d_{h-j}(\delta, F_{h-j,h}(\gamma_h)) \le 2$$

a contradiction.

Therefore, δ is contained in $X_{h-j,h} - X_{h-j} \subset X_{h+1-j,h}$. We first claim that δ must be an essential curve in $X_{h+1-j,h}$. If not, then it is contained in the boundary. However, any boundary component of $X_{h+1-j,h}$ which is essential in $X_{h-j,h}$ is contained (and essential) in X_{h-j}. This is a contradiction.

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Now since δ is essential in $X_{h+1-j,h}$, by the hypothesis of the induction we have

$$0 \neq i(\delta, \gamma_{h+1-j}) + i(\delta, F_{h+1-j,h}(\gamma_h)) = i(\delta, \gamma_{h+1-j}) + i(\delta, F_{h-j,h}(\gamma_h)).$$

The last equality follows from the fact that $F_{h-j,h}$ differs from $F_{h+1-j,h}$ only in X_{h-j}, which is disjoint from δ. Finally, we note that $\gamma_{h+1-j} \subseteq \partial X_{h-j}$, and hence $i(\delta, \gamma_{h+1-j}) = 0$. Consequently,

$$i(\delta, F_{h-j,h}(\gamma_h)) \neq 0$$

contradicting our choice of δ. Therefore, $\{\gamma_{h-j}, F_{h-j,h}(\gamma_h)\}$ fills $X_{h-j,h}$. This completes the induction, and hence the proof of the lemma. □

Lemma 7.3. *For all* $0 \leq k \leq m-1$,

$$i(\gamma_k, F_{k,k-1} f_k(\gamma_k)) = i(\gamma_k, f_k f_{k+1} \cdots f_{k-1} f_k(\gamma_k)) \neq 0.$$

Proof. We recall from the previous proof that $\{\gamma_{k+1}, F_{k+1,k}(\gamma_k)\}$ not only fills S, but satisfies

$$d_{k+1}(\gamma_{k+1}, F_{k+1,k}(\gamma_k)) \geq 15.$$

Since $\gamma_{k+1} \subseteq \partial X_k$ and $\gamma_k \subseteq \partial X_{k+1}$ and X_k and X_{k+1} overlap, it follows from Theorem 2.9 (see also Remark 2.10) that

$$d_k(\gamma_k, F_{k+1,k}(\gamma_k)) \leq 4.$$

Since f_k translates at least 16 on $\mathcal{AC}(X_k)$, we have

$$\begin{aligned} d_k(\gamma_k, f_k F_{k+1,k}(\gamma_k)) &\geq d_k(F_{k+1,k}(\gamma_k), f_k F_{k+1,k}(\gamma_k)) - d_k(\gamma_k, F_{k+1,k}(\gamma_k)) \\ &\geq 16 - 4 \geq 12. \end{aligned}$$

Since $f_k F_{k+1,k} = F_{k,k-1} f_k$, the lemma follows. □

7.2. General construction. Let $\{e_k\}_{k=0}^{\infty}$ be a sequence of integers satisfying inequality (3.1) for $a > 2$ sufficiently large as so as to satisfy (5.4) and hence (5.3) in Theorem 5.1 (see the convention at the end of Section 5.1).

For $k \geq 0$, let $\bar{k} \in \{0, \dots, m-1\}$ be the residue mod m, and for $k \geq m$ define

$$\mathcal{D}_k = \mathcal{D}_{\gamma_{\bar{k}}}^{e_{k-m}} \quad \text{and} \quad \phi_k = \mathcal{D}_k f_{\bar{k}}.$$

The sequence of curves $\{\gamma_k\}_{k=0}^{\infty}$ is defined as follows:

(1) The first m curves are $\gamma_0, \dots, \gamma_{m-1}$, as above.

(2) For $k \geq m$, set

$$\gamma_k = \phi_m \phi_{m+1} \cdots \phi_k(\gamma_{\bar{k}}).$$

Remark 7.4. We could have avoided having the first m curves as special cases and alternatively defined a sequence $\{\delta_k\}_{k \geq 0}$ by $\delta_k = \phi_0 \cdots \phi_k(\gamma_{\bar{k}})$ for all $k \geq 0$. This sequence differs from ours by applying the homeomorphism $\phi_0 \cdots \phi_{m-1}$. This is a useful observation when it comes to describing consecutive elements in the sequence, but our choice allows us to keep $\gamma_0, \dots, \gamma_{m-1}$ as the first m curves.

Proposition 7.5. *With the conditions above, the sequence* $\{\gamma_k\}_{k=0}^{\infty}$ *satisfies* $\mathcal{P}$ *for some* $0 < b_1 \leq b \leq b_2$ *(where b is the constant assumed from the start).*

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To simplify the proof, we begin with the following lemma.

Lemma 7.6. *For any $2m$ consecutive curves $\gamma_{k-m},\dots,\gamma_{k+m-1}$, there is a homeomorphism $H_k : S \to S$ taking these curves to the curves*

$$\gamma_{\bar{k}}\,,\;\dots\;,\;\gamma_{\overline{k+m-1}}\,,\;f_{\bar{k}}(\gamma_{\bar{k}})\,,\;\dots\;,\;f_{\bar{k}}\cdots f_{\overline{k+m-1}}(\gamma_{\overline{k+m-1}})$$

(in the same order). Furthermore, the homeomorphism can be chosen to take γ_{k+m} to

$$\mathcal{D}^{e_k}_{f_{\bar{k}}(\gamma_{\bar{k}})}(f_{\bar{k}}\cdots f_{\overline{k+m-1}}f_{\bar{k}}(\gamma_{\bar{k}})).$$

Proof. We prove the lemma assuming $k \geq 2m$ to avoid special cases (the general case can be easily derived from Remark 7.4, for example). We define

$$H_k = (\phi_m \cdots \phi_{k-1}\mathcal{D}_k\mathcal{D}_{k+1}\cdots\mathcal{D}_{k+m-1})^{-1}.$$

Let $h,h' \in \{0,\dots,m-1\}$ and note that since $i(\gamma_h,\gamma_{h'}) = 0$, $\mathcal{D}_{\gamma_{h'}}(\gamma_h) = \gamma_h$. Furthermore, if $h \neq h'$, from the fact that f_h is supported on X_h and $\gamma_{h'}$ is disjoint from X_h we easily deduce $\mathcal{D}_{\gamma_{h'}}$ and f_h commute, and $\phi_h(\gamma_{h'}) = \gamma_{h'}$.

From these facts we observe that for $k-m \leq j \leq k-1$, we have

$$\begin{aligned} H_k^{-1}(\gamma_j) &= \phi_m\cdots\phi_{k-1}\mathcal{D}_k\mathcal{D}_{k+1}\cdots\mathcal{D}_{k+m-1}(\gamma_{\bar{j}}) \\ &= \phi_m\cdots\phi_{k-1}(\gamma_{\bar{j}}) \\ &= \phi_m\cdots\phi_j(\gamma_{\bar{j}}) = \gamma_j, \end{aligned}$$

while for $k \leq j \leq k+m-1$, we have

$$\begin{aligned} H_k^{-1}(f_{\bar{k}}\cdots f_{\bar{j}}(\gamma_{\bar{j}})) &= \phi_m\cdots\phi_{k-1}\mathcal{D}_k\cdots\mathcal{D}_{k+m-1}f_{\bar{k}}\cdots f_{\bar{j}}(\gamma_j) \\ &= \phi_m\cdots\phi_{k-1}\mathcal{D}_k f_{\bar{k}}\cdots\mathcal{D}_j f_{\bar{j}}\mathcal{D}_{j+1}\cdots\mathcal{D}_{k+m-1}(\gamma_{\bar{j}}) \\ &= \phi_m\cdots\phi_j\mathcal{D}_{j+1}\cdots\mathcal{D}_{k+m-1}(\gamma_{\bar{j}}) \\ &= \phi_m\cdots\phi_j(\gamma_{\bar{j}}) = \gamma_j. \end{aligned}$$

This completes the proof of the first statement.

Next, since $\mathcal{D}_{k+m} = \mathcal{D}^{e_k}_{\gamma_{\bar{k}}}$, we have

$$\begin{aligned} f_{\bar{k}}\cdots f_{\overline{k+m-1}}\phi_{k+m}(\gamma_{\bar{k}}) &= f_{\bar{k}}\cdots f_{\overline{k+m-1}}\mathcal{D}_{k+m}f_{\bar{k}}(\gamma_k) \\ &= f_{\bar{k}}\mathcal{D}_{k+m}f_{\overline{k+1}}\cdots f_{\overline{k+m-1}}f_{\bar{k}}(\gamma_{\bar{k}}) \\ &= f_{\bar{k}}\mathcal{D}_{k+m}f_{\bar{k}}^{-1}f_{\bar{k}}\cdots f_{\overline{k+m-1}}f_{\bar{k}}(\gamma_{\bar{k}}) \\ &= f_{\bar{k}}\mathcal{D}^{e_k}_{\gamma_{\bar{k}}}f_{\bar{k}}^{-1}f_{\bar{k}}\cdots f_{\overline{k+m-1}}f_{\bar{k}}(\gamma_{\bar{k}}) \\ &= \mathcal{D}^{e_k}_{f_{\bar{k}}(\gamma_{\bar{k}})}f_{\bar{k}}\cdots f_{\overline{k+m-1}}f_{\bar{k}}(\gamma_{\bar{k}}). \end{aligned} \tag{7.1}$$

Applying H_k^{-1} to the left-hand side gives γ_{k+m}, proving the last statement. □

Proof of Proposition 7.5. Let $\gamma_{k-m},\dots,\gamma_{k+m-1}$ be any $2m$ consecutive curves in our sequence, and let $H_k\colon S \to S$ be the homeomorphism from Lemma 7.6 putting these curves into the standard form described by that lemma. Since H_k sends the first m to $\gamma_{\bar{k}},\dots,\gamma_{\overline{k+m-1}}$, it follows that these curves are pairwise disjoint. Moreover, the set of all $2m$ curves fills S by Lemma 7.1 and Remark 7.2 (in fact, the first and last alone fill S). Therefore, the sequence satisfies conditions (i) and (ii) of $\mathcal{P}$.

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To prove that condition (iii) is also satisfied, we need to define γ'_{k+m} so that

$$\gamma_{k+m} = \mathcal{D}^{e_k}_{\gamma_k}(\gamma'_{k+m}),$$

and verify the intersection conditions. We fix $k \geq 2m$ and define

$$\gamma'_{k+m} = \phi_m \cdots \phi_{m+k-1} f_{\bar{k}}(\gamma_{\bar{k}})$$

(the case of general $k \geq m$ is handled by special cases or by appealing to Remark 7.4). Note that by definition, $\gamma_{k+m} = \phi_m \cdots \phi_{m+k-1}\phi_{m+k}(\gamma_{\bar{k}})$ and applying H_k to γ_k and γ_{k+m}, Lemma 7.6 gives us

$$H_k(\gamma_k) = f_{\bar{k}}(\gamma_{\bar{k}}) \quad \text{and} \quad H_k(\gamma_{k+m}) = \mathcal{D}^{e_k}_{f_{\bar{k}}(\gamma_{\bar{k}})}(f_{\bar{k}} \cdots f_{\overline{k+m-1}} f_{\bar{k}}(\gamma_{\bar{k}})).$$

Then, as in the proof of Lemma 7.6 (compare (7.1)), we have

$$H_k(\gamma'_{k+m}) = f_{\bar{k}} \cdots f_{\overline{k+m-1}} f_{\bar{k}}(\gamma_{\bar{k}}).$$

Therefore,

$$H_k(\gamma_{k+m}) = \mathcal{D}^{e_k}_{H_k(\gamma_k)}(H_k(\gamma'_{k+m})) = H_k(\mathcal{D}^{e_k}_{\gamma_k}(\gamma'_{k+m})),$$

so $\gamma_{k+m} = \mathcal{D}^{e_k}_{\gamma_k}(\gamma'_{k+m})$.

To prove the intersection number conditions on $i(\gamma'_{k+m}, \gamma_j)$ from property (iii) of $\mathcal{P}$, it suffices to prove them for the H_k-images. Thus, for $j \in \{k+1, \ldots, k+m-1\}$ we note that by Lemma 7.6, $H_k(\gamma_j) = f_{\bar{k}} \cdots f_{\bar{j}}(\gamma_{\bar{j}})$, and hence

$$\begin{aligned} i(\gamma_j, \gamma'_{k+m}) &= i(f_{\bar{k}} \cdots f_{\bar{j}}(\gamma_{\bar{j}}), f_{\bar{k}} \cdots f_{\overline{k+m-1}} f_{\bar{k}}(\gamma_{\bar{k}})) \\ &= i(\gamma_{\bar{j}}, f_{\overline{j+1}} \cdots f_{\overline{k+m-1}} f_{\bar{k}}(\gamma_{\bar{k}})) \\ &= i(\gamma_{\bar{j}}, \gamma_{\bar{k}}) = 0. \end{aligned}$$

The second-to-last equality is obtained by applying $(f_{\overline{j+1}} \cdots f_{\overline{k+m-1}} f_{\bar{k}})^{-1}$ to both entries, and observing that this fixes $\gamma_{\bar{j}}$ (cf. the proof of Lemma 7.6).

On the other hand, for $j = k$, the same basic computation shows

$$i(\gamma_k, \gamma'_{k+m}) = i(\gamma_{\bar{k}}, f_{\bar{k}}(\gamma_{\bar{k}})) = b$$

by assumption (4).

Finally, similar calculations show that for $j \in \{k-m, \ldots, k-1\}$, by Lemmas 7.1 and 7.3, we have

$$i(\gamma_j, \gamma'_{k+m}) = i(\gamma_{\bar{j}}, f_{\bar{k}} \cdots f_{\overline{k+m-1}} f_{\bar{k}}(\gamma_{\bar{k}})) \neq 0.$$

There are only finitely many possible choices of $\bar{j}$ and $\bar{k}$, so the values are uniformly bounded between two constants $b_1 < b_2$. Without loss of generality, we may assume $b_1 \leq b \leq b_2$. This completes the proof. □

While any sequence of curves as above satisfies the conditions in sections in $\mathcal{P}$ from Definition 3.1, we will need one more condition when analyzing the limits of Teichmüller geodesics. It turns out that any construction as above also satisfies this property. We record this property here for later use.

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Lemma 7.7. *Suppose that the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is constructed as above. If γ_k, γ_h are any two curves with $m \le h - k < 2m - 1$, then γ_k and γ_h fill a subsurface whose boundary consists entirely of curves in the sequence. Furthermore, for any $k \le j \le h$, γ_j is either contained in this subsurface, or is disjoint from it. If $h - k \ge 2m - 1$, then γ_k and γ_h fill S.*

Proof. First assume $m \le h - k \le 2m - 1$. Applying the homeomorphism $H_k\colon S \to S$ from Lemma 7.6, γ_k and γ_h are sent to $\gamma_{\bar{k}}$ and $f_{\bar{k}} \cdots f_{\bar{h}}(\gamma_{\bar{h}}) = F_{\bar{k},\bar{h}}(\gamma_{\bar{h}})$, respectively. This fills the surface $X_{\bar{k},\bar{h}}$ which has boundary contained in $\gamma_0 \cup \cdots \cup \gamma_{m-1}$. By Lemma 7.1 it follows that $H_k^{-1}(X_{\bar{k},\bar{h}})$ is filled by $\{\gamma_k, \gamma_h\}$ and has boundary in $H_k(\gamma_0) \cup \cdots \cup H_k(\gamma_{m-1})$. All the components of this multicurve are in our sequence, as required for the first statement.

For each $k \le j \le h - m$ and $k + m \le j \le h$, we hace $\bar{j} \in \mathcal{J}(\bar{k}, \bar{h})$, and as pointed out in Remark 7.2, $\gamma_{\bar{j}}$ and $F_{\bar{k},\bar{j}}(\gamma_{\bar{j}})$ are contained in $X_{\bar{k},\bar{h}}$. Consequently, for these values of j, $\gamma_j \in H_k(X_{\bar{k},\bar{h}})$. On the other hand, if $k < j < h$, and j does not fall into one of the above two cases, then $h - m + 1 \le j \le k + m - 1$, which implies $0 \le j - k, h - j \le m - 1$ and hence $i(\gamma_j, \gamma_k) = i(\gamma_j, \gamma_h) = 0$, and hence γ_j is disjoint from $H_k(X_{\bar{k},\bar{h}})$. This completes the proof of the second statement.

When $h - k = 2m - 1$, we have $X_{\bar{k},\bar{h}} = S$, and hence $\{\gamma_{\bar{k}}, F_{\bar{k},\bar{h}}(\gamma_{\bar{h}})\}$ fills S. Consequently, $\{\gamma_k, \gamma_h\}$ also fills S.

Now we must prove that for $h - k \ge 2m - 1$, that γ_k and γ_h fill S. The proof is by induction, but we need a little more information in the induction. For simplicity, we assume that $k \ge m + 1$ to avoid special cases.

To describe the additional conditions, for $k < l$, let $\Phi_l = \phi_m \cdots \phi_l$, so that Φ_{k+m-1}^{-1} sends the curves $\gamma_k, \ldots, \gamma_h$ (in order) to the curves

$$\gamma_{\bar{k}}, \ldots, \gamma_{\overline{k+m-1}}, \phi_{k+m}(\gamma_{\overline{k+m}}), \ldots, \phi_{k+m} \cdots \phi_h(\gamma_{\bar{h}}).$$

With this notation, we now wish to prove by double induction (on k and $h - k$) that for all $m + 1 \le k < h$ with $h - k \ge 2m - 1$ we have

$$\{\gamma_k, \gamma_h\} \text{ fills } S \quad \text{and} \quad d_{\Phi_{k+m-1}(X_{\bar{k}})}(\gamma_k, \gamma_h) \ge 12.$$

The base case is $h - k = 2m - 1$ and any $k \ge m + 1$. We have already pointed out that $\{\gamma_k, \gamma_h\}$ fills S. We note that applying Φ_{k+m}^{-1} takes $\gamma_{k+1}, \ldots, \gamma_h$ to

$$\gamma_{\overline{k+1}}, \ldots, \gamma_{\overline{k+m}}, \phi_{k+m+1}(\gamma_{\overline{k+m+1}}), \ldots, \phi_{k+m} \cdots \phi_{k+2m-1}(\gamma_{\overline{k+2m-1}}).$$

For the first and last curves $\{\gamma_{\overline{k+1}}, \phi_{k+m+1} \cdots \phi_{k+2m-1}(\gamma_{\overline{k+2m-1}})\}$ we see that these fill

$$X_{\overline{k+1},\overline{k+2m-1}} = X_{\overline{k+1},\overline{k-1}}$$

which has $\gamma_{\bar{k}}$ as a boundary component. Since $\pi_{\bar{k}}(\phi_{k+m+1} \cdots \phi_{k+2m-1}(\gamma_{\overline{k+2m-1}}))$ is disjoint from $\gamma_{\bar{k}}$, it follows that applying ϕ_{k+m} to this last curve $\phi_{k+m+1} \cdots \phi_{k+2m-1}(\gamma_{\overline{k+2m-1}})$ we have

$$d_{\bar{k}}(\gamma_{\bar{k}}, \phi_{k+m}\phi_{k+m+1} \cdots \phi_{k+2m-1}(\gamma_{\overline{k+2m-1}})) \ge 14 > 12.$$

But notice that $\Phi_{k+m-1}^{-1}(\gamma_{k+2m-1}) = \phi_{k+m} \cdots \phi_{k+2m-1}(\gamma_{\overline{k+2m-1}})$ while on the other hand $\Phi_{k+m-1}^{-1}(\gamma_k) = \gamma_{\bar{k}}$, hence

$$d_{\Phi_{k+m-1}(X_{\bar{k}})}(\gamma_k, \gamma_{k+2m-1}) \ge 12,$$

as required for the base case.

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For the induction step, the proof is quite similar. We assume that the statement holds for all $k \geq m+1$ and all $2m-1 \leq h-k \leq N$, and prove it for $h-k = N+1$. Since $h-(k+1) = N$ and $k+1 \geq m+2 \geq m+1$, by the inductive assumption it follows that $\{\gamma_{k+1}, \gamma_h\}$ fills S and that

$$d_{\Phi_{k+m}(X_{\overline{k+1}})}(\gamma_{k+1}, \gamma_h) \geq 12.$$

Therefore, applying Φ_{k+m}^{-1}, we have

$$d_{\overline{k+1}}(\gamma_{\overline{k+1}}, \phi_{k+m+1} \cdots \phi_h(\gamma_{\bar{h}})) \geq 12.$$

The homeomorphism Φ_{k+m}^{-1} sends $\gamma_k, \dots, \gamma_h$ to the sequence

$$\phi_{k+m}^{-1}(\gamma_{\bar{k}}), \gamma_{\overline{k+1}}, \dots, \gamma_{\overline{k+m}}, \phi_{k+m+1}(\gamma_{\overline{k+1}}), \dots, \phi_{k+m+1} \cdots \phi_h(\gamma_{\bar{h}}).$$

Since $\gamma_{\bar{k}} \subset \partial X_{\overline{k+1}}$ and $\gamma_{\overline{k+1}} \subset \partial X_{\bar{k}}$, Theorem 2.9 (see also Remark 2.10) ensures that we have

$$d_{\bar{k}}(\gamma_{\bar{k}}, \phi_{k+m+1} \cdots \phi_h(\gamma_{\bar{h}})) \leq 4.$$

Applying ϕ_{k+m} (which translates by at least 16 on $\mathcal{C}(X_{\bar{k}})$) to the second curve, we get

$$d_{\bar{k}}(\gamma_{\bar{k}}, \phi_{k+m}\phi_{k+m+1} \cdots \phi_h(\gamma_{\bar{h}}) \geq 12.$$

In particular, we have

$$d_{\Phi_{k+m-1}(X_{\bar{k}})}(\gamma_k, \gamma_h) \geq 12.$$

This proves part of the requirement on γ_k, γ_h.

We must also show that $\{\gamma_k, \gamma_h\}$ fills the surface S. We will show that the Φ_{k+m-1}-image $\{\gamma_{\bar{k}}, \phi_{k+m} \cdots \phi_h(\gamma_{\bar{h}})\}$ fills S, which will suffice. To see this, take any essential curve δ and suppose it is disjoint from both $\gamma_{\bar{k}}$ and $\phi_{k+m} \cdots \phi_h(\gamma_{\bar{h}})$. Then note that δ must have empty projection to $X_{\bar{k}}$, for otherwise the triangle inequality implies that the distance from $\pi_{\bar{k}}(\gamma_{\bar{k}})$ to $\pi_{\bar{k}}(\phi_{k+m} \cdots \phi_h(\gamma_{\bar{h}}))$ is at most 4, a contradiction to the fact that

$$d_{\bar{k}}(\gamma_{\bar{k}}, \phi_{k+m+1} \cdots \phi_h(\gamma_{\bar{h}})) = d_{\Phi_{k+m-1}(X_{\bar{k}})}(\gamma_k, \gamma_h) \geq 12.$$

Since $\{\gamma_{\overline{k+1}}, \phi_{k+m+1} \cdots \phi_h(\gamma_{\bar{h}})\}$ fills S, δ must intersect one of these curves. However, $\gamma_{\overline{k+1}}$ is contained in the boundary of $X_{\bar{k}}$, and hence δ is disjoint from this. Consequently, δ must intersect $\phi_{k+m+1} \cdots \phi_h(\gamma_{\bar{h}})$. Since ϕ_{k+m} is supported on $X_{\bar{k}}$ which is disjoint from δ, we have

$$\begin{aligned} 0 \neq i(\delta, \phi_{k+m+1} \cdots \phi_h(\gamma_{\bar{h}})) &= i(\phi_{k+m}^{-1}(\delta), \phi_{k+m+1} \cdots \phi_h(\gamma_{\bar{h}})) \\ &= i(\delta, \phi_{k+m} \cdots \phi_h(\gamma_{\bar{h}})). \end{aligned}$$

This contradicts our initial assumption on δ, hence no such δ exists and $\{\gamma_{\bar{h}}, \phi_{k+m} \cdots \phi_h(\gamma_{\bar{h}})\}$ fills S as required. This completes the proof. □

7.3. Specific examples. Here we provide two specific families of examples of the general construction, but it is quite flexible and easy to build many more examples. We need to describe $\gamma_0, \dots, \gamma_{m-1}$, together with the rest of the data from the beginning of Section 7.1. For this, we will first ensure that all of our subsurfaces X_k have the property that $\gamma_{k\pm 1} \subseteq \partial X_k$ (indices mod m). This is the first of the four conditions required. For the other three conditions,

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it will be enough to choose the sequence so that for any $0 \leq k, h \leq m - 1$, there is a homeomorphism of pairs $(X_k, \gamma_k) \cong (X_h, \gamma_h)$. For then, we can choose $f_0 \colon S \to S$ any homeomorphism which is the identity on $S \setminus X_k$, pseudo-Anosov on $\mathcal{AC}(X_k)$ with translation distance at least 15, and then use the homeomorphisms $(X_0, \gamma_0) \cong (X_k, \gamma_k)$ to conjugate f_0 to homeomorphisms $f_k \colon S \to S$.

7.3.1. Maximal-dimensional simplices. For the first family of examples, we can choose a pants decomposition on $S_{g,0}$ a closed genus $g \geq 3$ surface as shown in Figure 2. Each X_k is homeomorphic to a 4-holed sphere, and $\gamma_k \subset X_k$ is an essential curve. Any two (X_k, γ_k) and (X_h, γ_h) are clearly homeomorphic pairs. In this case $m = 3g - 3$, and the limiting lamination ν from Proposition 4.4 defines a simplex of measures with maximal possible dimension in $\mathcal{PML}(S)$ by Theorem 6.5. One can also construct examples in genus 2 by taking $\gamma_0, \gamma_1, \gamma_2$ to be a pants decomposition of non-separating curves.

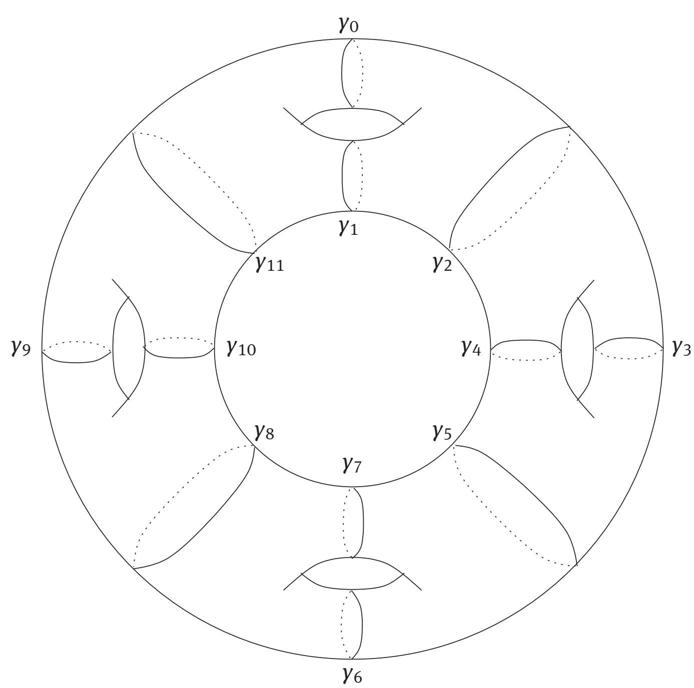

Figure 2. The pairwise disjoint curves $\gamma_0, \ldots, \gamma_{m-1}$ for the first family of examples in the case of genus 5 (and hence $m = 12$).

7.3.2. Non-maximal examples. For our second family, we choose $m = g - 1$, and take a sequence $\gamma_0, \ldots, \gamma_{m-1}$ as shown in Figure 3. Here each X_k is homeomorphic to a surface of genus 2 with two boundary components and γ_k is a curve that cuts X_k into two genus 1 surfaces with two boundary components.

8. Teichmüller geodesics and active intervals

In [36–38] the fourth author has developed techniques to control the length-functions and twist parameters along Teichmüller geodesics in terms of subsurface coefficients. In [22] this

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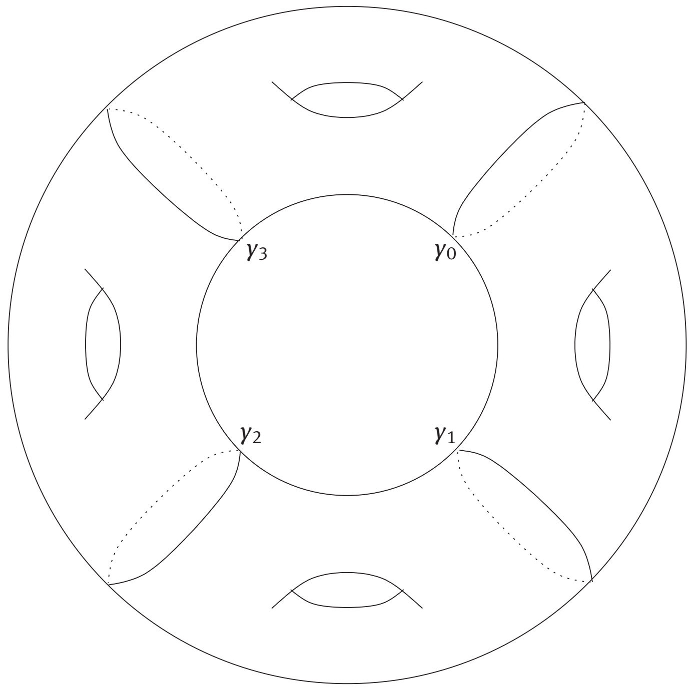

Figure 3. The pairwise disjoint curves $\gamma_0, \dots, \gamma_{m-1}$ for the second family in the case of genus 5 (and hence $m = 4$).

control was used to study the limit sets of Teichmüller geodesics in the Thurston compactification of Teichmüller space. Here we also appeal to this control. Most of the estimates in this section are similar to the ones in [22, Section 6].

For the remainder of this section and the next we assume that $\{\gamma_k\}_{k=0}^{\infty}$ is a sequence of curves satisfying the condition $\mathcal{P}$ from Definition 3.1 with $a > 1$ large enough to satisfy (5.4) and consequently so that (5.3) in Theorem 5.1 holds, and the sequence of powers $\{e_k\}_{k=0}^{\infty}$ satisfy the growth condition (3.1) for this a. For $h = 0, \dots, m-1$, let $\gamma_i^h = \gamma_{im+h}$, as usual.

Let ν be the nonuniquely ergodic lamination determined by the sequence (see Theorem 4.3 and Corollary 6.2). Furthermore let $\bar{\nu}^h$, for $h = 0, \dots, m-1$, be the ergodic measures from Theorems 5.10 and 6.5, so that $\gamma_i^h \to \bar{\nu}^h$ in $\mathcal{PML}(S)$, for each h. Let

$$\bar{\nu} = \sum_{h=0}^{m-1} x_h \bar{\nu}^h,$$

for any $x_h > 0$ for each $h = 0, \dots, m-1$.

Let $X \in \mathrm{Teich}(S)$ and μ be a short marking at X. By [18], there is a unique Teichmüller geodesic ray starting at X with vertical foliation $\bar{\nu}$, and we let $\bar{\eta}$ be the horizontal foliation (with support η). Denote the Teichmüller geodesic ray by $r : [0, \infty) \to \mathrm{Teich}(S)$. For a $t \in \mathbb{R}$, we sometimes denote $r(t) = X_t$ and denote the quadratic differential at X_t by q_t. We write $v_t(\alpha), h_t(\alpha), \ell_t(\alpha)$ for the q_t-vertical variation, q_t-horizontal variation, and q_t-length of α, respectively. In particular,

$$\begin{aligned} v_t(\alpha) &= \exp(-t) i(\alpha, \bar{\eta}), \\ h_t(\alpha) &= \exp(t) i(\alpha, \bar{\nu}), \\ \ell_t(\alpha) &\stackrel{*}{\asymp} v_t(\alpha) + h_t(\alpha). \end{aligned}$$

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We write $\mathrm{Hyp}_t(\alpha) = \mathrm{Hyp}_{X_t}(\alpha)$, the X_t-hyperbolic length of α and $w_t(\alpha) = w_{X_t}(\alpha)$ for the X_t-width, and recall from (2.1) that

$$w_t(\alpha) \stackrel{+}{\asymp} 2\log\left(\frac{1}{\mathrm{Hyp}_t(\alpha)}\right).$$

We also recall that $\epsilon_0 > 0$ is the Margulis constant, and that any two hyperbolic geodesics of length at most ϵ_0 must be embedded and disjoint.

For any curve α let $\mathrm{cyl}_t(\alpha)$ be the maximal flat cylinder foliated by all geodesic representatives of α in the q_t metric, as in Section 6.1, and let $\mathrm{mod}(\mathrm{cyl}_t(\alpha))$ denote its modulus. Fix $M > 0$ sufficiently large so that for any curve α with $\mathrm{mod}(\mathrm{cyl}_t(\alpha)) \geq M$, for some $t \in \mathbb{R}$, then $\mathrm{Hyp}_t(\alpha) \leq \epsilon_0$. For any $k \in \mathbb{N}$, let J_{γ_k}, also denoted J_k, be the *active interval of* γ_k

$$J_k = \{t \in [0,\infty) \mid \mathrm{mod}(\mathrm{cyl}_t(\gamma_k)) \geq M\}.$$

Write $J_k = [\underline{a}_k, \bar{a}_k]$ and denote the midpoint of J_k by a_k (the balance time of γ_k along the geodesic, i.e. the unique t when $v_t(\gamma_k) = h_t(\gamma_k)$). For each $h \in \{0,\dots,m-1\}$ and $i \geq 0$, we also write $J_{im+h} = J_i^h$, $a_i^h = a_{im+h}$, $\underline{a}_i^h = \underline{a}_{im+h}$, and $\bar{a}_i^h = \bar{a}_{im+h}$, to denote the data associated to $\gamma_i^h = \gamma_{im+h}$.

Proposition 8.1 (Active intervals of curves in the sequence). *With the assumptions and notation as above, we have the following:*

(i) *For k sufficiently large, $J_k \neq \emptyset$. Moreover, $J_k \cap J_l = \emptyset$ whenever $i(\gamma_k,\gamma_l) \neq 0$.*

(ii) *For $0 \leq f < k$ sufficiently large with $k - f \geq m$, J_f occurs before J_k. Consequently, some tail of each subsequence $\{J_i^h\}_{i=0}^{\infty}$ appears in order.*

(iii) *For k sufficiently large and a multiplicative constant depending only on ν and X,*

$$\mathrm{Hyp}_{a_k}(\gamma_k) \stackrel{*}{\asymp} \frac{1}{d_{\gamma_k}(\mu,\nu)} \stackrel{*}{\asymp} \frac{1}{e_k}.$$

(iv) *For an additive constant depending only on ν, X, and M, we have*

$$|J_k| \stackrel{+}{\asymp} \log d_{\gamma_k}(\mu,\nu) \stackrel{+}{\asymp} \log(e_k).$$

The following will be convenient for the proof of Proposition 8.1.

Lemma 8.2. *With notation and assumptions above, there exists $k_0 \geq 0$ sufficiently large so that if $Y \subseteq S$ is a subsurface such that for some $k \geq k_0$, $d_S(\gamma_k, \partial Y) \leq 2$, then*

$$d_Y(\mu,\nu) \stackrel{+}{\asymp}_{G+1} d_Y(\eta,\nu),$$

where G is the constant from Theorem 2.11 *(for a geodesic).*

Proof. Let g be a geodesic in $\mathcal{C}(S)$ from (any curve in) μ limiting to η if η is an ending lamination, or from μ to any curve α with $i(\alpha,\bar{\eta}) = 0$ otherwise. Since η and ν fill S, and $\gamma_k \to \nu \in \partial\mathcal{C}(S)$, the distance from γ_k to g tends to infinity with k. For Y and γ_k as in the statement of the lemma, $d_S(\partial Y, \gamma_k) \leq 2$, and hence for k sufficiently large, ∂Y has distance at least 4 from g. Consequently, ∂Y intersects every curve on g, and Theorem 2.11 guarantees that $\mathrm{diam}_Y(g) \leq G$. Thus for all $\beta \in g$, $d_Y(\beta,\mu) \leq G$. Since g limits to η (or one of it is

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curves is disjoint from η), it follows that $d_Y(\eta,\mu) \le G+1$, and so the lemma follows from the triangle inequality in $\mathcal{C}(Y)$. □

Proof of Proposition 8.1. From [36], if $d_{\gamma_k}(\eta,\nu)$ is sufficiently large, then at the balance time a_k, $\mathrm{cyl}_{a_k}(\gamma_k)$ has modulus at least M. For all k sufficiently large, (4.7) and Lemma 8.2 imply

$$d_{\gamma_k}(\eta,\nu) \stackrel{+}{\asymp} d_{\gamma_k}(\mu,\nu) \stackrel{+}{\asymp} e_k.$$

By construction, $e_k \to \infty$ as $k \to \infty$, and hence $J_k \neq \emptyset$ for all sufficiently large k. Furthermore, for all $t \in J_k$, we have $\mathrm{Hyp}_t(\gamma_k) \le \epsilon_0$. Since two curves with length bounded by ϵ_0 are disjoint, part (i) follows.

By (4.5) in Proposition 4.5 we have for all $0 \le f < k < l$ with $l-k, k-f \ge m$ that

$$d_{\gamma_k}(\gamma_f,\gamma_l) \stackrel{+}{\asymp} e_k.$$

Let $N \ge 0$ be such that for all $k \ge N$, $e_k > B_0$, where B_0 is the constant from Proposition 2.9. Thus for all $N < f < k < l$ with $l-k, k-f \ge m$ we have

$$d_{\gamma_f}(\gamma_k,\gamma_l) \le B_0.$$

Since $\gamma_k \to \nu \in \partial\mathcal{C}(S)$, the triangle inequality in $\mathcal{C}(\gamma_k)$ implies that

$$d_{\gamma_f}(\gamma_k,\nu) \stackrel{+}{\asymp} 0$$

for all $N \le f < k$ with $k-f \ge m$. Let $N_0 \ge N$ be sufficiently large so that if $f \ge N_0$, then $d_{\gamma_f}(\eta,\nu) \stackrel{+}{\asymp} e_f$. Thus, for $k-f \ge m$, $f \ge N_0$, at the balance time $t = a_f$ of γ_f, the q_t-geodesic representative of γ_k is more vertical than horizontal, and hence $a_f < a_k$. By part (i), the intervals J_f and J_k are disjoint, so part (ii) holds. (See also the discussion in [36, Proposition 5.6].)

For part (iv), observe that by [36], the modulus of $\mathrm{cyl}_t(\gamma_k)$ satisfies

$$\mathrm{mod}(\mathrm{cyl}_t(\gamma_k)) \stackrel{*}{\asymp} \frac{d_{\gamma_k}(\eta,\nu)}{\cosh^2(t-a_k)}. \tag{8.1}$$

For k is sufficiently large, Lemma 8.2 implies $d_{\gamma_k}(\eta,\nu) \stackrel{+}{\asymp} d_{\gamma_k}(\mu,\nu) \stackrel{+}{\asymp} e_k$. At the endpoint $\bar{a}_k$ of J_k, $\mathrm{mod}(\mathrm{cyl}_{\bar{a}_k}(\gamma_k)) = M$. Since $|J_k| = 2(\bar{a}_k - a_k)$, we have

$$M \stackrel{*}{\asymp} \frac{e_k}{\cosh^2(\frac{1}{2}|J_k|)}.$$

Taking logarithms we obtain $\log(e_k) - |J_k| \stackrel{+}{\asymp} \log(M)$, proving part (iv).

We proceed to the proof of part (iii). Following Rafi in [36, Section 6], we introduce the following constants associated to a curve $\alpha \in \mathcal{C}(S)$ and an essential subsurface $Y \subseteq S$ with $\alpha \subseteq \partial Y$ (when Y is an annulus, recall that $\alpha \subseteq \partial Y$ means that α is the core curve of Y).

- If Y is a nonannular subsurface, an arc β in Y is a *common K-quasi-parallel of $\pi_Y(\eta)$ and $\pi_Y(\nu)$ for α and Y* if β transversely intersects α and

  $$\max\{i(\beta,\pi_Y(\eta)), i(\beta,\pi_Y(\nu))\} \le K.$$

  Here $\pi_Y(\eta)$ denotes the arc-and-curve projection of η: the union of arcs and curves obtained by intersecting η with Y (likewise for ν). Define $K(Y) = \log K$, where K is the smallest number so that η and ν have a common K-quasi-parallel.
- If Y is an annular subsurface, let $K(Y) = d_Y(\eta,\nu)$.

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Now define K_α to be the largest $K(Y)$ where $\alpha \subseteq \partial Y$. Then [36, Theorem 6.1] implies that

$$\mathrm{Hyp}_a(\alpha) \overset{*}{\asymp} \frac{1}{K_\alpha},$$

where a is the balance time of α along the geodesic ray r.

In what follows we show that for all sufficiently large k, K_{γ_k} is approximately equal to e_k. Since we will be interested in subsurfaces Y with $\gamma_k \subseteq \partial Y$ (or subsurfaces of those, $Z \subseteq Y$), we can apply to Lemma 8.2 deducing that

$$d_Y(\eta, \nu) \overset{+}{\asymp} d_Y(\mu, \nu).$$

We will assume that k is sufficiently large for this to hold, and will use this without further mention.

First suppose Y is the annulus with core curve γ_k, and observe that by Proposition 4.5 and Lemma 8.2,

$$d_Y(\eta, \nu) \overset{+}{\asymp} d_Y(\mu, \nu) \overset{+}{\asymp} e_k,$$

thus $K(Y) \overset{+}{\asymp} e_k$. So we consider the case that Y is a nonannular subsurface with $\gamma_k \subseteq \partial Y$, and prove that for sufficiently large k, $K(Y) \prec e_k$.

If Y contains no curves γ_k from the sequence as essential curves, then for every subsurface $Z \subseteq Y$, by Proposition 4.5 and Lemma 8.2 we have

$$d_Z(\eta, \nu) \overset{+}{\asymp} d_Z(\mu, \nu) \overset{+}{\asymp} 0.$$

Then choosing the threshold A in Theorem 2.7 larger than the upper bound on these projections, and applying the theorem to $\pi_Y(\eta), \pi_Y(\nu)$, we see that

$$i(\pi_Y(\eta), \pi_Y(\nu)) \overset{+}{\asymp} 0.$$

In this case we have $K(Y) \overset{+}{\asymp} 0$, and so $K(Y) \prec e_k$ for all sufficiently large k.

Next we suppose that there are curves from our sequence contained in Y. Let

$$\{\gamma_l\}_{l \in \mathcal{L}} \subseteq \{\gamma_f\}_{f=0}^{\infty},$$

where $\mathcal{L}$ is an ordered subset of $\mathbb{N}$ which is the set of curves from our sequence which are contained in Y. From (4.1) in Theorem 4.1 we see that $\mathcal{L} \subseteq \{k-m+1, \dots, k+m-1\}$ since any other curve in the sequence intersects γ_k. We proceed to find an upper bound for the factor $K(Y)$. For this purpose let $\beta \subseteq \pi_Y(\gamma_{k+m})$ be any component arc of the projection. Then from Theorem 2.7 and Lemma 8.2 we have

$$i(\beta, \pi_Y \nu) \asymp \sum_{\substack{W \subseteq Y,\\ \text{nonannular}}} \{d_W(\gamma_{k+m}, \nu)\}_A + \sum_{\substack{W \subseteq Y,\\ \text{annular}}} \log\{d_W(\gamma_{k+m}, \nu)\}_A$$

and

$$\begin{aligned} i(\beta, \pi_Y \eta) &\asymp \sum_{\substack{W \subseteq Y,\\ \text{nonannular}}} \{d_W(\gamma_{k+m}, \eta)\}_A + \sum_{\substack{W \subseteq Y,\\ \text{annular}}} \log\{d_W(\gamma_{k+m}, \eta)\}_A \\ &\asymp \sum_{\substack{W \subseteq Y,\\ \text{nonannular}}} \{d_W(\gamma_{k+m}, \mu)\}_A + \sum_{\substack{W \subseteq Y,\\ \text{annular}}} \log\{d_W(\gamma_{k+m}, \mu)\}_A. \end{aligned}$$

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Choose the threshold constant A from Theorem 2.7 larger than the constant $R(\mu)$ from Proposition 4.5. Appealing to that proposition and the fact that any $l \in \mathcal{L}$ is less than $k+m$, the first of these equations implies that $i(\beta, \pi_Y \nu) \asymp 0$. For the second set of equations, note that any $l \in \mathcal{L}$ with $\gamma_l \pitchfork \gamma_{k+m}$ has $l \leq k$. Therefore, by Theorem 2.7 and the fact that $\{e_f\}$ is increasing, we have

$$\begin{aligned} i(\beta, \pi_Y \mu) &\asymp \sum_{l \in \mathcal{L}} \log\{d_{\gamma_l}(\gamma_{k+m}, \mu)\}_A \\ &\prec \sum_{l=k-m+1}^{k} \log(d_{\gamma_l}(\gamma_{k+m}, \mu)) \\ &\asymp \sum_{l=k-m+1}^{k} \log(e_l) \prec m \log(e_k) \prec e_k. \end{aligned}$$

Therefore, β is a K-quasi-parallel with $K \prec e_k$. Consequently,

$$K(Y) \leq \log(K) \prec \log(e_k) \prec e_k.$$

This completes the proof of part (iii), and hence the proposition. □

Next we list some estimates for the locations of the intervals $J_i^h \subseteq [0, \infty)$, and provide more information on the relative positions of the intervals.

Let $h \in \{0, \ldots, m-1\}$. From part (i) and (iv) of Proposition 8.1, together with the definitions, we have that for i sufficiently large

$$\underline{a}_i^h \stackrel{+}{\asymp} a_i^h - \frac{\log e_i^h}{2}, \tag{8.2}$$

$$\bar{a}_i^h \stackrel{+}{\asymp} a_i^h + \frac{\log e_i^h}{2}. \tag{8.3}$$

Together with these estimates, the next lemma tells us the location of the active intervals, up to an additive error.

Lemma 8.3. *For any $h = \{0, \ldots, m-1\}$ and i sufficiently large,*

$$a_i^h \stackrel{+}{\asymp} \sum_{j=0}^{i-1} \log be_j^h + \frac{\log e_i^h}{2} - \frac{\log x_h}{2}. \tag{8.4}$$

The additive error depends on X, γ_0^h, and ν.

Proof. The proof of this lemma is similar to that of [22, Lemma 6.3], so we just sketch the proof. Choose i sufficiently large so that $J_i^h \neq \emptyset$ and $a_i^h > 0$, and so that we may estimate $i(\gamma_i^h, \mu)$ using Lemma 5.11 (since μ is a finite set of curves). Then appealing to the fact that X is a fixed surface and μ a short marking, we have

$$v_0(\gamma_i^h) \stackrel{*}{\asymp} l_0(\gamma_i^h) \stackrel{*}{\asymp} \mathrm{Hyp}_0(\gamma_i^h) \stackrel{*}{\asymp} i(\gamma_i^h, \mu) \stackrel{*}{\asymp} A(0, h+im) = \prod_{j=0}^{i-1} be_j^h. \tag{8.5}$$

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Since $v_t(\gamma_i^h)h_t(\gamma_i^h)$ is constant in t, and $v_{a_i^h}(\gamma_i^h) = h_{a_i^h}(\gamma_i^h)$, we have, for i sufficiently large,

$$\begin{aligned} v^2_{a_i^h}(\gamma_i^h) &= v_{a_i^h}(\gamma_i^h)h_{a_i^h}(\gamma_i^h) \\ &= v_0(\gamma_i^h)h_0(\gamma_i^h) \\ &\stackrel{*}{\asymp} i(\gamma_i^h, \mu)i(\gamma_i^h, \bar{\nu}) \\ &\stackrel{*}{\asymp} i(\gamma_i^h, \mu)\left(\sum_{d=0}^{m-1} x_d i(\gamma_i^h, \bar{\nu}^d)\right). \end{aligned}$$

Since μ is a fixed set of curves and γ_0^h a fixed curve, $i(\gamma_0^h, \gamma_i^h) \stackrel{*}{\asymp} i(\mu, \gamma_i^h)$ for all i sufficiently large. Thus from (6.1), for $h \neq d$, $d \in \{0, \dots, m-1\}$, we have

$$i(\gamma_i^h, \bar{\nu}^h) \stackrel{*}{\asymp} \frac{1}{i(\gamma_{i+1}^h, \mu)} \quad \text{and} \quad i(\gamma_i^h, \bar{\nu}^d)i(\gamma_{i+1}^h, \mu) \to 0.$$

The above estimates and Lemma 5.11 imply that for i sufficiently large,

$$v^2_{a_i^h}(\gamma_i^h) \stackrel{*}{\asymp} x_h \frac{i(\gamma_i^h, \mu)}{i(\gamma_{i+1}^h, \mu)} \stackrel{*}{\asymp} \frac{x_h}{be_i^h}.$$

Combining this with (8.5), we have

$$\exp(a_i^h) = \frac{v_0(\gamma_i^h)}{\exp(-a_i^h)v_0(\gamma_i^h)} = \frac{v_0(\gamma_i^h)}{v_{a_i^h}(\gamma_i^h)} \stackrel{*}{\asymp} \frac{\prod_{j=0}^{i-1} be_j^h}{\sqrt{\frac{x_h}{be_i^h}}}.$$

Solving for a_i^h and taking logarithms (discarding a constant $\log b$) proves (8.4), completing the proof. □

Lemma 8.4. *For any k sufficiently large, we have $\bar{a}_k \stackrel{+}{\asymp} \underline{a}_{k+m}$, with additive error depending on X, M, γ_0^h, and ν.*

Proof. Let $k = im + h$, where $h \in \{0, \dots, m-1\}$. From (8.2), (8.3) and (8.4) we calculate

$$\begin{aligned} \underline{a}_{k+m} - \bar{a}_k &= \underline{a}_{i+1}^h - \bar{a}_i^h \\ &\stackrel{+}{\asymp} \sum_{j=0}^{i} \log be_j^h + \frac{\log e_{i+1}^h}{2} - \frac{\log x_h}{2} - \frac{\log e_{i+1}^h}{2} \\ &\qquad - \left(\sum_{j=0}^{i-1} \log be_j^h + \frac{\log e_i^h}{2} - \frac{\log x_h}{2} + \frac{\log e_i^h}{2}\right) \\ &= \log be_i^h - \log e_i^h = \log b. \end{aligned}$$

Therefore $\bar{a}_k \stackrel{+}{\asymp} \underline{a}_{k+m}$ since $\log b$ is a constant. □

Let $k, l \in \mathbb{N}$ and $0 < l - k \leq m$. Suppose that $k \equiv h \bmod m$ and $l \equiv d \bmod m$, where $h, d \in \{0, \dots, m-1\}$. Then for the pair (k, l) one of the following two hold:

(8.6) $h < d$ and there exists an $i \in \mathbb{N}$, so that $k = mi + h$ and $l = mi + d$,

(8.7) $h > d$ and there exists an $i \in \mathbb{N}$, so that $k = mi + h$ and $l = m(i+1) + d$.

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Notation 8.5. Let $\{x_i\}_{i=0}^{\infty}$ and $\{y_i\}_{i=0}^{\infty}$ be sequences of real numbers. We write $x_i \ll y_i$ if $x_i < y_i$ for all i sufficiently large and $y_i - x_i \to \infty$ as $i \to \infty$.

Lemma 8.6. *For $k, l \in \mathbb{N}$ sufficiently large, where $0 \le l - k < m$, the following holds:*

$$\bar{a}_{k-m} < \underline{a}_l \ll \bar{a}_k.$$

Proof. The proof is similar to the proof of [22, Lemma 7.3]. For the first inequality, note that $l - (k - m) \ge m$. By Proposition 8.1 (i)–(ii), J_{k-m} occurs before J_l, and so we have $\bar{a}_{k-m} < \underline{a}_l$.

We show that $\underline{a}_l \ll \bar{a}_k$. If $l = k$, then since $|J_k| \to \infty$ as $k \to \infty$, we have $\underline{a}_k \ll \bar{a}_k$. Now assume that $k < l$ and let $k \equiv h \bmod m$ and $l \equiv d \bmod m$ with $h, d \in \{0, \dots, m-1\}$. First, suppose that (8.6) holds so $h < d$. Using (3.1), (8.2), (8.3) and (8.4), and the fact that $e_k \ge a^{k-f} e_f$ for $k > f$, we have

$$\begin{aligned}
\bar{a}_k - \underline{a}_l &= \bar{a}_i^h - \underline{a}_i^d \\
&\overset{+}{\asymp} \sum_{j=0}^{i-1} \log b e_j^h + \log e_i^h - \frac{1}{2}\log x_h - \sum_{j=0}^{i-1} \log b e_j^d + \frac{1}{2}\log x_d \\
&= \sum_{j=0}^{i-1} \log \frac{e_j^h}{e_j^d} + \log e_i^h + \frac{1}{2}\log\frac{x_d}{x_h} \\
&= \sum_{j=1}^{i} \log \frac{e_j^h}{e_{j-1}^d} + \log e_0^h + \frac{1}{2}\log\frac{x_d}{x_h} \\
&\ge \sum_{j=1}^{i} (m + h - d)\log a + \frac{1}{2}\log\frac{x_d}{x_h} \\
&= i(m + h - d)\log a + \frac{1}{2}\log\frac{x_d}{x_h}.
\end{aligned}$$

Now since $m + h - d > 0$, the last term goes to ∞ as $i \to \infty$.

Next suppose that (8.7) holds so $h > d$. Then we similarly have

$$\begin{aligned}
\bar{a}_k - \underline{a}_l &= \bar{a}_i^h - \underline{a}_{i+1}^d \\
&\overset{+}{\asymp} \sum_{j=1}^{i-1} \log b e_j^h + \log e_i^h - \sum_{j=1}^{i} \log b e_j^d + \frac{1}{2}\log\frac{x_d}{x_h} \\
&= \sum_{j=1}^{i} \log \frac{e_j^h}{e_j^d} + \frac{1}{2}\log\frac{x_d}{x_h} - \log b \\
&= \sum_{j=1}^{i} \log \frac{e_j^h}{e_j^d} + \frac{1}{2}\log\frac{x_d}{x_h} - \log b \\
&= i(h - d)\log a + \frac{1}{2}\log\frac{x_d}{x_h} - \log b.
\end{aligned}$$

Now since $h - d > 0$, the last term goes to ∞ as $i \to \infty$. $\square$

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To obtain a greater control over the arrangement of intervals J_k along the Teichmüller geodesic ray (see Lemma 8.8 below) we consider the following growth conditions, in addition to (3.1):

$$e_{k+1} \geq \left(\prod_{j=0}^{k} e_j \right)^2. \tag{8.8}$$

Such sequences exist simply by setting $e_0 \geq a$ and defining e_k recursively, ensuring at every step that (8.8) is satisfied.

Condition (8.8) has the following consequence.

Lemma 8.7. *Suppose that a sequence* $\{e_k\}_k$ *satisfies* (3.1) *and* (8.8)*.*

(i) *If* (8.6) *holds, then*

$$\frac{(e_i^d)^{\frac{1}{2}}}{e_i^h} \prod_{j=0}^{i-1} \frac{e_j^d}{e_j^h} \to \infty.$$

(ii) *If* (8.7) *holds, then*

$$(e_{i+1}^d)^{\frac{1}{2}} \prod_{j=0}^{i} \frac{e_j^d}{e_j^h} \to \infty.$$

Proof. Let $k \equiv d \bmod m$ and $l \equiv h \bmod m$, where $d, h \in \{0, \dots, m-1\}$. First suppose that (8.6) holds so $h < d$. Since $\{e_k\}$ is increasing (more than) exponentially fast, we have

$$\prod_{j=0}^{i-1} \frac{e_j^d}{e_j^h} \to \infty.$$

Moreover, by (8.8) we have

$$(e_i^d)^{\frac{1}{2}} \geq e_i^h,$$

that is,

$$\frac{(e_i^d)^{\frac{1}{2}}}{e_i^h} \geq 1.$$

Thus (i) follows.

Now suppose that (8.7) holds so $h > d$. Then

$$(e_{i+1}^d)^{\frac{1}{2}} \geq \prod_{j=0}^{m(i+1)+d-1} e_j \geq \prod_{j=0}^{i} e_j^h,$$

where the second inequality holds because $m(i+1) + d > mi + h$. Therefore, condition (ii) easily follows in this case as well. □

Lemma 8.8. *Suppose that the growth condition* (8.8) *holds. Then for* $k, l \in \mathbb{N}$ *sufficiently large with* $0 < l - k < m$ *we have*

$$\bar{a}_k \ll a_l.$$

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Proof. Let $f \equiv h \bmod m$ and $l \equiv d \bmod m$, where $h, d \in \{0, \dots, m-1\}$. First suppose that (8.6) holds so $h < d$. Then from (8.3) and (8.4) we calculate

$$\begin{aligned}
a_l - \bar{a}_k &= a_i^d - \bar{a}_i^h \\
&\overset{+}{\asymp} \sum_{j=0}^{i-1} \log b e_j^d + \frac{\log e_i^d}{2} - \frac{\log x_d}{2} - \left(\sum_{j=0}^{i-1} \log b e_j^h + \log e_i^h - \frac{\log x_h}{2} \right) \\
&= \log \left(\frac{(e_i^d)^{\frac{1}{2}}}{e_i^h} \prod_{j=0}^{i-1} \frac{e_j^d}{e_j^h} \right) + \frac{1}{2} \log \frac{x_h}{x_d} \to \infty,
\end{aligned}$$

where the sequence tends to infinity as $i \to \infty$ by Lemma 8.7.

Now suppose that (8.7) holds so $h > d$. Then we have

$$\begin{aligned}
a_l - \bar{a}_k &= a_{i+1}^d - \bar{a}_i^h \\
&\overset{+}{\asymp} \sum_{j=0}^{i} \log b e_i^d + \frac{\log e_{i+1}^d}{2} - \frac{\log x_d}{2} - \left(\sum_{j=0}^{i-1} \log b e_j^h + \log e_i^h - \frac{\log x_h}{2} \right) \\
&= \log \left((e_{i+1}^d)^{\frac{1}{2}} \prod_{j=0}^{i} \frac{e_j^d}{e_j^h} \right) + \log b + \frac{1}{2} \log \frac{x_h}{x_d} \to \infty,
\end{aligned}$$

where again the convergence to infinity as $i \to \infty$ is by Lemma 8.7. □

The following conveniently summarizes the relative positions of intervals for large indices. See Figure 4.

Lemma 8.9. *For $k < l$ sufficiently large and $l < k + m$, we have*

$$\underline{a}_k \ll \underline{a}_l \ll a_k \ll \bar{a}_k < \underline{a}_{k+m} \ll a_l \ll \bar{a}_l < \underline{a}_{l+m} \ll a_{k+m}.$$

Furthermore,

$$\bar{a}_k \overset{+}{\asymp} \underline{a}_{k+m}.$$

Proof. This is immediate from Lemmas 8.4, 8.6 and 8.8. □

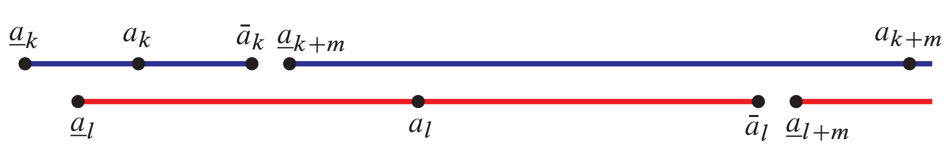

Figure 4. Relative positions of active intervals, $k < l < k + m$.

9. Limit sets of Teichmüller geodesics

In this section, we continue with the assumptions from the previous section on the sequences $\{\gamma_k\}_{k=0}^{\infty}$ and $\{e_k\}_{k=0}^{\infty}$ (including both condition (3.1) and condition (8.8)), limiting lamination $\nu \in \partial C(S)$ of $\{\gamma_k\}_{k=0}^{\infty}$, Teichmüller geodesic ray $r(t) = X_t$ with quadratic

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differential q_t at time $t \in [0, \infty)$, vertical foliations $\bar{\nu} = \sum_{h=0}^{m-1} x_i \bar{\nu}^h$ and horizontal foliation $\bar{\eta}$ for $(X, q) = (X_0, q_0)$, short marking μ for X, and active intervals $J_k = [\underline{a}_k, \bar{a}_k]$ with midpoint a_k. We will also be appealing to all the estimates from the previous sections regarding this data.

In addition, we will need one more condition on $\{\gamma_k\}_{k=0}^{\infty}$, which we add to the properties $\mathcal{P}$ assumed already: For any $k \geq 0$, let

$$\sigma_k = \gamma_k \cup \gamma_{k+1} \cup \cdots \cup \gamma_{k+m-1}.$$

The additional condition is

$\mathcal{P}$ (iv) Let α be any essential curve in $S \backslash \sigma_k$. Then there is no subsurface $Y \subseteq S$ with $\alpha \subseteq \partial Y$ which is filled by a collection of the curves in the sequence $\{\gamma_k\}_{k=0}^{\infty}$.

Recall that when Y is an annular subsurface by $\alpha \subseteq \partial Y$, we mean that α is the core curve of Y.

Remark 9.1. Note that when σ_k is a pants decomposition of S, condition $\mathcal{P}$ (iv) holds vacuously because there are no essential curves in $S \backslash \sigma_k$. Together with the other conditions in $\mathcal{P}$, the new condition $\mathcal{P}$ (iv) is equivalent to requiring that any subsurface filled by a subset of $\{\gamma_k\}_{k=0}^{\infty}$ has as boundary a union of curves in $\{\gamma_k\}_{k=0}^{\infty}$. According to Lemma 7.7 condition $\mathcal{P}$ (iv) holds for the sequences constructed in Section 7.

Under these assumptions, Theorem 1.4 from the introduction, which describes the limit set of $r(t)$ in the Thurston compactification $\overline{\mathrm{Teich}}(S) = \mathrm{Teich}(S) \cup \mathcal{PML}(S)$, can be restated as follows. Recall that the set of projective classes of measures on ν is a simplex $\Delta(\nu)$ spanned by the projective classes of the ergodic measures $[\bar{\nu}^0], \ldots, [\bar{\nu}^{m-1}]$.

Theorem 9.2. *The accumulation set of $r(t)$ in $\mathcal{PML}(S)$ is the simple closed curve in the simplex $\Delta(\nu)$ that is the concatenation of edges*

$$\big[[\bar{\nu}^0], [\bar{\nu}^1]\big] \cup \big[[\bar{\nu}^1, \bar{\nu}^2]\big] \cup \cdots \cup \big[[\bar{\nu}^{m-1}], [\bar{\nu}^0]\big].$$

We begin by reducing this theorem to a more manageable statement (Theorem 9.3), which also provides more information about how the sequence limits to the simple closed curve. We then briefly sketch the idea of the proof, and describe some of the necessary estimates. After that we reduce the theorem further to a technical version (Theorem 9.17), providing even more detailed information about what the limit looks like, and which allows for a more concise proof. After supplying the final estimates necessary, we carry out the proof.

9.1. First reduction and sketch of proof. By Proposition 8.1, the intervals J_k are nonempty for all k sufficiently large. Combining this with Lemma 8.8, it follows that for all $k < l$ sufficiently large, $\bar{a}_k < \bar{a}_l$, and that $\bar{a}_l \to \infty$ with l. Therefore, the set of intervals $[\bar{a}_k, \bar{a}_{k+1}]$ for all sufficiently large k, cover all but a compact subset of $[0, \infty)$, and consecutive segments intersect only in their endpoints. Theorem 9.2 easily follows from

Theorem 9.3. *Fix $h, h' \in \{0, \ldots, m-1\}$ with $h' \equiv h+1 \bmod m$ and suppose that $\{t_i\}$ is a sequence with $t_i \in [\bar{a}_{im+h}, \bar{a}_{im+h+1}]$ for all sufficiently large i. Then $r(t_i) = X_{t_i}$ accumulates on the edge $[[\bar{\nu}^h], [\bar{\nu}^{h'}]] \subset \Delta(\nu)$.*

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Furthermore, if $\{t_i - \bar{a}_{im+h}\}$ *is bounded independent of* i, *then*

$$\lim_{i\to\infty} X_{t_i} = [\bar{\nu}^h].$$

Proof of Theorem 9.2 *assuming Theorem* 9.3. From the second part of Theorem 9.3 applied to $t_i = \bar{a}_{im+h}$, it follows that for all $h \in \{0, \dots, m-1\}$,

$$\lim_{i\to\infty} X_{\bar{a}_{im+h}} = [\bar{\nu}^h]$$

If $h' \equiv h+1$ as in Theorem 9.3, then combining this with the first part of that theorem, we see that the accumulation set of the sequence of subsets $\{r([\bar{a}_{im+h}, \bar{a}_{im+h+1}])\}_{i=0}^{\infty} \subset \mathrm{Teich}(S)$ is contained in $[[\bar{\nu}^h], [\bar{\nu}^{h'}]]$ and contains the endpoints. Consequently, any Hausdorff limit of this sequence of *connected* sets is a connected subset of $[[\bar{\nu}^h], [\bar{\nu}^{h'}]]$ containing the endpoints, and hence is equal to $[[\bar{\nu}^h], [\bar{\nu}^{h'}]]$. The accumulation set of this sequence of sets therefore contains $[[\bar{\nu}^h], [\bar{\nu}^{h'}]]$, and is thus equal to it. Since this holds for every $h \in \{0, \dots, m-1\}$, and the intervals $\{[\bar{a}_k, \bar{a}_{k+1}]\}$ cover all but a compact subset of $[0, \infty)$, this completes the proof. □

Remark 9.4. Before proceeding we note that the assumptions on $\{\gamma_k\}_{k=0}^{\infty}$ and $\{e_k\}_{k=0}^{\infty}$ are "shift invariant", meaning that if we start the sequence at any $k_0 \geq 0$, and reindex (without changing the order), the resulting sequence will also satisfy all the required conditions. Consequently, it suffices to prove Theorem 9.3 for $h = 0$ and $h' = 1$. This greatly simplifies the notation, and allows us to avoid duplicating essentially identical arguments.

To sketch the proof, we recall that a sequence $\{Z_i\} \subset \mathrm{Teich}(S)$ converges to a point $[\bar{\lambda}] \in \mathcal{PML}(S)$ if and only if

$$\lim_{i\to\infty} \frac{\mathrm{Hyp}_{Z_i}(\delta)}{\mathrm{Hyp}_{Z_i}(\delta')} = \frac{i(\bar{\lambda}, \delta)}{i(\bar{\lambda}, \delta')}$$

for all simple closed curves δ, δ' with $i(\bar{\lambda}, \delta') \neq 0$; see Section 2. Thus we must provide sufficient control over the hyperbolic lengths of curves and relate these to intersection numbers with measures on ν.

Now the idea of the proof of this theorem is as follows. For any sufficiently large t, we estimate hyperbolic lengths $\mathrm{Hyp}_{X_t}(\delta)$ in terms of "contributions" from the intersections of δ with the curves in a bounded length pants decomposition (Proposition 9.6). When t is in the interval $[\bar{a}_k, \bar{a}_{k+1}]$, we choose a bounded length pants decomposition containing either σ_k or σ_{k+1}, depending on more precise information about t. The contributions from the curves in these sub-multicurves dominate the contributions from the other curves (the ratios tend to zero), and so the key is to understand these contributions.

On the active interval J_l, the contribution from γ_l grows linearly in the first half of the interval (Lemma 9.10), but during the second half, they speed up. Thus near $\bar{a}_k$, the contribution from γ_k will be greater than from the rest of σ_k, since $\bar{a}_k$ is still in the first half of J_l, for $l = k+1, \dots, k+m-1$. As we proceed far beyond $\bar{a}_k$, the bounded length pants decomposition eventually changes to become σ_{k+1}. The contribution from γ_k transitions to the contribution from γ_{k+m} and until the contribution from γ_{k+1} speeds up, this is the dominating term. However, as the contribution from γ_{k+1} speeds up, its contribution eventually takes over. During the transition, the contribution from γ_l, for $2 \leq l \leq m-1$ is still dominated by either γ_{k+m} or γ_{k+1}.

With this sketch in mind, we now start to discuss the details.

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9.2. General hyperbolic geometry estimates. For a curve $\alpha \in \mathcal{C}(S)$ and a point $Z \in \mathrm{Teich}(S)$, we have the length and width $\mathrm{Hyp}_Z(\alpha)$ and $w_Z(\alpha)$, respectively, as defined in Section 2. Given two curves $\alpha, \delta \in \mathcal{C}(S)$ and $Z \in \mathrm{Teich}(S)$, we will also need the *twist of δ about α with respect to Z*, denoted $\mathrm{tw}_\alpha(\delta, Z)$. This is defined as

$$\mathrm{tw}_\alpha(\delta, Z) = \mathrm{diam}_\alpha(\pi_\alpha(\delta) \cup \alpha^{\perp Z}) \geq 0,$$

where $\alpha^{\perp Z}$ is the set of Z-geodesics in the annular cover Y_α meeting (the lift of the geodesic representative of) α orthogonally.

Remark 9.5. There are different definitions of $\mathrm{tw}_\alpha(\delta, Z)$ in the literature (see e.g. [9, 10, 33]). Some of these come equipped with a sign which we have no need of, and our definition agrees with (the absolute values of) the other definitions, up to a uniformly bounded additive error (at least those we will be appealing to).

For curves $\alpha, \delta \in \mathcal{C}(S)$ and $Z \in \mathrm{Teich}(S)$ define the *contribution to the Z-length of δ coming from α* by

$$\mathrm{Hyp}_Z(\delta, \alpha) := i(\delta, \alpha)[w_Z(\alpha) + \mathrm{tw}_\alpha(\delta, Z)\,\mathrm{Hyp}_Z(\alpha)]. \tag{9.1}$$

The next fact, from [10, Lemma 7.2], provides our primary means of control on hyperbolic lengths.

Proposition 9.6. *Given $L > 0$ and $Z \in \mathrm{Teich}(S)$, suppose that P is an L-bounded length pants decomposition ($\mathrm{Hyp}_Z(\alpha) \leq L$ for all $\alpha \in P$). Then for any curve $\delta \in \mathcal{C}(S)$ we have*

$$\Big|\mathrm{Hyp}_Z(\delta) - \sum_{\alpha \in P} \mathrm{Hyp}_Z(\delta, \alpha)\Big| = O\Big(\sum_{\alpha \in P} i(\delta, \alpha)\Big),$$

where the constant of the O-notation depends only on L.

To effectively use this proposition to analyze lengths of curves in X_t as $t \to \infty$, we must develop a better picture of the hyperbolic geometry of bounded length curves in X_t.

9.3. Hyperbolic estimates for $\{\gamma_k\}$. As in Section 8, we will write

$$\mathrm{Hyp}_t(\alpha) = \mathrm{Hyp}_{X_t}(\alpha), \quad \mathrm{Hyp}_t(\delta, \alpha) = \mathrm{Hyp}_{X_t}(\delta, \alpha) \quad \text{and} \quad w_t(\alpha) = w_{X_t}(\alpha).$$

By a result of Wolpert [41], hyperbolic lengths change (grow/shrink) at most exponentially in Teichmüller distance, and hence we have:

Lemma 9.7. *For any curve α and any $t, s \in \mathbb{R}$, we have*

$$\mathrm{Hyp}_t(\alpha) \leq \exp(2(|t - s|))\,\mathrm{Hyp}_s(\alpha).$$

From Lemma 8.9, all sufficiently large t are either contained in exactly m intervals $J_k, \dots, J_{k+m-1}$ or in exactly $m - 1$ intervals $J_{k+1}, \dots, J_{k+m-1}$ and the *bounded length* interval $[\bar{a}_k, \underline{a}_{k+m}]$ (the interval after J_k but before J_{k+m}). In the former case, every curve in σ_k has length at most ϵ_0, the Margulis constant. In the latter case, we can use Lemma 9.7 to bound the length of curves in σ_k. It will be useful to have a slight generalization of that, which we state here.

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Lemma 9.8. *For any $W > 0$, if t is sufficiently large (depending on W), is contained in $J_{k+1}, J_{k+2}, \dots, J_{k+m-1}$, and satisfies $0 < t - \bar{a}_k < W$, then every curve in σ_k has X_t-length at most* $\exp(2W)\epsilon_0$.

Proof. Since $\bar{a}_k$ is in all the intervals $J_k, \dots, J_{k+m-1}$, we have $\mathrm{Hyp}_{\bar{a}_k}(\gamma_l) \le \epsilon_0$ for $k \le l \le k+m-1$. Now apply Lemma 9.7. □

In particular, note that once k is sufficiently large, Lemma 8.4 guarantees that $\underline{a}_{k+m} - \bar{a}_k$ is uniformly bounded by some constant W_0, and so setting $L_0 = \exp(2W_0)\epsilon_0$, we see that for any sufficiently large t, there is always some k so that all curves of σ_k have length at most L_0. In addition, this gives us lower bounds on lengths as well.

Lemma 9.9. *For all k sufficiently large,* $\mathrm{Hyp}_{\underline{a}_k}(\gamma_k) \overset{*}{\asymp} 1 \overset{*}{\asymp} \mathrm{Hyp}_{\bar{a}_k}(\gamma_k)$.

The multiplicative constant here depends only on W_0, the constants in property $\mathcal{P}$, and the Margulis constant ϵ_0.

Proof. We already have $\mathrm{Hyp}_{\underline{a}_k}(\gamma_k) \le \epsilon_0$, so we need to prove a uniform lower bound. Since $i(\gamma_k, \gamma_{k-m}) \in [b_1, b_2]$ from $\mathcal{P}$, and $\mathrm{Hyp}_{\underline{a}_k}(\gamma_{k-m}) \le L_0 = \exp(2W_0)\epsilon_0$, according to Lemma 2.12 we have

$$\mathrm{Hyp}_{\underline{a}_k}(\gamma_k) \ge w_{\underline{a}_k}(\gamma_{k-m}) i(\gamma_k, \gamma_{k-m}) \ge 2\sinh^{-1}\left(\frac{1}{\sinh(\frac{L_0}{2})}\right) b_1.$$

A similar argument applies for the estimate on $\mathrm{Hyp}_{\bar{a}_k}(\gamma_k)$. □

We will also need good estimates on $w_t(\gamma_k)$, especially on the first half of the interval when γ_k initially becomes short.

Lemma 9.10. *For all sufficiently large k and $t \in [\underline{a}_k, a_k]$, we have*

$$w_t(\gamma_k) \overset{+}{\asymp} 4(t - \underline{a}_k).$$

The implicit constant depends on the constant from Lemma 9.9.

Remark 9.11. There is a mistake in [22, Lemma 8.3], which claims that the width grows at most linearly with coefficient 1 (instead of 4). This does not affect any of the proofs. It is also worth noting that only an upper bound was proved there, whereas here there are both upper and lower bounds.

Proof. We first prove the upper bound on $w_t(\gamma_k)$. For this, we note that by Lemma 9.7,

$$1 \overset{*}{\asymp} \mathrm{Hyp}_{\underline{a}_k}(\gamma_k) \le \exp(2(t - \underline{a}_k))\,\mathrm{Hyp}_t(\gamma_k).$$

Dividing by $\mathrm{Hyp}_t(\gamma_k)$ and taking logarithms, we get

$$\log\left(\frac{1}{\mathrm{Hyp}_t(\gamma_k)}\right) \overset{+}{\prec} 2(t - \underline{a}_k).$$

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Multiplying by 2 and applying (2.1) proves

$$w_t(\gamma_k) \overset{+}{\prec} 4(t - \underline{a}_k).$$

For the lower bound, we will appeal to (8.1), which for k sufficiently large implies

$$\operatorname{mod}(\mathrm{cyl}_t(\gamma_k)) \overset{*}{\asymp} \frac{e_k}{\cosh^2(t - a_k)}.$$

Lifting $\mathrm{cyl}_t(\gamma_k)$ to the annular cover Y_{γ_k}, the modulus of the former is bounded above by the modulus of the latter by monotonicity of modulus of annuli. The latter on the other hand can be computed explicitly as $\pi/\mathrm{Hyp}_t(\gamma_k)$ (see e.g. [27]). Thus, taking logs and noting that

$$\log(\cosh^2(t - a_k)) \overset{+}{\asymp} 2|t - a_k| = 2(a_k - t),$$

we have

$$\log(e_k) - 2(a_k - t) \overset{+}{\prec} \log\left(\frac{\pi}{\mathrm{Hyp}_t(\gamma_k)}\right).$$

Then by Proposition 8.1 we have $\log(e_k) \overset{+}{\asymp} 2(a_k - \underline{a}_k)$ and hence

$$2(t - \underline{a}_k) \overset{+}{\prec} \log\left(\frac{1}{\mathrm{Hyp}_t(\gamma_k)}\right).$$

Appealing to (2.1) again we have $4(t - \underline{a}_k) \overset{+}{\prec} w_t(\gamma_k)$. □

We will also want to estimate $\mathrm{tw}_{\gamma_k}(\delta, X_t)$, for an arbitrary curve δ. This is given by the following formula from [37].

Theorem 9.12. *Given a curve $\delta \in \mathcal{C}_0(S)$ and large enough $k \in \mathbb{N}$ we have*

$$\mathrm{tw}_{\gamma_k}(\delta, X_t) = \begin{cases} 0 \pm O(\frac{1}{\mathrm{Hyp}_{X_t}(\gamma_k)}), & t \leq a_k, \\ e_k \pm O(\frac{1}{\mathrm{Hyp}_{X_t}(\gamma_k)}), & t \geq a_k. \end{cases}$$

This theorem shows, in particular, that the twisting is independent of δ (up to an error). In fact, arguing as in Lemma 8.2, we can easily prove that this is the case in general.

Lemma 9.13. *For any two curves δ, δ' and constant L, there exists $T > 0$ with the following property. If $\alpha \in \mathcal{C}(S)$ is a curve and $t_0 \geq T$ with $\mathrm{Hyp}_{t_0}(\alpha) \leq L$, then for all t,*

$$\mathrm{tw}_\alpha(\delta, X_t) \overset{+}{\asymp}_G \mathrm{tw}_\alpha(\delta', X_t),$$

where G is the constant from Theorem 2.11 *(for geodesics).*

Proof. For sufficiently large t_0, a curve α with bounded length must have bounded distance from some γ_k in $\mathcal{C}(S)$. As in the proof of Lemma 8.2, this can be assumed to be very far from the geodesic in $\mathcal{C}(S)$ between δ and δ' (by assuming t_0, and hence k, is very large). Appealing to Theorem 2.11, we see that $d_\alpha(\delta, \delta') \leq G$. Since $\mathrm{tw}_\alpha(\delta, X_t)$ is defined in terms of distance in $\mathcal{C}(\alpha)$, the lemma follows from the triangle inequality in $\mathcal{C}(\alpha)$. □

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9.4. Bounded length pants decompositions. When $m = \xi(S)$, then for all sufficiently large times t, there exists k so that σ_k is a bounded length pants decomposition for X_t. In this case, the estimates from the previous subsection then provide many of the necessary ingredients to apply Proposition 9.6 to control $\mathrm{Hyp}_t(\delta)$, for an arbitrary curve δ.

If $m < \xi(S)$, then a bounded length pants decompositions will contain other curves not in the sequence $\{\gamma_k\}$, and in this subsection, we describe the necessary estimates to handle the contribution to length from these. The reader only interested in the case $m = \xi(S)$ may skip this subsection.

We begin by bounding from below the length of the other curves in a bounded length pants decomposition.

Lemma 9.14. *There exists $\epsilon > 0$ depending on $R(\mu)$ from Proposition* 4.5 *such that for all sufficiently large t, if $\mathrm{Hyp}_t(\alpha) \leq \epsilon$, then $\alpha \in \{\gamma_k\}_{k=0}^{\infty}$.*

Proof. Let α be a curve not in $\{\gamma_k\}_{k=0}^{\infty}$. We will show that K_α is uniformly bounded. This requires us to bound $K(Z)$ for all essential subsurfaces Z with $\alpha \subseteq \partial Z$; see the proof of Proposition 8.1 for the definition of K_α and $K(Z)$.

By Proposition 4.5 and Lemma 8.2, $d_\alpha(\eta,\nu) \leq R(\mu) + G + 1$. Consider the set of curves in $\{\gamma_k\}_{k=0}^{\infty}$ that are contained in and fill an essential subsurface Z with the property that $\alpha \subseteq \partial Z$. Then, by $\mathcal{P}$ (iv), this set of curves is contained in a subsurface $Y \subset Z$ such that α is not a boundary component of Y.

Let $W \subset S - (Y \cup \alpha)$ be the (possibly disconnected) union of components meeting α (so two components of ∂W are isotopic to α in S). Since W contains no curves in $\{\gamma_k\}$, it follows from Proposition 4.5 and Lemma 8.2 that for all connected subsurfaces $V \subset W$, $d_V(\eta,\nu) \leq R(\mu) + G + 1$. By Theorem 2.7, $i(\pi_W(\eta), \pi_W(\nu))$ is bounded above (depending only on $R(\mu)$ and G). Consequently, there exists a simple closed curve ω in S intersecting α at most twice with $i(\pi_W(\omega), \pi_W(\eta))$ and $i(\pi_W(\omega), \pi_W(\nu))$ uniformly bounded (again depending on $R(\mu)$ and G). Therefore, $i(\pi_Z(\omega), \pi_Z(\eta))$ and $i(\pi_Z(\omega), \pi_Z(\nu))$ are uniformly bounded, hence so is $K(Z)$.

According to [36, Theorem 6.1], there is a uniform lower bound for $\mathrm{Hyp}_t(\alpha)$. The lemma is completed by setting $\epsilon > 0$ to be any number less than this uniform lower bound. □

In what follows, we will assume $L \geq L_0 = \exp(2W_0)\epsilon_0$ as in Section 9.3.

Theorem 9.15. *Let $\delta \in \mathcal{C}(S)$ be any curve and $L \geq L_0$. Then there exist constants $K, C, T > 0$, depending on L, δ, and $R(\mu)$ from Proposition* 4.5*, with the following properties. Suppose $t \geq T$ and that P is an L-bounded length pants decomposition of S containing σ_k, for some k. Then for all $\alpha \in P \setminus \sigma_k$, we have*

$$i(\delta,\alpha) \overset{*}{\prec}_K A(0, k+m-1) \quad \textit{and} \quad \mathrm{tw}_\alpha(\delta, X_t) \leq C.$$

Proof. We first prove the bound on intersection numbers. For any t, suppose α is part of an L-bounded length pants decomposition. Then [38, Theorem 6.1] and the triangle inequality imply that for every subsurface $Z \neq Y_\alpha$, we have

$$d_Z(\eta,\alpha) + d_Z(\alpha,\nu) \overset{+}{\asymp} d_Z(\eta,\nu),$$

where the additive error depends on S and L.

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We assume that $T_0 > 0$ is large enough so that for all $t \geq T_0$ there exists k so that every curve in σ_k has length at most L at time t. We write $k(t)$ for such a k. As in the proof of Lemma 8.2 and Lemma 9.13, we may take $T \geq T_0$ so that for all $t \geq T$,

$$d_Z(\delta, \nu) \stackrel{+}{\asymp} d_Z(\eta, \nu)$$

for surfaces Z with $d_S(\partial Z, \gamma_{k(t)}) \leq 2$.

Now let $t \geq T$ and let P be an L-bounded length pants decomposition containing $\sigma_{k(t)}$, and let Y be the component of $S \setminus \sigma_{k(t)}$ containing α and $Z \subseteq Y$ any subsurface. According to Proposition 4.5 and Lemma 8.2 we have $d_Z(\eta, \nu) \leq R(\mu) + G + 1$, and so combining the inequalities above, there exists R' (depending on $R(\mu)$ and L) so that for all surfaces $Z \subseteq Y$, we have

$$d_Z(\delta, \alpha) \leq R'.$$

Therefore, taking the threshold sufficiently large in Theorem 2.7 for the subsurface Y, there exists a constant I (depending on R' and Theorem 2.7) so that

$$i(\pi_Y(\delta), \alpha) \leq I.$$

Now, every arc of $\pi_Y(\delta)$ comes from a pair of intersection points with curves in $\sigma_{k(t)}$. Consequently, taking $\kappa(\delta)$ to be the constant from Lemma 5.11, we have

$$i(\delta, \alpha) \leq I \sum_{d=k(t)}^{k(t)+m-1} i(\delta, \gamma_d) \stackrel{*}{\asymp}_{\kappa(\delta)} I \sum_{d=k(t)}^{k(t)+m-1} A(0, d) \leq mIA(0, k(t) + m - 1).$$

Thus, setting $K = mI\kappa(\delta)$ proves the first statement.

For the bound on twist number, we again appeal to [37]—the same estimate in Theorem 9.12. Since $\alpha \notin \{\gamma_k\}_{k=0}^{\infty}$ (and α has bounded length at time $t \geq T$), we have

$$d_\alpha(\eta, \nu) \leq R(\mu) + G + 1,$$

where $R(\mu)$ is from Proposition 4.5 and G the constant appearing in Lemma 8.2 (from Theorem 2.11). Since the length of α is bounded below by ϵ, according to Lemma 9.14, it follows from [37] that

$$\mathrm{tw}_\alpha(\delta, X_t) \leq C$$

for some $C > 0$ depending on $R(\mu), G, \epsilon$ and the surface S. □

9.5. Second reduction and division into cases. We now consider the setup as in Theorem 9.3. As mentioned in Remark 9.4, to simplify the notation we assume $h = 0$ and $h' = 1$. It is convenient to switch to the notation $\gamma_i^h = \gamma_{im+h}$, $a_i^h = a_{im+h}$, $\sigma_i^h = \sigma_{im+h}$, etc.

We consider sequences $\{t_i\}$ with $t_i \in [\bar{a}_i^0, \bar{a}_i^1]$ for all sufficiently large i, falling into one of two possible cases:

Case 1. There exists $W > 0$ so that $t_i \in [\bar{a}_i^0, \bar{a}_i^0 + W]$.

Case 2. We have $\lim_{i\to\infty} t_i - \bar{a}_i^0 = \infty$.

For any curve $\delta \in \mathcal{C}(S)$ define

$$U_i^h(t, \delta) = w_t(\gamma_i^h) + \mathrm{tw}_{\gamma_i^h}(\delta, X_t)\, \mathrm{Hyp}_t(\gamma_i^h).$$

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We will also fix a curve δ_0 for reference and write

$$U_i^h(t) = U_i^h(t, \delta_0).$$

The next lemma is not needed for the reduction, but for later use we make note of it now.

Lemma 9.16. *For any curve $\delta \in \mathcal{C}(S)$ and $L > 0$, there exists $T > 0$ so that for all $t \geq T$ and i, h with $\mathrm{Hyp}_t(\gamma_i^h) \leq L$, we have*

$$U_i^h(t, \delta) \overset{+}{\asymp}_{GL} U_i^h(t).$$

Here the constant G is from Theorem 2.11 appearing in Lemma 9.13.

Proof. Given L, Lemma 9.13 provides $T > 0$ so that for all $t \geq T$, if $\mathrm{Hyp}_t(\gamma_i^h) \leq L$, then

$$|\mathrm{tw}_{\gamma_i^h}(\delta, X_t) - \mathrm{tw}_{\gamma_i^h}(\delta_0, X_t)| \leq G.$$

Therefore, we have

$$|U_i^h(t, \delta) - U_i^h(t)| = |\mathrm{tw}_{\gamma_i^h}(\delta, X_t) - \mathrm{tw}_{\gamma_i^h}(\delta_0, X_t)| \,\mathrm{Hyp}_t(\gamma_i^h) \leq GL,$$

as desired. □

We now turn to our second reduction.

Theorem 9.17. *Suppose that $\{t_i\}$ is a sequence with $t_i \in [\bar{a}_i^0, \bar{a}_i^1]$ for all sufficiently large i and δ is any curve (not necessarily δ_0).*

- *If $\{t_i\}$ falls into Case 1, then*

$$\lim_{i\to\infty} \frac{U_i^0(t_i)\, i(\delta, \gamma_i^0)}{\mathrm{Hyp}_{t_i}(\delta)} = 1.$$

- *If $\{t_i\}$ falls into Case 2, then*

$$\lim_{i\to\infty} \frac{U_i^1(t_i)\, i(\delta, \gamma_i^1) + U_{i+1}^0(t_i)\, i(\delta, \gamma_{i+1}^0)}{\mathrm{Hyp}_{t_i}(\delta)} = 1.$$

Note in this theorem, the terms $U_j^h(t_i)$ do not depend on δ (cf. Lemma 9.16).

Proof of Theorem 9.3 *assuming Theorem* 9.17. Suppose $\{t_j\}_{j=0}^{\infty}$ with $t_j \in [\bar{a}_{i_j}^0, \bar{a}_{i_j}^1]$ for all sufficiently large j and some i_j, so that X_{t_j} converges to some point in $\mathcal{PML}(S)$. We may pass to a subsequence so that either $t_j - \bar{a}_{i_j}^0 \leq W$ for some W, or else $t_j - \bar{a}_{i_j}^0 \to \infty$ with j. This subsequence can be viewed as a subsequence of a sequence falling into Case 1 or Case 2, respectively, and hence the conclusion of Theorem 9.17 holds for $\{t_j\}$.

Now let $\delta, \delta' \in \mathcal{C}(S)$ be any two curves. If we are in Case 2, then by Theorem 9.17 and Theorem 5.10 we have

$$\lim_{j\to\infty} \frac{\mathrm{Hyp}_{t_j}(\delta)}{\mathrm{Hyp}_{t_j}(\delta')} = \lim_{j\to\infty} \frac{\mathrm{Hyp}_{t_j}(\delta) \frac{U_{i_j}^0(t_j)\, i(\delta, \gamma_{i_j}^0)}{\mathrm{Hyp}_{t_j}(\delta)}}{\mathrm{Hyp}_{t_j}(\delta') \frac{U_{i_j}^0(t_j)\, i(\delta', \gamma_{i_j}^0)}{\mathrm{Hyp}_{t_j}(\delta')}} = \lim_{j\to\infty} \frac{i(\delta, \gamma_{i_j}^0)}{i(\delta', \gamma_{i_j}^0)} = \frac{i(\delta, \bar{\nu}^0)}{i(\delta', \bar{\nu}^0)}.$$

Since δ and δ' were arbitrary, it follows that $X_{t_j} \to [\bar{\nu}^0]$.

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Now suppose we are in the second case. Compactness of $\mathcal{PML}(S)$ implies that by passing to a further subsequence (of the same name) the sequence

$$\{[U^1_{i_j}(t_j)\gamma^1_{i_j} + U^0_{i_j+1}(t_j)\gamma^0_{i_j+1}]\}_{j=0}^{\infty}$$

converges in $\mathcal{PML}(S)$. Note that this limit is necessarily of the form

$$[y_0\bar{\nu}^0 + y_1\bar{\nu}^1] \in \big[[\bar{\nu}^0],[\bar{\nu}^1]\big]$$

by Theorem 5.10. Now observe that for all j, the numerator from Case 2 of Theorem 9.17 is given by

$$U^1_{i_j}(t_j)i(\delta,\gamma^1_{i_j}) + U^0_{i_j+1}(t_j)i(\delta,\gamma^0_{i_j+1}) = i(\delta, U^1_{i_j}(t_j)\gamma^1_{i_j} + U^0_{i_j+1}(t_j)\gamma^0_{i_j+1}).$$

Therefore, similar to the above calculation, appealing to Theorem 9.17 we have

$$\lim_{j\to\infty}\frac{\mathrm{Hyp}_{t_j}(\delta)}{\mathrm{Hyp}_{t_j}(\delta')} = \lim_{j\to\infty}\frac{i(\delta, U^1_{i_j}(t_j)\gamma^1_{i_j} + U^0_{i_j+1}(t_j)\gamma^0_{i_j+1})}{i(\delta', U^1_{i_j}(t_j)\gamma^1_{i_j} + U^0_{i_j+1}(t_j)\gamma^0_{i_j+1})} = \frac{i(\delta, y_0\bar{\nu}^0 + y_1\bar{\nu}^1)}{i(\delta', y_0\bar{\nu}^0 + y_1\bar{\nu}^1)}.$$

Again, because δ,δ' were arbitrary we see that X_{t_j} limits to $[y_0\bar{\nu}^0 + y_1\bar{\nu}^1]$. This completes the proof. □

9.6. Final estimates and proof of Theorem 9.17. Here we provide the final estimates necessary for the proof of Theorem 9.17 (and hence the main theorem). The proof for each of the two cases are similar, and many of the estimates can be made simultaneously.

We assume for the remainder of the paper that $\{t_i\}$ is a sequence so that $t_i \in [\bar{a}^0_i, \bar{a}^1_i]$ for all sufficiently large i and that δ is an arbitrary curve (not necessarily our reference curve δ_0).

If we are in Case 1 with $t_i - \bar{a}^0_i \le W$, then by Lemma 9.8, for all sufficiently large i there exist $L \ge \exp(2W)\epsilon$ and an L-bounded length pants decomposition P_i for X_{t_i} containing σ^0_i. Let

$$P^c_i = P_i \setminus \sigma^0_i.$$

If we are in Case 2, then by Lemma 8.4, for i sufficiently large, we have $t_i \in [\underline{a}^0_{i+1}, a^0_{i+1}]$, and there exist $L \ge 0$ (depending only on S) and an L-bounded pants decomposition P_i for X_{t_i} containing σ^1_i. Similar to Case 1, we let

$$P^c_i = P_i \setminus \sigma^1_i.$$

We use Proposition 9.6 to estimate $\mathrm{Hyp}_{t_i}(\delta)$. Appealing to Theorem 9.15 together with Lemma 5.11 and monotonicity of $\{A(0,k)\}_{k=0}^{\infty}$ (Lemma 5.6) to group together all the intersection number errors in Proposition 9.6, this takes a somewhat simpler form. To write it, recall that for all $h \in \{0,\dots,m-1\}$ and $i \ge 0$, we have

$$c^h_i = A(0, im+h) = \prod_{j=0}^{i-1} be^h_j.$$

The estimates are then similar, but depend on the case:

Case 1. We have

$$\mathrm{Hyp}_{t_i}(\delta) = \sum_{h=0}^{m-1}\mathrm{Hyp}_{t_i}(\delta,\gamma^h_i) + \sum_{\alpha\in P^c_i}\mathrm{Hyp}_{t_i}(\delta,\alpha) + O(c^{m-1}_i). \tag{9.2}$$

The O-error term depends on L (hence W) and δ, but is independent of i.

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Case 2. We have

$$\operatorname{Hyp}_{t_i}(\delta) = \sum_{h=1}^{m-1} \operatorname{Hyp}_{t_i}(\delta, \gamma_i^h) + \operatorname{Hyp}_{t_i}(\delta, \gamma_{i+1}^0) + \sum_{\alpha \in P_i^c} \operatorname{Hyp}_{t_i}(\delta, \alpha) + O(c_{i+1}^0). \tag{9.3}$$

In this case, the O-error term depends on L (which depends only on S) and δ, but is again independent of i.

We will appeal to the various estimates previously made, specifically those in Section 8, Section 9.3, and Section 9.4. The first estimate involves the contributions to (9.2) and (9.3) from the curves of P_i^c.

Lemma 9.18. *For all i sufficiently large and $\alpha \in P_i^c$, we have*

$$\operatorname{Hyp}_{t_i}(\delta, \alpha) = \begin{cases} O(c_i^{m-1}) & \text{in Case } 1, \\ O(c_{i+1}^m) & \text{in Case } 2. \end{cases}$$

Here the implicit constant in the O-notation depends on δ.

Proof. From (9.1) we have

$$\operatorname{Hyp}_{t_i}(\delta, \alpha) = \big(w_{t_i}(\alpha) + \operatorname{tw}_\alpha(\delta, X_{t_i}) \operatorname{Hyp}_{t_i}(\alpha)\big) i(\delta, \alpha).$$

By Lemma 9.14 and Theorem 9.15, every term on the right except $i(\delta, \alpha)$ is bounded, depending on δ and L (and the resulting constants from those statements). The lemma follows. □

Corollary 9.19. *For all i sufficiently large we have*

$$\operatorname{Hyp}_{t_i}(\delta) = \sum_{h=0}^{m-1} \operatorname{Hyp}_{t_i}(\delta, \gamma_i^h) + O(c_i^{m-1}) \qquad \textit{in Case } 1, \tag{9.4}$$

$$\operatorname{Hyp}_{t_i}(\delta) = \sum_{h=1}^{m-1} \operatorname{Hyp}_{t_i}(\delta, \gamma_i^h) + \operatorname{Hyp}_{t_i}(\delta, \gamma_{i+1}^0) + O(c_{i+1}^0) \quad \textit{in Case } 2. \tag{9.5}$$

We write the remaining terms using the notation set in the previous section as

$$\operatorname{Hyp}_{t_i}(\delta, \gamma_j^h) = U_j^h(t_i, \delta) i(\delta, \gamma_j^h).$$

Estimates for these terms are given in the next four lemmas.

Lemma 9.20. *For all sufficiently large i and all $1 < h \le m-1$, we have*

$$U_i^h(t_i, \delta) \overset{+}{\asymp} 4\left(\sum_{j=1}^{i} \log\left(\frac{e_j^0}{e_{j-1}^h}\right) + t_i - \bar{a}_i^0\right).$$

In Case 1, this also holds for $h = 1$.

Proof. Note that for $1 < h \le m-1$ (as well as $h = 1$ in Case 1), we have $\underline{a}_i^h < t_i < a_i^h$, for all sufficiently large i. Therefore, $\operatorname{Hyp}_t(\gamma_i^h) \le \epsilon_0 < L$ and so Theorem 9.12 implies

$$\operatorname{tw}_{\gamma_i^h}(\delta, X_{t_i}) \operatorname{Hyp}_{t_i}(\gamma_i^h) \overset{*}{\prec} 1.$$

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On the other hand by Lemma 9.10,

$$w_{t_i}(\gamma_i^h) \overset{+}{\asymp} 4(t_i - \underline{a}_i^h) = 4(\bar{a}_i^0 - \underline{a}_i^h + (t - \bar{a}_i^0)),$$

since $\underline{a}_i^h \ll \bar{a}_i^0 \le t_i \le a_i^h$ (for sufficiently large i and all $1 < h \le m-1$ in both cases, and also $h = 1$ in Case 1) by Lemma 8.9. The lemma now follows from this by substituting in from (8.2), (8.3), and (8.4) and dropping constants. □

Lemma 9.21. *Suppose that $\{t_i\}$ falls into Case 1 with constant W. Then for all sufficiently large i, we have*

$$U_i^0(t_i, \delta) \overset{*}{\asymp} e_i^0,$$

where the multiplicative error depends on W, δ, (and all resulting constants), but not i.

Proof. Because $t_i - \bar{a}_i^0 \le W$, $\mathrm{Hyp}_{t_i}(\gamma_i^0)$ is bounded above and below by Lemma 9.9 and Lemma 9.7, the bound depending on W. By Lemma 2.12, $w_{t_i}(\gamma_i^0)$ is also bounded. To complete the proof, we note that by Theorem 9.12,

$$\mathrm{tw}_{\gamma_i^0}(\delta, t_i) \overset{*}{\asymp} e_i^0.$$ □

Lemma 9.22. *Suppose that $\{t_i\}$ falls into Case 2. Then for all large i, we have*

$$U_{i+1}^0(t_i, \delta) \overset{+}{\asymp} 4(t - \bar{a}_i^0).$$

Proof. This is almost identical to the proof of Lemma 9.20, so we omit it. □

For the only remaining situation, a very coarse estimate will suffice.

Lemma 9.23. *Suppose that $\{t_i\}$ falls into Case 2. Then*

$$U_i^1(t_i, \delta) \to \infty.$$

Proof. Since we are in Case 2, we have $t_i - \underline{a}_i^1 \ge t_i - \bar{a}_i^0 \to \infty$. Then either $t_i \le a_i^1$ or $a_i^1 \le t_i \le \bar{a}_i^1$. In the former case, Lemma 9.10 shows that $w_{t_i}(\gamma_i^1) \to \infty$. In the latter case, either $w_{t_i}(\gamma_i^1) \to \infty$, and we are done, or else $w_{t_i}(\gamma_i^1)$ is bounded. If $w_{t_i}(\gamma_i^1)$ is bounded, then (2.1) implies $\mathrm{Hyp}_{t_i}(\gamma_i^1)$ is bounded below. Since $e_i^1 \to \infty$, Theorem 9.12 implies that $\mathrm{tw}_{\gamma_i^1}(\delta, \gamma_i^1) \to \infty$, completing the proof. □

From these, we deduce the following:

Corollary 9.24. *If the sequence $\{t_i\}$ falls into Case 1 (and hence $t_i - \bar{a}_i^0 \le W$), then for all i sufficiently large and $1 \le h \le m-1$ we have*

$$\mathrm{Hyp}_{t_i}(\delta, \gamma_i^h) \overset{*}{\asymp} \left(\sum_{j=1}^{i} \log\left(\frac{e_j^0}{e_{j-1}^h} \right) \right) \prod_{j=0}^{i-1} b e_j^h, \tag{9.6}$$

$$\mathrm{Hyp}_{t_i}(\delta, \gamma_i^0) \overset{*}{\asymp} \prod_{j=0}^{i} b e_j^0. \tag{9.7}$$

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If the sequence $\{t_i\}$ falls into Case 2 *(and hence $t_i - \bar{a}_i^0 \to \infty$), then for all i sufficiently large and* $2 \le h \le m-1$ *we have*

$$\mathrm{Hyp}_{t_i}(\delta, \gamma_i^h) \overset{*}{\asymp} \left(\sum_{j=1}^{i} \log\left(\frac{e_j^0}{e_{j-1}^h}\right) + t_i - \bar{a}_i^0 \right) \prod_{j=0}^{i-1} be_j^h, \tag{9.8}$$

$$\mathrm{Hyp}_{t_i}(\delta, \gamma_{i+1}^0) \overset{*}{\asymp} (t_i - \bar{a}_i^0) \prod_{j=0}^{i} be_j^0. \tag{9.9}$$

The multiplicative constants depend on W (in Case 1) and δ, and all constants that depend on these.

Proof. By Lemma 5.11, there exists $\kappa(\delta) > 0$ so that

$$i(\delta, \gamma_i^h) \overset{*}{\asymp}_{\kappa(\delta)} A(0, im + h) = c_i^h = \prod_{j=0}^{i-1} be_i^h.$$

Since

$$\mathrm{Hyp}_{t_i}(\delta, \gamma_j^h) = U_j^h(t_i, \delta) i(\delta, \gamma_j^h),$$

the corollary follows from Lemmas 9.20, 9.21, 9.22, and 9.23. □

Proof of Theorem 9.17. Observe that from Lemmas 9.20, 9.21, 9.22, and 9.23, we see that for all h, as $i \to \infty$ we have

$$U_i^h(\delta, t_i) \to \infty \quad \text{and} \quad U_{i+1}^0(\delta, t_i) \to \infty,$$

where the second limit is only true in Case 2, and the first is only relevant for $h = 0$ in Case 1. By Lemma 9.16, it suffices to prove Theorem 9.17 replacing all terms of the form $U_j^h(t_i)$ with terms $U_j^h(t_i, \delta)$.

The proof will use the estimates (9.4) and (9.5) from Corollary 9.19 and we divide it into the two cases.

Proof in Case 1. We look at each term on the right-hand side of (9.4) and divide by the term $\mathrm{Hyp}_{t_i}(\delta, \gamma_i^0)$. Doing this for the terms $\mathrm{Hyp}_{t_i}(\delta, \gamma_i^h)$ for $1 \le h \le m-1$, equations (9.6) and (9.7) imply

$$\frac{\mathrm{Hyp}_{t_i}(\delta, \gamma_i^h)}{\mathrm{Hyp}_{t_i}(\delta, \gamma_i^0)} \overset{*}{\asymp} be_0^h \left(\sum_{j=1}^{i} \log\left(\frac{e_j^0}{e_{j-1}^h}\right)\right) \prod_{j=1}^{i} \frac{e_{j-1}^h}{e_j^0} = \log\left(\prod_{j=1}^{i} \frac{e_j^0}{e_{j-1}^h} \right) \prod_{j=1}^{i} \frac{e_{j-1}^h}{e_j^0}.$$

Since $jm > (j-1)m + h$ implies $e_j^0 \ge a e_{j-1}^h$, we have

$$\prod_{j=1}^{i} \frac{e_{j-1}^h}{e_j^0} \le a^{-i},$$

and since $a > 1$,

$$\lim_{i \to \infty} \frac{\mathrm{Hyp}_{t_i}(\delta, \gamma_i^h)}{\mathrm{Hyp}_{t_i}(\delta, \gamma_i^0)} = 0.$$

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The only remaining term, other than $\mathrm{Hyp}_{t_i}(\delta, \gamma_i^0)$, is $O(c_i^{m-1})$. For this, we note that by definition

$$c_i^h = \prod_{j=0}^{i-1} be_j^h,$$

and therefore, for the same reason as above, we have

$$\frac{O(c_i^h)}{\mathrm{Hyp}_{t_i}(\gamma_i^0)} \stackrel{*}{\asymp} be_0^h \prod_{j=1}^{i} \frac{e_{j-1}^h}{e_j^0} \to 0$$

as $i \to \infty$. Now combining all these estimates into (9.4) we have

$$\lim_{i\to\infty} \frac{\mathrm{Hyp}_{t_i}(\delta)}{\mathrm{Hyp}_{t_i}(\delta, \gamma_i^0)} = \lim_{i\to\infty} \sum_{h=0}^{m-1} \frac{\mathrm{Hyp}_{t_i}(\delta, \gamma_i^h)}{\mathrm{Hyp}_{t_i}(\delta, \gamma_i^0)} + \frac{O(c_i^{m-1})}{\mathrm{Hyp}_{t_i}(\delta, \gamma_i^0)} = 1.$$

This completes the proof since

$$\mathrm{Hyp}_{t_i}(\delta, \gamma_i^0) = U_i^0(t_i, \delta) i(\delta, \gamma_i^0).$$

Proof in Case 2. We again look at each term on the right-hand side of (9.5) and this time begin by dividing most of the terms by $\mathrm{Hyp}_{t_i}(\delta, \gamma_{i+1}^0)$. Doing this for the terms $\mathrm{Hyp}_{t_i}(\delta, \gamma_i^h)$ for $2 \le h \le m-1$, equations (9.8) and (9.9), together with the fact that $t_i - \bar{a}_i^0 \to \infty$, imply

$$\begin{aligned} \frac{\mathrm{Hyp}_{t_i}(\delta, \gamma_i^h)}{\mathrm{Hyp}_{t_i}(\delta, \gamma_{i+1}^0)} &\stackrel{*}{\asymp} \frac{be_0^h}{t_i - \bar{a}_i^0} \left(\sum_{j=1}^{i} \log\left(\frac{e_j^0}{e_{j-1}^h} \right) + t_i - \bar{a}_i^0 \right) \prod_{j=1}^{i} \frac{e_{j-1}^h}{e_j^0} \\ &\le be_0^h \left(1 + \log\left(\prod_{j=1}^{i} \frac{e_j^0}{e_{j-1}^h} \right) \right) \prod_{j=1}^{i} \frac{e_{j-1}^h}{e_j^0}. \end{aligned}$$

Now as above, the right-hand side tends to 0 as $i \to \infty$, and hence

$$\lim_{i\to\infty} \frac{\mathrm{Hyp}_{t_i}(\delta, \gamma_i^h)}{\mathrm{Hyp}_{t_i}(\delta, \gamma_{i+1}^0)} = 0.$$

Next we consider the $O(c_{i+1}^0)$ term of (9.5). By the definition of c_{i+1}^0, together with (9.9) and the fact that $t_i - \bar{a}_i^0 \to \infty$, as $i \to \infty$ we have

$$\frac{O(c_{i+1}^0)}{\mathrm{Hyp}_{t_i}(\delta, \gamma_{i+1}^0)} \stackrel{*}{\asymp} \frac{\prod_{j=0}^{i} be_j^0}{(t_i - \bar{a}_i^0) \prod_{j=0}^{i} be_j^0} = \frac{1}{t_i - \bar{a}_i^0} \to 0.$$

Since $\mathrm{Hyp}_{t_i}(\delta, \gamma_i^1) + \mathrm{Hyp}_{t_i}(\delta, \gamma_{i+1}^0) > \mathrm{Hyp}_{t_i}(\delta, \gamma_{i+1}^0)$, we could have divided by this larger quantity, and the above limits would still be zero. Plugging into (9.5), we deduce

$$\lim_{i\to\infty} \frac{\mathrm{Hyp}_{t_i}(\delta)}{\mathrm{Hyp}_{t_i}(\delta, \gamma_i^1) + \mathrm{Hyp}_{t_i}(\delta, \gamma_{i+1}^0)} = 1.$$

Since

$$\mathrm{Hyp}_{t_i}(\delta, \gamma_i^1) + \mathrm{Hyp}_{t_i}(\delta, \gamma_{i+1}^0) = U_i^1(t_i, \delta) i(\delta, \gamma_i^1) + U_{i+1}^0(t_i, \delta) i(\delta, \gamma_{i+1}^0),$$

this completes the proof of Case 2, and hence of the theorem. □

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