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Geodesics in the mapping class group

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We construct explicit examples of geodesics in the mapping class group and show that the shadow of a geodesic in the mapping class group to the curve graph does not have to be a uniform quality quasigeodesic. We also show that the quasiaxis of a pseudo-Anosov element of the mapping class group may not have the strongly contracting property. Specifically, we show that, after choosing a generating set carefully, one can find a pseudo-Anosov homeomorphism ϕ, a sequence of points w_k and a sequence of radii r_k such that the ball $B(w_k, r_k)$ is disjoint from a quasiaxis a_ϕ of ϕ, but, for any projection from the mapping class group to a_ϕ, the diameter of the image of $B(w_k, r_k)$ grows like $\log(r_k)$.

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1 Introduction

Let S be a surface of finite type and let $\mathrm{Map}(S)$ denote the (pure) mapping class group of S, that is, the group of orientation-preserving self-homeomorphisms of S fixing the punctures of S, up to isotopy. This is a finitely generated group (see Dehn [9]) and, after choosing a generating set, the word length turns $\mathrm{Map}(S)$ into a metric space. The geometry of $\mathrm{Map}(S)$ has been a subject of extensive study. Most importantly, in [16], Masur and Minsky gave an estimate for the word length of a mapping class using the subsurface projection distances and constructed efficient quasigeodesics in the mapping class group, called hierarchy paths, connecting the identity to any given mapping class. The starting point of the construction of a hierarchy path is a geodesic in the curve graph of S which is known to be a Gromov hyperbolic space; see Masur and Minsky [15]. Hence, by construction, the shadow of a hierarchy path to the curve graph is nearly a geodesic.

It may seem intuitive that any geodesic in the mapping class group should also have this property, considering that similar statements have been shown to be true in other settings. For example, it is known that the shadow of a geodesic in Teichmüller space with respect to the Teichmüller metric is a reparametrized quasigeodesic in the curve

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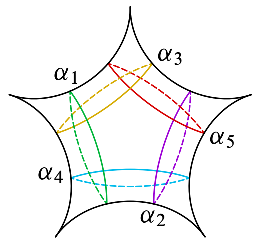

Figure 1: The curves $\alpha_1, \dots, \alpha_5$ used to generate $\mathcal{S}_n$.

graph [15]. The same is true for any geodesic in Teichmüller space with respect to the Thurston metric (see Lenzhen, Rafi and Tao [13]), for any line of minima in Teichmüller space (see Choi, Rafi and Series [7]), for a grafting ray (see Choi, Dumas and Rafi [6]), and for the set of short curves in a hyperbolic 3–manifold homeomorphic to $S \times \mathbb{R}$ (see Minsky [18]). However, it is difficult to construct explicit examples of geodesics in $\mathrm{Map}(S)$ and, so far, all estimates for the word length of an element have been up to a multiplicative error.

Here, we argue that one should not expect geodesics in $\mathrm{Map}(S)$ to be well behaved in general. Changing the generating set changes the metric on $\mathrm{Map}(S)$ significantly and a geodesic with respect to one generating set is only a quasigeodesic with respect to another generating set. Since $\mathrm{Map}(S)$ is not Gromov hyperbolic (it contains flats), its quasigeodesics are not well behaved in general. Similarly, one should not expect that the geodesics with respect to an arbitrary generating set behave well either.

We make this explicit in the case where $S = S_{0,5}$ is the five-times punctured sphere. Consider the curves $\alpha_1, \dots, \alpha_5$ depicted in Figure 1. Fix an integer $n \gg 1$ (to be determined in the proof of Theorem 1.4), and consider the generating set for $\mathrm{Map}(S)$

$$\mathcal{S}_n = \{D_{\alpha_i}, s_{i,j} : i, j \in \mathbb{Z}_5,\ j = i \pm 1 \bmod 5\},$$

where $s_{i,j} = D^n_{\alpha_i} D^{-1}_{\alpha_j}$ and D_α is a Dehn twist around a curve α. Since we are considering the pure mapping class group, the set $\{D_{\alpha_i}\}_{i=1}^5$ already generates $\mathrm{Map}(S)$. We denote the distance on $\mathrm{Map}(S)$ induced by the generating set $\mathcal{S}_n$ by $d_{\mathcal{S}_n}$. By an $\mathcal{S}_n$–geodesic, we mean a geodesic with respect to this metric.

Theorem 1.1 *There exists an $N > 1$ such that for all $n \ge N$, for every $K, C > 0$, there exists an $\mathcal{S}_n$–geodesic*

$$\mathcal{G}\colon [0, m] \to \mathrm{Map}(S)$$

such that the shadow of $\mathcal{G}$ to the curve graph $\mathcal{C}(S)$ is not a reparametrized (K, C)–quasigeodesic.

Even though the mapping class group is not Gromov hyperbolic, it does have hyperbolic directions. There are different ways to make this precise. For example, Behrstock [3] proved that in the direction of every pseudo-Anosov, the divergence function in $\mathrm{Map}(S)$ is superlinear. Another way to make this notion precise is to examine whether geodesics in $\mathrm{Map}(S)$ have the *contracting property*.

This notion is defined analogously with Gromov hyperbolic spaces, where, for every geodesic $\mathcal{G}$ and any ball disjoint from $\mathcal{G}$, the closest point projection of the ball to $\mathcal{G}$ has a uniformly bounded diameter. However, often it is useful to work with a different projection map. We call a map

$$\mathrm{Proj}\colon X \to \mathcal{G}$$

from a metric space X to a subset $\mathcal{G} \subset X$ a (d_1, d_2)*–projection map*, where $d_1, d_2 > 0$, if, for every $x \in X$ and $g \in \mathcal{G}$, we have

$$d_X(\mathrm{Proj}(x), g) \le d_1 \cdot d_X(x, g) + d_2.$$

This is a weak notion of projection since Proj is not even assumed to be coarsely Lipschitz. By the triangle inequality, the closest point projection is always a $(2,0)$–projection.

Definition 1.2 A subset $\mathcal{G}$ of a metric space X is said to have the *contracting property* if there is a constant $\rho < 1$, a constant $B > 0$ and a projection map $\mathrm{Proj}\colon X \to \mathcal{G}$ such that, for any ball $B(x, R)$ of radius R disjoint from $\mathcal{G}$, the projection of a ball $B(x, \rho R)$ of radius ρR has a diameter at most B:

$$\mathrm{diam}_X\big(\mathrm{Proj}(B(x, \rho R))\big) \le B.$$

We say $\mathcal{G}$ has the *strong contracting property* if ρ can be taken to be 1.

Remark 1.3 There are several closely related notions of contracting or strongly contracting in the literature (see for example Arzhantseva, Cashen and Tao [2, Section 2.1] for several such notions). Often, in these definitions, the projection map is assumed to be the closest point projection map. However, there are many situations where the closest point projection is not the most natural projection map. For example, in Duchin and Rafi [11], the projection in $\mathrm{Map}(S)$ is made to be compatible with the closest projection in the curve graph, which is not the same as taking the closest point projection in $\mathrm{Map}(S)$ itself (see also Masur and Minsky [15], Eskin, Masur and Rafi [12] and Clay, Rafi and Schleimer [8] for other such examples). But they always satisfy the definition above. Our weaker assumption on the projection map makes Theorem 1.4 stronger.

The axis of a pseudo-Anosov element has the contracting property in many settings. This has been shown to be true in the setting of Teichmüller space by Minsky [17], in the setting of the pants complex by Brock, Masur and Minsky [5] and in the setting of the mapping class group by Duchin and Rafi [11].

Arzhantseva, Cashen and Tao asked if the axis of a pseudo-Anosov element in the mapping class group has the strong contracting property and showed that a positive answer would imply that the mapping class group is growth tight [2]. Additionally, using work of Yang [20], one can show that if one pseudo-Anosov element has a strongly contracting axis with respect to some generating set, then a generic element in mapping class group has a strongly contracting axis with respect to this generating set. Similar arguments would also show that the mapping class group with respect to this generating set has purely exponential growth.

However, using our specific generating set, we show that this does not always hold:

Theorem 1.4 *For every $d_1, d_2 > 0$ there exists an $N > 1$ such that, for all $n \geq N$, there exists a pseudo-Anosov map ϕ, a constant $c_n > 0$, a sequence of elements $w_k \in \mathrm{Map}(S)$ and a sequence of radii $r_k > 0$ with $r_k \to \infty$ as $k \to \infty$ such that the following holds: Let a_ϕ be a quasiaxis for ϕ in $\mathrm{Map}(S)$ and let $\mathrm{Proj}_{a_\phi} \colon \mathrm{Map}(S) \to a_\phi$ be any (d_1, d_2)–projection map. Then the ball $B(w_k, r_k)$ of radius r_k centered at w_k is disjoint from a_ϕ and*

$$\mathrm{diam}_{\mathcal{S}_n}\big(\mathrm{Proj}_{a_\phi}(B(w_k, r_k))\big) \geq c_n \log(r_k).$$

We remark that, since a_ϕ has the contracting property [11], the diameter of the projection can grow at most logarithmically with respect to the radius r_k (see Corollary 5.7), hence the lower bound achieved by the above theorem is sharp.

Outline of proof

To find an exact value for the word length of an element $f \in \mathrm{Map}(S)$, we construct a homomorphism

$$h \colon \mathrm{Map}(S) \to \mathbb{Z},$$

where a large value for $h(f)$ guarantees a large value for the word length of f. At times, this lower bound is realized and an explicit geodesic in $\mathrm{Map}(S)$ is constructed (see Section 2). The pseudo-Anosov element ϕ is defined as

$$\phi = D_{\alpha_5} D_{\alpha_4} D_{\alpha_3} D_{\alpha_2} D_{\alpha_1}.$$

In Section 3, we find an explicit invariant train track for ϕ to show that ϕ is a pseudo-Anosov. In Section 4, we use the geodesics constructed in Section 2 to show that the shadows of geodesics in $\mathrm{Map}(S)$ are not necessarily uniform quality quasigeodesics in the curve complex. In Section 5, we begin by showing that ϕ acts loxodromically on $\mathrm{Map}(S)$, that is, it has a quasiaxis a_ϕ which fellow-travels the path $\{\phi^i\}$. We finish Section 5 by showing that the bound in our main theorem is sharp. In Section 6, we set up and complete the proof of Theorem 1.4.

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2 Finding explicit geodesics

In this section, we develop the tools needed to show that certain paths in $\mathrm{Map}(S)$ are geodesics. We emphasize again that, in our paper, S is the five-times punctured sphere and $\mathrm{Map}(S)$ is the pure mapping class group. That is, all homeomorphisms are required to fix the punctures pointwise.

By a *curve* on S we mean a free homotopy class of a nontrivial, nonperipheral simple closed curve. Fix a labeling of the five punctures of S with elements of $\mathbb{Z}_5$, the cyclic group of order 5. Any curve γ on S cuts the surface into two surfaces: one copy of $S_{0,3}$ containing two of the punctures from S, and one copy of $S_{0,4}$ which contains three of the punctures from S.

Definition 2.1 We say that a curve γ on S is an (i,j)*–curve* with $i,j \in \mathbb{Z}_5$ if the component of $(S-\gamma)$ that is a three-times punctured sphere contains the punctures labeled i and j. Furthermore, if $j = i \pm 1 \bmod 5$, we say that γ *separates two consecutive punctures*, and, if $j = i \pm 2 \bmod 5$, we say that γ *separates two nonconsecutive punctures*.

In [14], Luo gave a simple presentation of the mapping class group where the generators are the set of all Dehn twists

$$\mathcal{S} = \{D_\gamma : \gamma \text{ is a curve}\}$$

and the relations are of a few simple types. In our setting, we only have the following two relations:

- **Conjugating relation** For any two curves β and γ,
$$D_{D_\gamma(\beta)} = D_\gamma D_\beta D_\gamma^{-1}.$$
- **The lantern relation** Let i, j, k, l and m be distinct elements in $\mathbb{Z}_5$ and $\gamma_{i,j}$, $\gamma_{j,k}$ $\gamma_{k,i}$ and $\gamma_{l,m}$ be curves of the type indicated by the indices. Further assume that each pair of curves among $\gamma_{i,j}$, $\gamma_{j,k}$ and $\gamma_{k,i}$ intersect twice and that they are all disjoint from $\gamma_{l,m}$. Then
$$D_{\gamma_{i,j}} D_{\gamma_{j,k}} D_{\gamma_{k,i}} = D_{\gamma_{l,m}}.$$

Using this presentation, we construct a homomorphism from $\mathrm{Map}(S)$ into $\mathbb{Z}$.

Theorem 2.2 *There exists a homomorphism $h\colon \mathrm{Map}(S) \to \mathbb{Z}$ whose restriction to the generating set $\mathcal{S}$ is as follows:*
$$\begin{aligned} D_\gamma &\mapsto 1 && \text{if } \gamma \text{ separates two consecutive punctures,}\\ D_\gamma &\mapsto -1 && \text{if } \gamma \text{ separates two nonconsecutive punctures.}\end{aligned}$$

Proof To show that h extends to a homomorphism, it suffices to show that h preserves the relations stated above.

First, we check the conjugating relation. Let β and γ be a pair of curves. Since D_γ is a homeomorphism fixing the punctures, if β is an (i,j)–curve, so is $D_\gamma(\beta)$. In particular, $h(D_{D_\gamma(\beta)}) = h(D_\beta)$. Hence,
$$\begin{aligned} h(D_{D_\gamma(\beta)}) = h(D_\beta) &= h(D_\gamma) + h(D_\beta) - h(D_\gamma)\\ &= h(D_\gamma) + h(D_\beta) + h(D_\gamma^{-1}) = h(D_\gamma D_\beta D_\gamma^{-1}).\end{aligned}$$

We now show that h preserves the lantern relation. For any three punctures of S, labeled $i, j, k \in \mathbb{Z}_5$, two of these punctures are consecutive. Without loss of generality, suppose $j = i \pm 1 \bmod 5$. There are two cases:

(1) Assume k is consecutive to one of i or j. That is, without loss of generality, suppose $k = j \pm 1 \bmod 5$. Then $k = i \pm 2 \bmod 5$ and the remaining two punctures, l and m, are consecutive: $m = l \pm 1 \bmod 5$. Thus,
$$\begin{aligned} h(D_{\gamma_{i,j}} D_{\gamma_{j,k}} D_{\gamma_{k,i}}) &= h(D_{\gamma_{i,j}}) + h(D_{\gamma_{j,k}}) + h(D_{\gamma_{k,i}})\\ &= 1 + 1 + (-1)\\ &= 1 = h(D_{\gamma_{l,m}}).\end{aligned}$$

(2) Otherwise, $k = j \pm 2 \bmod 5$ and $k = i \pm 2 \bmod 5$, so that the remaining two punctures, l and m, are nonconsecutive: $m = l \pm 2 \bmod 5$. Thus,

$$\begin{aligned} h(D_{\gamma_{i,j}} D_{\gamma_{j,k}} D_{\gamma_{k,i}}) &= h(D_{\gamma_{i,j}}) + h(D_{\gamma_{j,k}}) + h(D_{\gamma_{k,i}}) \\ &= 1 + (-1) + (-1) \\ &= (-1) = h(D_{\gamma_{l,m}}). \end{aligned}$$

Thus, h preserves the lantern relation. □

Now, we switch back to the generating set $\mathcal{S}_n$ given in the introduction. The homomorphism of Theorem 2.2 gives a lower bound on the word length of elements in $\mathrm{Map}(S)$. Note that

$$h(s_{i,j}) = (n-1) \quad \text{and} \quad h(D_{\alpha_i}) = 1.$$

Lemma 2.3 *Let $n > 2$. For any $f \in \mathrm{Map}(S)$, let*

$$h(f) = q(n-1) + r$$

for integers q and r, where $0 \le |r| < \frac{1}{2}n - 1$. Then $\|f\|_{\mathcal{S}_n} \ge |q| + |r|$.

Proof First we show that, if $h(f) = a(n-1) + b$ for integers a and b, then $|a| + |b| \ge |q| + |r|$. To see this, consider such a pair a and b where $|a| + |b|$ is minimized. We argue in two cases:

Case 1 Assume $a < q$. Then

$$b = (q-a)(n-1) + r > (n-1) - \left(\tfrac{1}{2}n - 1\right) = \tfrac{1}{2}n. \tag{1}$$

We claim that, if we increase a by 1 and decrease b by $n-1$, we decrease the quantity $|a| + |b|$ which is a contradiction. This is clear if $b \ge n-1$. Otherwise, using (1), we have

$$|b - (n-1)| \le (n-1) - b < (n-1) - \tfrac{1}{2}n = \tfrac{1}{2}n - 1 \le |b| - 1.$$

Case 2 Assume $a > q$. Then

$$b = (q-a)(n-1) + r < -(n-1) + \tfrac{1}{2}n - 1 = -\tfrac{1}{2}n.$$

Therefore, we can decrease a by 1 and increase b by $n-1$ to decrease the quantity $|a| + |b|$, which again is a contradiction. Hence, $a = q$ and consequently $b = r$.

Now, write $f = g_1 g_2 \cdots g_k$, where $g_i \in \mathcal{S}_n$ or $g_i^{-1} \in \mathcal{S}_n$ and $k = \|f\|_{\mathcal{S}_n}$. For each g_i, $h(g_i)$ takes one of the values 1, -1, $n-1$ or $1-n$. Hence, there are integers a' and b'

such that

$$h(f) = h(g_1) + h(g_2) + \cdots + h(g_k) = a'(n-1) + b',$$

where $k \geq |a'| + |b'|$. But, as we saw before, we also have $|a'| + |b'| \geq |q| + |r|$. Hence $k \geq |q| + |r|$. □

This lemma allows us to find explicit geodesics in $\mathrm{Map}(S)$. We demonstrate this with an example.

Example 2.4 Let $f = D_{\alpha_1}^{n^k-1} \in \mathrm{Map}(S)$. We have

$$h(f) = n^k - 1 = (n-1)(n^{k-1} + n^{k-2} + \cdots + n^2 + n + 1).$$

Therefore, by Lemma 2.3 $\|f\|_{\mathcal{S}_n} \geq n^{k-1} + n^{k-2} + \cdots + n^2 + n + 1$. On the other hand (assuming k is even to simplify notation), we have

$$\begin{aligned} D_{\alpha_1}^{n^k-1} &= \big(D_{\alpha_1}^{n^k} D_{\alpha_2}^{-n^{k-1}}\big)\big(D_{\alpha_2}^{n^{k-1}} D_{\alpha_1}^{-n^{k-2}}\big) \cdots \big(D_{\alpha_1}^{n^2} D_{\alpha_2}^{-n}\big)\big(D_{\alpha_2}^{n} D_{\alpha_1}^{-1}\big) \\ &= s_{1,2}^{n^{k-1}} s_{2,1}^{n^{k-2}} \cdots s_{1,2}^{n} s_{2,1}. \end{aligned}$$

Since we used exactly $n^{k-1} + n^{k-2} + \cdots + n + 1$ elements in $\mathcal{S}_n$, we have found a geodesic path. However, notice there is a second geodesic path from the identity to f (which works for every k), namely

$$\begin{aligned} D_{\alpha_1}^{n^k-1} &= \big(D_{\alpha_1}^{n^k} D_{\alpha_2}^{-n^{k-1}}\big)\big(D_{\alpha_2}^{n^{k-1}} D_{\alpha_3}^{-n^{k-2}}\big) \cdots \big(D_{\alpha_{k-1}}^{n^2} D_{\alpha_k}^{-n}\big)\big(D_{\alpha_k}^{n} D_{\alpha_{k+1}}^{-1}\big) \\ &= s_{1,2}^{n^{k-1}} s_{2,3}^{n^{k-2}} \cdots s_{k-1,k}^{n} s_{k,k+1}, \end{aligned}$$

where the indices are considered to be in $\mathbb{Z}_5$. This shows that geodesics are not unique in $\mathrm{Map}(S)$. Either way, we have established that

$$\|D_{\alpha_1}^{n^k-1}\|_{\mathcal{S}_n} = n^{k-1} + n^{k-2} + \cdots + n + 1. \tag{2}$$

We now use a similar method to compute certain word lengths that will be useful later in the paper. Define

$$\phi = D_{\alpha_5} D_{\alpha_4} D_{\alpha_3} D_{\alpha_2} D_{\alpha_1}.$$

We will show in the next section that ϕ is a pseudo-Anosov element of $\mathrm{Map}(S)$. We also use the notation

$$\phi^{k/5} = D_{\alpha_k} D_{\alpha_{k-1}} \cdots D_{\alpha_1},$$

where again the indices are considered to be in $\mathbb{Z}_5$. This is accurate when k is divisible by 5 but we use it for any integer k. For a positive integer k, define

$$m_k = n^k + n^{k-1} + \cdots + n + 1 \quad \text{and} \quad \ell_k = n^k - n^{k-1} - n^{k-2} - \cdots - n - 1,$$

and let $w_k = D_{\alpha_1}^{m_k}$ and $u_k = D_{\alpha_1}^{\ell_k}$. Additionally, for k odd, we will define

$$v_k = D_{\alpha_1}^{-(k+1)/2} D_{\alpha_2}^{-(k+1)/2}.$$

In fact, for the rest of the paper, we always assume k is odd. We will show that u_k and w_k are closer to a large power of ϕ than the identity even though they are both just a power of a Dehn twist.

Proposition 2.5 *Let $n > 3$. For u_k and w_k as above, we have*

$$\|w_k \phi^{-(k+1)/5}\|_{\mathcal{S}_n} = \|w_k v_k\|_{\mathcal{S}_n} = n^{k-1} + 2n^{k-2} + \cdots + (k-1)n + k$$

and

$$\|\phi^{k/5} u_k\|_{\mathcal{S}_n} = n^{k-1} - n^{k-3} - 2n^{k-4} - \cdots - (k-3)n - (k-2) + 1.$$

Proof Note that

$$\begin{aligned} h(w_k \phi^{-(k+1)/5}) &= (n^k + n^{k-1} + \cdots + n + 1) - (k+1) \\ &= (n-1)(n^{k-1} + 2n^{k-2} + \cdots + (k-1)n + k). \end{aligned}$$

Lemma 2.3 implies that

$$\|w_k \phi^{-(k+1)/5}\|_{\mathcal{S}_n} \geq n^{k-1} + 2n^{k-2} + \cdots + (k-1)n + k.$$

On the other hand, since $m_k - 1 = n m_{k-1}$, we have

$$\begin{aligned} w_k \phi^{-(k+1)/5} &= D_{\alpha_1}^{m_k} \big(D_{\alpha_1}^{-1} D_{\alpha_2}^{-1} \cdots D_{\alpha_{k+1}}^{-1}\big) \\ &= D_{\alpha_1}^{m_k - 1} \big(D_{\alpha_2}^{-1} D_{\alpha_3}^{-1} \cdots D_{\alpha_{k+1}}^{-1}\big) \\ &= s_{1,2}^{m_{k-1}} D_{\alpha_2}^{m_{k-1}} \big(D_{\alpha_2}^{-1} D_{\alpha_2}^{-1} \cdots D_{\alpha_{k+1}}^{-1}\big) \\ &= s_{1,2}^{m_{k-1}} D_{\alpha_2}^{m_{k-1}-1} \big(D_{\alpha_3}^{-1} D_{\alpha_4}^{-1} \cdots D_{\alpha_{k+1}}^{-1}\big) \\ &= s_{1,2}^{m_{k-1}} s_{2,3}^{m_{k-2}} D_{\alpha_3}^{m_{k-2}} \big(D_{\alpha_3}^{-1} D_{\alpha_4}^{-1} \cdots D_{\alpha_{k+1}}^{-1}\big) \\ &\;\vdots \\ &= s_{1,2}^{m_{k-1}} s_{2,3}^{m_{k-2}} \cdots s_{k-1,k}^{m_1} s_{k,k+1}. \end{aligned}$$

Therefore,

$$\|w_k \phi^{-(k+1)/5}\|_{\mathcal{S}_n} = m_{k-1} + \cdots + m_1 + 1 = n^{k-1} + 2n^{k-2} + \cdots + (k-1)n + k.$$

To show that

$$\|w_k v_k\|_{\mathcal{S}_n} = n^{k-1} + 2n^{k-2} + \cdots + (k-1)n + k$$

is as above, but in place of applying $s_{i,i+1}$ for $1 \le i \le k$, we alternate between applying $s_{1,2}$ and $s_{2,1}$ to find (recall that k is odd and D_{α_1} and D_{α_2} commute)

$$w_k v_k = s_{1,2}^{m_{k-1}} s_{2,1}^{m_{k-2}} \cdots s_{2,1}^{m_1} s_{1,2},$$

which proves our claim. Similarly, we have

$$\begin{aligned} h(\phi^{k/5} u_k) &= k + (n^k - n^{k-1} - \cdots - n - 1) \\ &= (n-1)(n^{k-1} - n^{k-3} - 2n^{k-4} - \cdots - (k-3)n - (k-2)) + 1, \end{aligned}$$

and Lemma 2.3 implies

$$\|\phi^{k/5} u_k\|_{\mathcal{S}_n} \ge n^{k-1} - n^{k-3} - 2n^{k-4} - \cdots - (k-3)n - (k-2) + 1.$$

On the other hand, since $\ell_k + 1 = n\ell_{k-1}$, we have

$$\begin{aligned} \phi^{k/5} u_k &= (D_{\alpha_k} \cdots D_{\alpha_2} D_{\alpha_1}) D_{\alpha_1}^{\ell_k} \\ &= (D_{\alpha_k} \cdots D_{\alpha_3} D_{\alpha_2}) D_{\alpha_1}^{\ell_k + 1} \\ &= (D_{\alpha_k} \cdots D_{\alpha_3} D_{\alpha_2}) D_{\alpha_2}^{\ell_{k-1}} s_{1,2}^{\ell_{k-1}} \\ &= (D_{\alpha_k} \cdots D_{\alpha_4} D_{\alpha_3}) D_{\alpha_2}^{\ell_{k-1}+1} s_{1,2}^{\ell_{k-1}} \\ &= (D_{\alpha_k} \cdots D_{\alpha_4} D_{\alpha_3}) D_{\alpha_3}^{\ell_{k-2}} s_{2,3}^{\ell_{k-2}} s_{1,2}^{\ell_{k-1}} \\ &\;\;\vdots \\ &= D_{\alpha_k} D_{\alpha_k}^{\ell_1} s_{k-1,k}^{\ell_1} \cdots s_{2,3}^{\ell_{k-2}} s_{1,2}^{\ell_{k-1}} \\ &= D_{\alpha_{k+1}} s_{k,k+1} s_{k-1,k}^{\ell_1} \cdots s_{2,3}^{\ell_{k-2}} s_{1,2}^{\ell_{k-1}}. \end{aligned}$$

Therefore,

$$\|u_k \phi^{k/5}\|_{\mathcal{S}_n} = \ell_{k-1} + \cdots + \ell_1 + 2 = n^{k-1} - n^{k-3} - 2n^{k-4} - \cdots - (k-3)n - (k-2) + 1.$$

This is because the coefficient of n^i is 1 in ℓ_i and is -1 in $\ell_k, \dots, \ell_{i+1}$. Summing up, we get $-(k-i-2)$ as the coefficient of n^i. □

3 The pseudo-Anosov map ϕ

In this section, we introduce the pseudo-Anosov map ϕ which will be used in the proof of Theorem 1.4. Define

$$\phi = D_{\alpha_5} D_{\alpha_4} D_{\alpha_3} D_{\alpha_2} D_{\alpha_1}.$$

We check that ϕ is, in fact, a pseudo-Anosov.

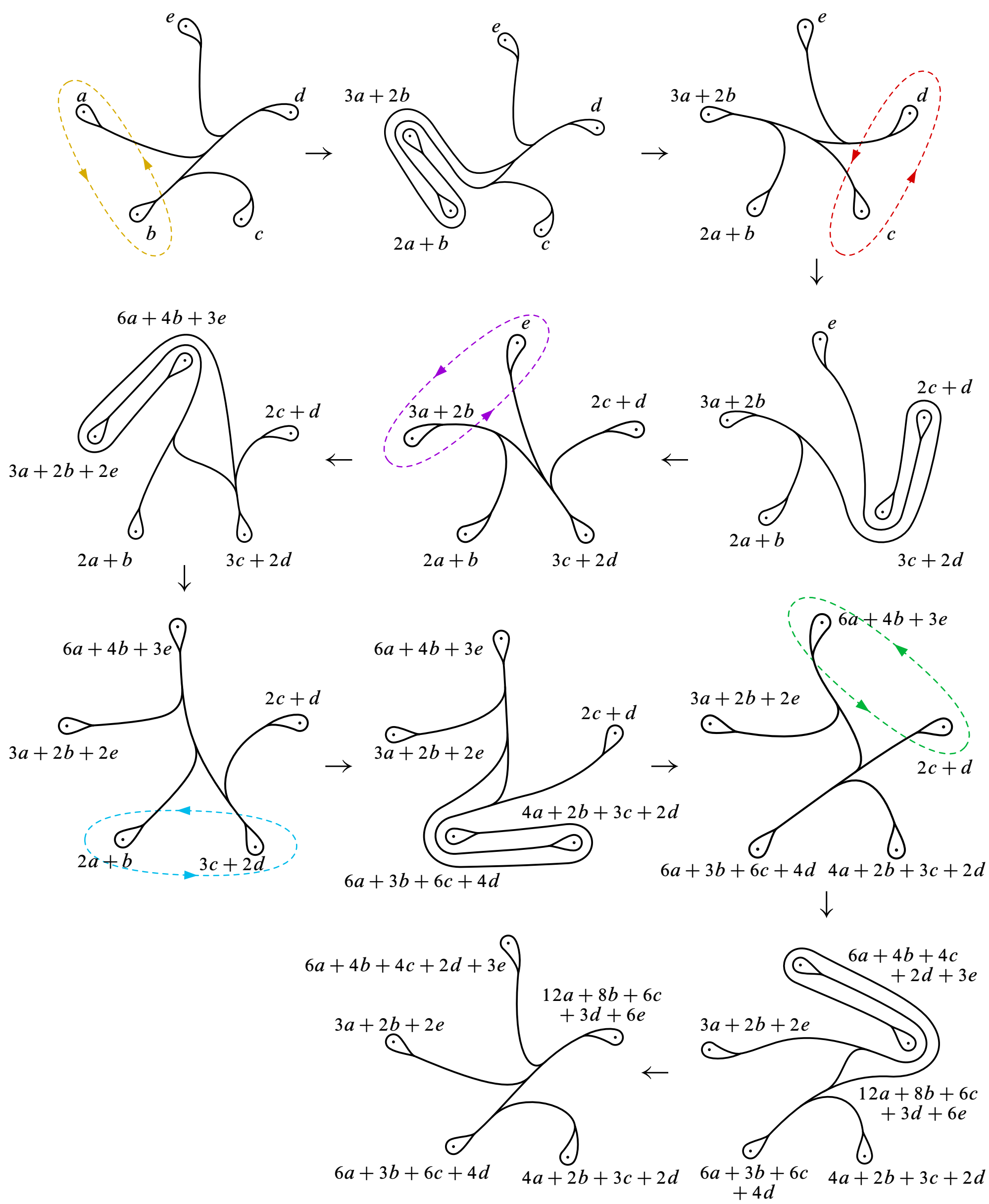

Figure 2: The train track $\phi(\tau)$ is carried by τ.

Theorem 3.1 *The map ϕ is pseudo-Anosov.*

Proof In order to prove that ϕ is a pseudo-Anosov map, we find a train track τ on S such that $\phi(\tau)$ is carried by τ and show that the matrix representation of ϕ in the coordinates given by τ is a Perron–Frobenius matrix (see [19] for basic information about train tracks).

The series of images in Figure 2 depict the train track τ and its images under successive applications of Dehn twists associated to ϕ. We see that $\phi(\tau)$ is indeed carried by τ and, keeping track of weights on τ, we calculate that the induced action on the space of weights on τ is given by the matrix

$$A = \begin{pmatrix} 3 & 2 & 0 & 0 & 2 \\ 6 & 3 & 6 & 4 & 0 \\ 4 & 2 & 3 & 2 & 0 \\ 12 & 8 & 6 & 3 & 6 \\ 6 & 4 & 4 & 2 & 3 \end{pmatrix}.$$

Note that the space of admissible weights on τ is the subset of $\mathbb{R}^5$ given by positive real numbers a, b, c, d and e such that $a+b+e=c+d$. The linear map described above preserves this subset. The square of the matrix A is strictly positive, which implies that the matrix is a Perron–Frobenius matrix. In fact, the top eigenvalue is

$$\lambda = \sqrt{13} + 2\sqrt{2\sqrt{13}+7} + 4,$$

which is associated to a unique irrational measured lamination F carried by τ that is fixed by ϕ. We now argue that F is filling. Note that curves on S are in one-to-one association with simple arcs connecting one puncture to another. We say an arc is carried by τ if the associated curve is carried by τ. If F is not filling, it is disjoint from some arc ω connecting two of the punctures. Modifying ω outside of a small neighborhood of τ, we can produce an arc that is carried by τ. In fact, for any two cusps of the train track τ, either an arc going clockwise or counterclockwise connecting these two cusps can be pushed into τ. Hence, we can replace the portion of ω that is outside of a small neighborhood of τ with such an arc to obtain an arc ω' that is still disjoint from F but is also carried by τ. Hence, if F is not filling, it is disjoint from some arc (and thus some curve) carried by τ. But F is the unique lamination carried by τ that is fixed under ϕ, which is a contradiction. This implies that ϕ is pseudo-Anosov. □

4 Shadow to curve complex not a uniform quality quasigeodesic

The curve graph $\mathcal{C}(S)$ is a graph whose vertices are curves on S and whose edges are pairs of curves with disjoint representatives. We assume every edge has length 1, turning $\mathcal{C}(S)$ into a metric space. This means that, for a pair of curves α and β, $d_{\mathcal{C}(S)}(\alpha,\beta)=n$ if

$$\alpha=\gamma_0,\dots,\gamma_n=\beta$$

is a shortest sequence of curves on S such that the successive γ_i are disjoint. Masur and Minsky showed that $\mathcal{C}(S)$ is an infinite diameter Gromov hyperbolic space [15].

We also talk about the distance between subsets of $\mathcal{C}(S)$ using the same notation. That is, for two sets of curves $\mu_0,\mu_1\subset\mathcal{C}(S)$ we define

$$d_{\mathcal{C}(S)}(\mu_0,\mu_1)=\max_{\substack{\gamma_0\in\mu_0\\ \gamma_1\in\mu_1}} d_{\mathcal{C}(S)}(\gamma_0,\gamma_1).$$

Definition 4.1 The *shadow map* from the mapping class group to the curve complex is the map defined as

$$\Upsilon\colon \mathrm{Map}(S)\to\mathcal{C}(S),\qquad f\mapsto f(\alpha_1).$$

The shadow map from $\mathrm{Map}(S)$ equipped with $d_{\mathcal{S}_n}$ to the curve complex is 4–Lipschitz:

Lemma 4.2 *For any $f\in\mathrm{Map}(S)$, we have*

$$d_{\mathcal{C}(S)}(\alpha_1,f\alpha_1)\le 4\|f\|_{\mathcal{S}_n}. \tag{3}$$

In particular, the Lipschitz constant of the shadow map is independent of n.

Proof It is sufficient to prove the lemma for elements of $\mathcal{S}_n$. Consider $D_{\alpha_i}\in\mathcal{S}_n$. If $i(\alpha_i,\alpha_1)=0$ then

$$d_{\mathcal{C}(S)}(\alpha_1,D_{\alpha_i}(\alpha_1))=d_{\mathcal{C}(S)}(\alpha_1,\alpha_1)=0.$$

If $i(\alpha_i,\alpha_1)=2$, then there is a curve α_j that is disjoint from both α_1 and α_i and hence α_j is also disjoint from $D_{\alpha_i}(\alpha_1)$. Therefore, $d_{\mathcal{C}(S)}(\alpha_1,D_{\alpha_i}(\alpha_1))=2$.

Now consider the element $s_{i,i+1}\in\mathcal{S}_n$. Note that $s_{i,i+1}^{-1}\alpha_i=\alpha_i$. Hence,

$$\begin{aligned} d_{\mathcal{C}(S)}(\alpha_1,s_{i,i+1}\alpha_1)&\le d_{\mathcal{C}(S)}(\alpha_1,\alpha_i)+d_{\mathcal{C}(S)}(\alpha_i,s_{i,i+1}\alpha_1)\\ &\le 2+d_{\mathcal{C}(S)}(s_{i,i+1}^{-1}\alpha_i,\alpha_1)\\ &\le 2+d_{\mathcal{C}(S)}(\alpha_i,\alpha_1)\le 2+2=4. \end{aligned}$$ □

Using this lemma and the theorems from Section 3, we show that the shadows of geodesics from the mapping class group to the curve complex are not always uniform quality quasigeodesics.

Theorem 4.3 *For all $K \geq 1$ and $C \geq 0$, there exists a geodesic in the mapping class group whose shadow to the curve complex is not a (K,C)–quasigeodesic.*

Proof Let k be a positive integer which is a multiple of 5. Recall that, for a positive integer k, we have

$$m_k = n^{k-1} + n^{k-2} + \cdots + n + 1, \quad \ell_k = n^k - n^{k-1} - n^{k-2} - \cdots - n - 1,$$

$w_k = D_{\alpha_1}^{m_k}$ and $u_k = D_{\alpha_1}^{\ell_k}$. Note that $m_{k-1} + \ell_k = n^k$. Hence, we can write

$$D_{\alpha_1}^{n^k} = (w_{k-1}\phi^{-k/5})(\phi^{k/5}u_k).$$

Also,

$$h(D_{\alpha_1}^{n^k}) = n^k = (n-1)(n^{k-1} + n^{k-2} + \cdots + n + 1) + 1.$$

Therefore, by Lemma 2.3,

$$\|D_{\alpha_1}^{n^k}\|_{\mathcal{S}_n} \geq n^{k-1} + n^{k-2} + \cdots + n + 2. \tag{4}$$

But, from Proposition 2.5, we have

$$\|w_{k-1}\phi^{-k/5}\|_{\mathcal{S}_n} = n^{k-2} + 2n^{k-3} + \cdots + (k-2)n + (k-1)$$

and

$$\|\phi^{k/5}u_k\|_{\mathcal{S}_n} = n^{k-1} - n^{k-3} - 2n^{k-4} - \cdots - (k-3)n - (k-2) + 1.$$

The sum of the word lengths of the two elements is

$$n^{k-1} + n^{k-2} + \cdots + n + 2,$$

which is equal to the lower bound found in (4). Thus,

$$\|D_{\alpha_1}^{n^k}\|_{\mathcal{S}_n} = \|w_{k-1}\phi^{-k/5}\|_{\mathcal{S}_n} + \|\phi^{k/5}u_k\|_{\mathcal{S}_n},$$

which means there is a geodesic connecting $D_{\alpha_1}^{n^k}$ to the identity that passes through $\phi^{k/5}u_k$.

Since ϕ is a pseudo-Anosov map, there is a lower bound on its translation distance along the curve graph (see Theorem 4.6 from [15]). Namely, there is a constant $\sigma > 0$ such that, for every m,

$$d_{\mathcal{C}(S)}(\alpha_1, \phi^m\alpha_1) \geq \sigma m. \tag{5}$$

Also, $u_k\alpha_1 = \alpha_1$, which implies

$$d_{\mathcal{C}(S)}(\alpha_1, \phi^{k/5}u_k\alpha_1) = d_{\mathcal{C}(S)}(\alpha_1, \phi^{k/5}\alpha_1) \geq \tfrac{1}{5}\sigma k.$$

That is,

$$\Upsilon(\mathrm{id}) = \Upsilon(D_{\alpha_1}^{n^k}) = \alpha_1.$$

However, $\Upsilon(\phi^{k/5}u_k)$ is at least distance $\frac{1}{5}\sigma k$ away from α_1. Therefore, choosing k large compared with σ, K and C, we see that the shadow of this geodesic (the one connecting id to $D_{\alpha_1}^{n^k}$ which passes through $\phi^{k/5}u_k$) to $\mathcal{C}(S)$ is not a (K,C)–quasigeodesic. □

5 Axis of a pseudo-Anosov in the mapping class group

Consider the path

$$\mathcal{A}_\phi\colon \mathbb{Z} \to \mathrm{Map}(S), \quad i \mapsto \phi^i.$$

Since $\|\phi\|_{\mathcal{S}_n} \le 5$, then $\|\phi^i\|_{\mathcal{S}_n} \le 5i$. Also, using Lemma 4.2 and (5) we get

$$\|\phi^i\|_{\mathcal{S}_n} \ge \tfrac{1}{4} d_{\mathcal{C}(S)}(\alpha_1, \phi^i\alpha_1) \ge \tfrac{1}{4} i\sigma.$$

Therefore,

$$\tfrac{1}{4} i\sigma \le \|\phi^i\|_{\mathcal{S}_n} \le 5i.$$

This proves the following lemma:

Lemma 5.1 *The path $\mathcal{A}_\phi$ is a quasigeodesic in $(\mathrm{Map}(S), d_{\mathcal{S}_n})$ for every n with uniform constants.*

We abuse notation and allow $\mathcal{A}_\phi$ to denote both the map and the image of the map in $\mathrm{Map}(S)$. For $i, j \in \mathbb{Z}$, let $g = g_{i,j}$ be a geodesic in $(\mathrm{Map}(S), d_{\mathcal{S}_n})$ connecting ϕ^i to ϕ^j. Let $\mathcal{G} = \Upsilon \circ g$ be the shadow of g to the curve complex and let

$$\mathrm{Proj}_{\mathcal{G}}\colon \mathrm{Map}(S) \to \mathcal{G}$$

be the composition of Υ and the closest point projection from $\mathcal{C}(S)$ to $\mathcal{G}$. The following theorem, proven in more generality by Duchin and Rafi [11, Theorem 4.2], is stated for geodesics $g_{i,j}$ and the path $\mathcal{G}$.

Theorem 5.2 *The path $\mathcal{G}$ is a quasigeodesic in $\mathcal{C}(S)$. Furthermore, there exists a constant B_n which depends on n and ϕ, and a constant B depending only on ϕ, such that the following holds: For $x \in \mathrm{Map}(S)$ with $d_{\mathcal{S}_n}(x, g) > B_n$, let $r = d_{\mathcal{S}_n}(x,g)/B_n$ and let $B(x,r)$ be the ball of radius r centered at x in $(\mathrm{Map}(S), d_{\mathcal{S}_n})$. Then*

$$\mathrm{diam}_{\mathcal{C}(S)}\big(\mathrm{Proj}_{\mathcal{G}}(B(x,r))\big) \le B.$$

Remark 5.3 In the proof of [11, Theorem 4.2], it can be seen that B_n (B_1 in their notation) is dependent on the generating set since B_n is taken to be large with respect to the constants from the Masur–Minsky distance formula, which depend on the generating set [16]. Let $\mathcal{S}$ be a fixed generating set for $\mathrm{Map}(S)$. Then the word lengths of elements in $\mathcal{S}_n$ in terms of $\mathcal{S}$ grow linearly in n with respect to $\mathcal{S}$. Hence, the constants involved in the Masur–Minsky distance formula also change linearly in n. That is, $B_n \asymp n$, where the symbol $\asymp$ means that the equality is true up to an additive constant and a multiplicative constant. Also, one can see that the constant B (B_2 in their proof) depends only on ϕ and the hyperbolicity constant of the curve graph, but not the generating set.

Remark 5.4 Since Υ is Lipschitz (Lemma 4.2) and the closest point projection is Lipschitz, the map $\mathrm{Proj}_{\mathcal{G}}$ is also Lipschitz. We assume B is larger than the Lipschitz constant of this map.

Since $\mathcal{A}_\phi$ is a quasigeodesic, Theorem 5.2 and the usual Morse argument (for example see [1]) implies the following:

Proposition 5.5 *The paths $\mathcal{A}_\phi[i,j]$ and $g_{i,j}$ fellow-travel each other and the constant depends only on n. That is, there is a bounded constant δ_n, depending on n, such that*

$$\delta_n \geq \max\Big(\max_{p\in\mathcal{A}_\phi[i,j]} \min_{q\in g_{i,j}} d_{\mathcal{S}_n}(p,q),\ \max_{p\in g_{i,j}} \min_{q\in\mathcal{A}_\phi[i,j]} d_{\mathcal{S}_n}(p,q)\Big).$$

We now show that ϕ acts loxodromically on $(\mathrm{Map}(S), d_{\mathcal{S}_n})$. That is, there exists a geodesic a_ϕ on $(\mathrm{Map}(S), d_{\mathcal{S}_n})$ that is preserved by a power of ϕ. This is a folklore theorem, but we were unable to find a reference for it in the literature. The proof given here follows the arguments in [4, Theorem 1.4], where Bowditch showed that ϕ acts loxodromically on the curve graph, which is more difficult since the curve graph is not locally finite. Bowditch's proof in turn follows the arguments of Delzant [10] for a hyperbolic group.

Proposition 5.6 *There is a geodesic*

$$a_\phi : \mathbb{Z} \to \mathrm{Map}(S)$$

that is preserved by some power of ϕ. We call the geodesic a_ϕ the quasiaxis for ϕ.

Proof The statement is true for the action of any pseudo-Anosov homeomorphism in any mapping class group equipped with any word metric coming from a finite generating set. We only sketch the proof since it is a simpler version of the argument given in [4].

Let $\mathcal{L}(i,j)$ be the set of all geodesics connecting ϕ^i to ϕ^j. Note that every point on every path in $\mathcal{L}(i,j)$ lies in the δ_n–neighborhood of $\mathcal{A}_\phi$. Letting $i \to \infty,\ j \to -\infty$ and using a diagonal limit argument ($\mathrm{Map}(S)$ is locally finite), we can find bi-infinite geodesics that are the limits of geodesic segments in sets $\mathcal{L}(i,j)$. Let $\mathcal{L}$ be the set of all such bi-infinite geodesics. Then $\phi(\mathcal{L}) = \mathcal{L}$ and every geodesic in $\mathcal{L}$ is also contained in the δ_n–neighborhood of $\mathcal{A}_\phi$. Let $\mathcal{L}/\phi$ represent the set of edges which appear in a geodesic in $\mathcal{L}$ up to the action of ϕ. Then $\mathcal{L}/\phi$ is a finite set.

Choose an order for $\mathcal{L}/\phi$. We say a geodesic $g \in \mathcal{L}$ is lexicographically least if, for all vertices $x, y \in g$, the sequence of ϕ–classes of directed edges in the segment $g_0 \subset g$ between x and y is lexicographically least among all geodesic segments from x to y that are part of a geodesic in $\mathcal{L}$. Let $\mathcal{L}_L$ be the set lexicographically least elements of $\mathcal{L}$. We will show that every element of $\mathcal{L}_L$ is preserved by a power of ϕ.

Let P be the cardinality of a ball of radius δ_n in $(\mathrm{Map}(S), d_{\mathcal{S}_n})$. We claim that $|\mathcal{L}_L| \le P^2$. Otherwise, we can find P^2+1 elements of $\mathcal{L}_L$ which all differ in some sufficiently large compact subset of $N_{\delta_n}(\mathcal{A}_\phi)$, the δ_n–neighborhood of $\mathcal{A}_\phi$. In particular, we can find $x, y \in N_{\delta_n}(\mathcal{A}_\phi)$ so that each of these P^2+1 geodesics has a subsegment connecting a point in $N_{\delta_n}(x)$ to a point in $N_{\delta_n}(y)$, and these subsegments are all distinct. But then at least two such segments must share the same endpoints, which means they cannot both be lexicographically least.

Since ϕ permutes elements of $\mathcal{L}_L$, each geodesic in $\mathcal{L}_L$ is preserved by $\phi^{(P^2)!}$. □

As before, we use the notation a_ϕ to denote both the map and the image of the map in $\mathrm{Map}(S)$. We now show that the projection of a ball that is disjoint from a_ϕ to a_ϕ grows at most logarithmically with the radius of the ball, proving that Theorem 1.4 is sharp.

Corollary 5.7 *There are uniform constants $c_1, c_2 > 0$ such that, for $x \in \mathrm{Map}(S)$ and $R = d_{\mathcal{S}_n}(x, a_\phi)$, we have*

$$\mathrm{diam}_{\mathcal{C}(S)}\big(\mathrm{Proj}_{\mathcal{G}}(\mathrm{Ball}(x,R))\big) \le c_1 n \cdot \log(R) + c_2 n.$$

Proof Consider $y \in \mathrm{Ball}(x, R - B_n)$. Let

$$x = x_0, x_1, \dots, x_N = y$$

be a sequence of points along the geodesic connecting x to y such that

$$d_{\mathcal{S}_n}(x_i, x_{i+1}) \le \frac{d_{\mathcal{S}_n}(x_i, a_\phi)}{B_n} + 1 \tag{6}$$

and, for $i = 0, \dots, N-2$,

$$d_{\mathcal{S}_n}(x_i, x_{i+1}) \ge \frac{d_{\mathcal{S}_n}(x_i, a_\phi)}{B_n}.$$

Note that

$$d_{\mathcal{S}_n}(x_i, a_\phi) \ge R - d_{\mathcal{S}_n}(x_0, x_i) \ge d_{\mathcal{S}_n}(x_i, y).$$

Therefore, for $i = 1, \dots, N-2$,

$$d_{\mathcal{S}_n}(x_i, x_{i+1}) \ge \frac{d_{\mathcal{S}_n}(x_i, a_\phi)}{B_n} \ge \frac{d_{\mathcal{S}_n}(x_i, y)}{B_n}.$$

This implies

$$d_{\mathcal{S}_n}(x_{i+1}, y) = d_{\mathcal{S}_n}(x_i, y) - d_{\mathcal{S}_n}(x_i, x_{i+1}) \le \Big(1 - \frac{1}{B_n}\Big) d_{\mathcal{S}_n}(x_i, y)$$

and hence

$$d_{\mathcal{S}_n}(x_{N-2}, y) \le \Big(1 - \frac{1}{B_n}\Big)^{N-2} d_{\mathcal{S}_n}(x_0, y) \le \Big(1 - \frac{1}{B_n}\Big)^{N-2} R.$$

But

$$d_{\mathcal{S}_n}(x_{N-2}, y) \ge d_{\mathcal{S}_n}(x_{N-2}, x_{N-1}) \ge \frac{d_{\mathcal{S}_n}(x_{N-2}, a_\phi)}{B_n} \ge \frac{B}{B_n}.$$

Therefore,

$$\Big(1 - \frac{1}{B_n}\Big)^{N-2} R \ge \frac{B}{B_n}.$$

This means, for some constant c_n' depending only on n, we have (see Remark 5.3)

$$N \le c_n' \log R \quad \text{with } c_n' \asymp \frac{1}{-\log(1 - 1/B_n)} \asymp B_n \asymp n, \tag{7}$$

where the symbol $\asymp$ means that the equalities are true up to an additive constant and a multiplicative constant.

Setting $r_i = d_{\mathcal{S}_n}(x_i, a_\phi)/B_n$, equation (6) implies that there exists $z \in B(x_i, r_i)$ such that $d_{\mathcal{S}_n}(z, x_{i+1}) \le 1$. Applying Theorem 5.2 to $B(x_i, r_i)$ and Remark 5.4 to z and x_{i+1}, we get

$$\begin{aligned} d_{\mathcal{C}(S)}(\mathrm{Proj}_{\mathcal{G}_\phi}(x_i), \mathrm{Proj}_{\mathcal{G}_\phi}(x_{i+1})) & \\ \le d_{\mathcal{C}(S)}(\mathrm{Proj}_{\mathcal{G}_\phi}(x_i), \mathrm{Proj}_{\mathcal{G}_\phi}(z)) &+ d_{\mathcal{C}(S)}(\mathrm{Proj}_{\mathcal{G}_\phi}(z), \mathrm{Proj}_{\mathcal{G}_\phi}(x_{i+1})) \\ \le 2B. & \end{aligned}$$

In view of (7), we have

$$d_{\mathcal{C}(S)}(\mathrm{Proj}_{\mathcal{G}_\phi}(x), \mathrm{Proj}_{\mathcal{G}_\phi}(y)) \le 2Bc'_n \log R. \tag{8}$$

Now, for any $y' \in \mathrm{Ball}(x, R)$ there is a $y \in \mathrm{Ball}(x, R - B_n)$ with $d_{\mathcal{S}_n}(y, y') \le B_n$. But Υ is 4–Lipschitz and the closest point projection from $\mathcal{C}(S)$ to $\mathcal{G}_\phi$ is also Lipschitz with a Lipschitz constant depending on the hyperbolicity constant of $\mathcal{C}(S)$. Therefore,

$$d_{\mathcal{C}(S)}(\mathrm{Proj}_{\mathcal{G}_\phi}(y), \mathrm{Proj}_{\mathcal{G}_\phi}(y')) \le c'' B_n, \tag{9}$$

where c'', the Lipschitz constant for $\mathrm{Proj}_{\mathcal{G}_\phi}$, is a uniform constant. The second part of (7) implies that there is a constant c_1 such that $2Bc'_n \le c_1 n$. Also, there is a constant c_2 with $c'' B_n \le c_2 n$. Corollary 5.7 now follows from (8), (9) and the triangle inequality. □

6 The logarithmic lower bound

In this section, we will show that the quasiaxis a_ϕ of the pseudo-Anosov map ϕ does not have the strongly contracting property, proving Theorem 1.4 from the introduction.

Definition 6.1 Given a metric space (X, d_X), a subset $\mathcal{G}$ of X and constants $d_1, d_2 > 0$, we call a map $\mathrm{Proj}\colon X \to \mathcal{G}$, a (d_1, d_2)–projection map if, for every $x \in X$ and $g \in \mathcal{G}$,

$$d_X(\mathrm{Proj}(x), g) \le d_1 \cdot d_X(x, g) + d_2.$$

To prove this theorem, notice first that the geodesic found in Section 2 may not determine the nearest point of $\mathcal{A}_\phi$ to $w_k = D_{\alpha_1}^{m_k}$, where $m_k = n^k + n^{k-1} + \cdots + n + 1$.

Lemma 6.2 *If ϕ^{p_k} is the nearest point of $\mathcal{A}_\phi$ to w_k, then $p_k \ge \frac{1}{5}(k+1)$.*

Proof Consider a point ϕ^m on $\mathcal{A}_\phi$ where $m < \frac{1}{5}k$. Applying the homomorphism h, we have

$$h(w_k \phi^{-m}) = (m_k - 5m) > (m_k - (k+1)) = h(w_k \phi^{-(k+1)/5}).$$

But $m_k - (k+1)$ is divisible by $n-1$. Hence, if we write $m_k - m = q(n-1) + r$ with $|r| \le \frac{1}{2}(n-1)$, we have

$$|q| \ge \frac{m_k - (k+1)}{n-1} \quad \text{and} \quad |r| \ge 0.$$

Lemma 2.3 implies that $\|w_k \phi^{-m}\|_{\mathcal{S}_n} > \|w_k \phi^{-(k+1)/5}\|_{\mathcal{S}_n}$, which means the closest point in $\mathcal{A}_\phi$ to w_k is some point ϕ^{p_k} with $p_k \ge \frac{1}{5}(k+1)$. □

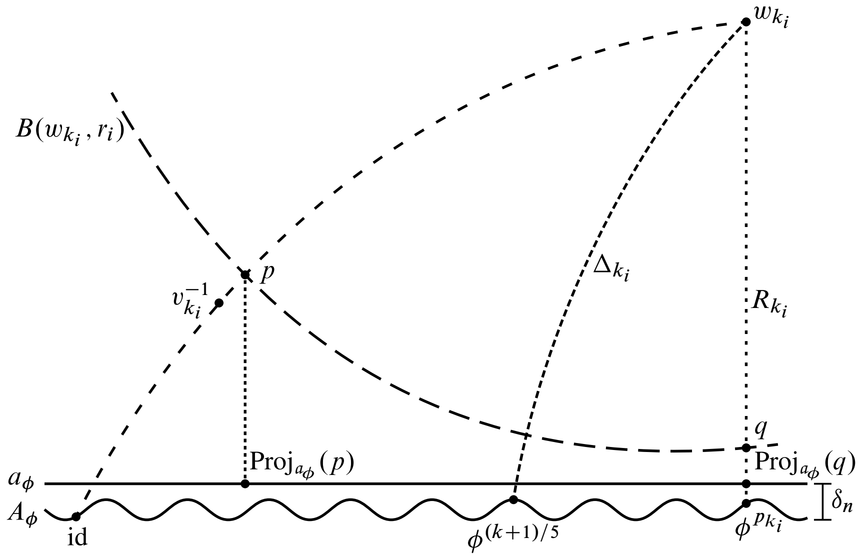

Figure 3: Setup for the proof of Theorem 1.4.

Let $R_k = d_{\mathcal{S}_n}(w_k, \phi^{p_k}) = d_{\mathcal{S}_n}(w_k, \mathcal{A}_\phi)$ and $\Delta_k = d_{\mathcal{S}_n}(w_k, \phi^{(k+1)/5})$.

Proof of Theorem 1.4 For fixed $d_1, d_2 > 0$, let $\mathrm{Proj}_{a_\phi} \colon \mathrm{Map}(S) \to a_\phi$ be any (d_1, d_2)–projection map. Fix n large enough that

$$\tfrac{1}{4}\sigma > \frac{5d_1}{n-1}. \tag{10}$$

Choose the sequence $\{k_i\} = \{2n^i - 3\}$ and recall that

$$v_{k_i} = D_{\alpha_1}^{-(k_i+1)/2} D_{\alpha_2}^{-(k_i+1)/2}.$$

By Example 2.4 $\big($notice that $\frac{1}{2}(k_i + 1) = n^i - 1\big)$,

$$d_{\mathcal{S}_n}(v_{k_i}, \mathrm{id}) = \|v_{k_i}\|_{\mathcal{S}_n} = \frac{k_i + 1}{n-1} = \|v_{k_i}^{-1}\|_{\mathcal{S}_n}, \tag{11}$$

and by Proposition 2.5 we have

$$d_{\mathcal{S}_n}(w_{k_i}, v_{k_i}^{-1}) = \Delta_{k_i}.$$

Consider a ball $B(w_{k_i}, r_i)$ of radius $r_i = R_{k_i} - (\delta_n + 1)$ around w_{k_i}. This ball is disjoint from a_ϕ since $\mathcal{A}_\phi$ and a_ϕ are δ_n–fellow-travelers by Proposition 5.5, and $R_{k_i} = d_{\mathcal{S}_n}(w_{k_i}, \mathcal{A}_\Phi)$. For the rest of the proof, we refer to Figure 3.

Since h is a homomorphism, we have

$$h(w_{k_i}\phi^{-k_i/5}) = h(w_{k_i}\phi^{-p_{k_i}}) + h(\phi^{p_{k_i}}\phi^{-k_i/5}).$$

Proposition 2.5 showed

$$h(w_{k_i}\phi^{-(k_i+1)/5}) = (n-1)\Delta_{k_i};$$

from Lemma 2.3, we have

$$h(w_{k_i}\phi^{-p_{k_i}}) \le (n-1)R_{k_i};$$

and since $\|\phi\|_{\mathcal{S}_n} = 5$, we have

$$h(\phi^{p_{k_i}}\phi^{-(k_i+1)/5}) = 5p_{k_i} - (k_i+1).$$

The above equations imply

$$\Delta_{k_i} - R_{k_i} \le \frac{5p_{k_i} - k_i}{n-1}.$$

Consider a point p on the geodesic from w_{k_i} to $v_{k_i}^{-1}$ such that $d_{\mathcal{S}_n}(w_{k_i}, p) = r_i$, ie such that

$$d_{\mathcal{S}_n}(p, v_{k_i}^{-1}) = \Delta_{k_i} - r_i = \Delta_{k_i} - (R_{k_i} - \delta_n - 1) \le \frac{5p_{k_i} - k_i}{n-1} + \delta_n + 1.$$

This and (11) imply

$$d_{\mathcal{S}_n}(\mathrm{id}, p) \le \frac{k_i+1}{n-1} + \frac{5p_{k_i} - k_i}{n-1} + \delta_n + 1 = \frac{5p_{k_i}+1}{n-1} + \delta_n + 1.$$

Since a_ϕ and $\mathcal{A}_\phi$ are δ_n–fellow-travelers by Proposition 5.5, there exists a point $x_0 \in a_\phi$ in the δ_n–neighborhood of the identity. Thus $d_{\mathcal{S}_n}(p, x_0) \le (5p_{k_i}+1)/(n-1) + 2\delta_n + 1$ and

$$\begin{aligned} d_{\mathcal{S}_n}(\mathrm{id}, \mathrm{Proj}_{a_\phi}(p)) &\le d_{\mathcal{S}_n}(\mathrm{id}, x_0) + d_{\mathcal{S}_n}(x_0, \mathrm{Proj}_{a_\phi}(p)) \\ &\le \delta_n + d_1 \cdot d_{\mathcal{S}_n}(x_0, p) + d_2 \\ &\le \frac{5d_1 p_{k_i}}{n-1} + A_p, \end{aligned} \tag{12}$$

where A_p is a constant depending on δ_n, d_1 and d_2 but independent of k_i. Similarly, we consider a point q on the geodesic from w_{k_i} to $\phi^{p_{k_i}}$ such that $d_{\mathcal{S}_n}(w_{k_i}, q) = r_i$. Again, since a_ϕ and $\mathcal{A}_\phi$ are δ_n–fellow-travelers by Proposition 5.5, there exists an $x_1 \in a_\phi$ such that $d_{\mathcal{S}_n}(\phi^{p_{k_i}}, x_1) \le \delta_n$, and thus $d_{\mathcal{S}_n}(q, x_1) \le 2\delta_n + 1$. Therefore,

$$\begin{aligned} d_{\mathcal{S}_n}(\phi^{p_{k_i}}, \mathrm{Proj}_{a_\phi}(q)) &\le d_{\mathcal{S}_n}(\phi^{p_{k_i}}, x_1) + d_{\mathcal{S}_n}(x_1, \mathrm{Proj}_{a_\phi}(q)) \\ &\le \delta_n + d_1 \cdot (2\delta_n + 1) + d_2 \le A_q, \end{aligned} \tag{13}$$

where, again, A_q depends on δ_n, d_1 and d_2 but is independent of k_i. Since $p, q \in B(w_{k_i}, r_i)$, we have

$$\begin{aligned}\operatorname{diam}_{\mathcal{S}_n}\bigl(\operatorname{Proj}_{a_\phi}(B(w_{k_i}, r_i))\bigr) & \\ &\geq d_{\mathcal{S}_n}(\operatorname{Proj}_{a_\phi}(p), \operatorname{Proj}_{a_\phi}(q)) \\ &\geq d_{\mathcal{S}_n}(\mathrm{id}, \phi^{p_{k_i}}) - d_{\mathcal{S}_n}(\mathrm{id}, \operatorname{Proj}_{a_\phi}(p)) - d_{\mathcal{S}_n}(\operatorname{Proj}_{a_\phi}(q), \phi^{p_{k_i}}).\end{aligned}$$

But $d_{\mathcal{S}_n}(\mathrm{id}, \phi^{p_{k_i}}) \geq \frac{1}{4}\sigma p_{k_i}$. By combining this fact and equations (12) and (13), we find

$$\begin{aligned}\operatorname{diam}_{\mathcal{S}_n}\bigl(\operatorname{Proj}_{a_\phi}(B(w_{k_i}, r_i))\bigr) &\geq \tfrac{1}{4}\sigma p_{k_i} - \frac{5d_1 p_{k_i}}{n-1} - A_p - A_q \\ &= p_{k_i}\left(\tfrac{1}{4}\sigma - \frac{5d_1}{n-1}\right) - A_p - A_q.\end{aligned} \tag{14}$$

By our assumption (10) on n, this expression is positive and goes to infinity as $p_{k_i} \to \infty$. But, for n large enough, $r_i \leq R_{k_i} \leq \Delta_{k_i} \leq n^{k_i}$. Also, $p_{k_i} \geq \frac{1}{5}k_i$. Hence,

$$5p_{k_i} \log n \geq \log r_i.$$

Hence, there is a constant c_n such that

$$\operatorname{diam}_{\mathcal{S}_n}\bigl(\operatorname{Proj}_{a_\phi}(B(w_{k_i}, r_i))\bigr) \geq c_n \log r_i. \qquad \square$$

References

[1] **G N Arzhantseva**, **C H Cashen**, **D Gruber**, **D Hume**, *Characterizations of Morse quasi-geodesics via superlinear divergence and sublinear contraction*, Doc. Math. 22 (2017) 1193–1224 MR Zbl

[2] **G N Arzhantseva**, **C H Cashen**, **J Tao**, *Growth tight actions*, Pacific J. Math. 278 (2015) 1–49 MR Zbl

[3] **J A Behrstock**, *Asymptotic geometry of the mapping class group and Teichmüller space*, Geom. Topol. 10 (2006) 1523–1578 MR Zbl

[4] **B H Bowditch**, *Tight geodesics in the curve complex*, Invent. Math. 171 (2008) 281–300 MR Zbl

[5] **J Brock**, **H Masur**, **Y Minsky**, *Asymptotics of Weil–Petersson geodesics, II: Bounded geometry and unbounded entropy*, Geom. Funct. Anal. 21 (2011) 820–850 MR Zbl

[6] **Y-E Choi**, **D Dumas**, **K Rafi**, *Grafting rays fellow travel Teichmüller geodesics*, Int. Math. Res. Not. 2012 (2012) 2445–2492 MR Zbl

[7] **Y-E Choi**, **K Rafi**, **C Series**, *Lines of minima and Teichmüller geodesics*, Geom. Funct. Anal. 18 (2008) 698–754 MR Zbl

[8] **M Clay**, **K Rafi**, **S Schleimer**, *Uniform hyperbolicity of the curve graph via surgery sequences*, Algebr. Geom. Topol. 14 (2014) 3325–3344 MR Zbl

[9] **M Dehn**, *Papers on group theory and topology*, Springer (1987) MR Zbl

[10] **T Delzant**, *Sous-groupes distingués et quotients des groupes hyperboliques*, Duke Math. J. 83 (1996) 661–682 MR Zbl

[11] **M Duchin**, **K Rafi**, *Divergence of geodesics in Teichmüller space and the mapping class group*, Geom. Funct. Anal. 19 (2009) 722–742 MR Zbl

[12] **A Eskin**, **H Masur**, **K Rafi**, *Large-scale rank of Teichmüller space*, Duke Math. J. 166 (2017) 1517–1572 MR Zbl

[13] **A Lenzhen**, **K Rafi**, **J Tao**, *The shadow of a Thurston geodesic to the curve graph*, J. Topol. 8 (2015) 1085–1118 MR Zbl

[14] **F Luo**, *A presentation of the mapping class groups*, Math. Res. Lett. 4 (1997) 735–739 MR Zbl

[15] **H A Masur**, **Y N Minsky**, *Geometry of the complex of curves, I: Hyperbolicity*, Invent. Math. 138 (1999) 103–149 MR Zbl

[16] **H A Masur**, **Y N Minsky**, *Geometry of the complex of curves, II: Hierarchical structure*, Geom. Funct. Anal. 10 (2000) 902–974 MR Zbl

[17] **Y N Minsky**, *Quasi-projections in Teichmüller space*, J. Reine Angew. Math. 473 (1996) 121–136 MR Zbl

[18] **Y N Minsky**, *Bounded geometry for Kleinian groups*, Invent. Math. 146 (2001) 143–192 MR Zbl

[19] **R C Penner**, **J L Harer**, *Combinatorics of train tracks*, Ann. of Math. Stud. 125, Princeton Univ. Press (1992) MR Zbl

[20] **W-y Yang**, *Genericity of contracting elements in groups*, Math. Ann. 376 (2020) 823–861 MR Zbl

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